CONES GENERATED BY REGULAR MODULES, II

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We continue with the proof of the following result.

Theorem. [3, 8.3] *Let Q be a wild quiver, U, V be indecomposable regular modules with U quasi-simple. There exists $m(U,V) \in \mathbb{N}$ such that $\tau_{kQ}^{-m(U,V)}V$ generates all preinjective modules and all modules in $(\to U)$.*

In view of [2, Theorem 6] we only have to consider the case that Q is a wild quiver with $|Q_0| = 2$, i.e. Q is a wild Kronecker quiver: Let $r \geq 3$. The quiver Γ_r with vertices $1, 2$ and r arrows $\gamma_1, \ldots, \gamma_r \colon 1 \to 2$ is called the (wild) Kronecker quiver with r arrows.

Let $r \geq 3$. One can show that an indecomposable module $M \in \operatorname{mod} k\Gamma_r$ has no self-extensions if and only if M is preprojective or preinjective. In particular, there is no regular tilting module for $k\Gamma_r$. Hence we can not extend the proof of [2, Theorem 6] to this situation. In [7], which we follow from now on, so-called elementary modules are used as a "replacement" for regular tilting modules.

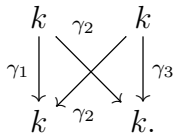
1. Elementary modules

Definition. Let Q be a wild quiver and $E \in \operatorname{mod} kQ$ be a non-zero regular module. We say that E is elementary, provided there is no short exact sequence $0 \to A \to E \to B \to 0$ with A and B regular and non-zero.

Examples. (*a*) Let $M \in \operatorname{mod} kQ$ be a non-zero regular module of minimal dimension, then M is elementary.

(*b*) We consider $r = 3$. Let E be the (regular indecomposable) $k\Gamma_r$-module corresponding to the representation $k \to k$ with $E(\gamma_1) = \mathrm{id}_k$ and $E(\gamma_2) = 0 = E(\gamma_3)$. Let $0 \to E \to F \to E \to 0$ be a short exact sequence that does not split. One can show that F is regular indecomposable. By construction F has dimension vector $\underline{\dim} F = (2, 2)$ and is not elementary.

(*c*) We consider the module G corresponding to the representation



One can show that G is elementary. Note that $\underline{\dim} G = (2, 2) = \underline{\dim} F$.

The proof of the following Lemma may be found in [5].

Lemma 1. *Let E and F be elementary modules and X, Y be regular modules.*

(*a*) *If $f \colon X \to E$ and $g \colon E \to Y$ are non-zero morphisms, then $g \circ f \neq 0$.*

(*b*) $\operatorname{End}_{kQ}(E) \cong k$.

(*c*) *If $\underline{\dim} E = \underline{\dim} F$ and $E \ncong F$, then $0 = \operatorname{Hom}_{kQ}(E, F)$.*

Lemma 2. *Let $E_1, \ldots, E_{l+1}$ be pairwise non-isomorphic elementary kQ-modules such that $\underline{\dim} E_i = \underline{\dim} E_j$ for all i, j and $\tau_{kQ} E_{l+1}$ is generated by $\mathcal{E} := \bigoplus_{i=1}^{l} E_i$. Let M be an indecomposable kQ-module with submodule $U \cong E_{l+1}$ such that $\tau_{kQ}(M/U)$ is generated by $\mathcal{E}$. Then $\tau_{kQ} M$ is generated by $\mathcal{E}$.*

Proof. Consider the short exact sequence $0 \to U \to M \xrightarrow{\pi} M/U \to 0$. Since $U \cong E_{l+1}$ is regular, we obtain the exact sequence

$$0 \to \tau_{kQ}U \xrightarrow{\iota} \tau_{kQ}M \xrightarrow{\tau_{kQ}(\pi)} \tau_{kQ}(M/U) \to 0.$$

Let $i \in \{1, \dots, l\}$ and consider a morphism $f\colon E_i \to \tau_{kQ}(M/U)$. Since $\underline{\dim}\, E_i = \underline{\dim}\, E_{l+1}$, we conclude with the Auslander-Reiten formula and Lemma 1

$$\operatorname{Ext}^1_{kQ}(E_i, \tau_{kQ}E_{l+1}) \cong \operatorname{Hom}_{kQ}(E_{l+1}, E_i) = 0.$$

Since $U \cong E_{l+1}$, we get an exact sequence

$$0 \to \operatorname{Hom}_{kQ}(E_i, \tau_{kQ}U) \to \operatorname{Hom}_{kQ}(E_i, \tau_{kQ}M) \to \operatorname{Hom}_{kQ}(E_i, \tau_{kQ}(M/U)) \to 0,$$

i.e. there is $g\colon E_i \to \tau_{kQ}M$ such that $\tau_{kQ}(\pi) \circ g = f$.

Since $\tau_{kQ}(M/U) \in \operatorname{Gen}(\mathcal{E})$, we find an epimorphism $h\colon \mathcal{E}^a \to \tau_{kQ}(M/U)$. The considerations above show that there is $\varphi\colon \mathcal{E}^a \to \tau_{kQ}M$ such that $\tau_{kQ}(\pi) \circ \varphi = h$. Since $\tau_{kQ}U \cong \tau_{kQ}E_{l+1} \in \operatorname{Gen}(\mathcal{E})$, we find an epimorphism $\psi\colon \mathcal{E}^b \to \tau_{kQ}U$. Now $\tau_{kQ}M = \operatorname{im}(\iota \circ \psi) + \operatorname{im}\varphi$ yields $\tau_{kQ}M \in \operatorname{Gen}(\mathcal{E})$. □

2. The Kronecker case

Now we specialize to the quiver Γ_r. By [7] there exist pairwise non-isomorphic elementary modules $E_1, \dots, E_{l+1}$ for $l := 2r - 1$ with the following properties:

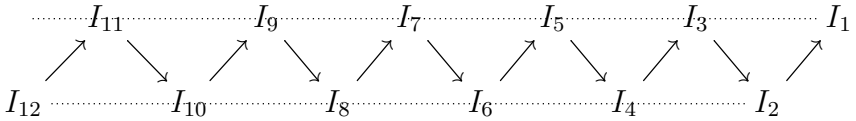
(*i*) $\underline{\dim}\, E_i = (1,1)$ for all $i \in \{1, \dots, l+1\}$. Therefore every proper factor module E_i is isomorphic to the simple injective module $I(1)$.
(*ii*) $\mathcal{E} := \bigoplus_{i=1}^{l} E_i$ generates the injective indecomposable modules $I(1), I(2)$ and $\tau_{k\Gamma_r}I(1)$.
(*iii*) $\tau_{k\Gamma_r}E_{l+1}$ is generated by $\mathcal{E}$.

Remark. Statement (*ii*) and (*iii*) were shown using covering theory of the Kronecker quiver. The proofs are too involved to be presented in these talks.

From now we fix a set of elementary modules in $\operatorname{mod} k\Gamma_r$ that satisfies $(i) - (iii)$.

Lemma 3. *Let I be an preinjective module, then $I \in \operatorname{Gen}(\mathcal{E}) \cap \ker \operatorname{Ext}^1_{k\Gamma_r}(\mathcal{E}, -)$.*

Proof. Clearly every preinjective module is in $\ker \operatorname{Ext}^1_{k\Gamma_r}(\mathcal{E}, -)$.
We define $I_1 := I(1)$ and $I_2 := I(2)$. Now we define inductively $I_t := \tau_{k\Gamma_r}I_{t-2}$ for all $t \geq 3$. The preinjective Auslander-Reiten component looks as follows (all arrows are of multiplicity r):



We do the proof by induction on $t \in \mathbb{N}$. The cases $t = 1, 2, 3$ follow immediatly by property (*ii*). Now assume that $t \geq 4$. We write $I_t = \tau^a_{k\Gamma_r}I(i)$ for some $a \in \mathbb{N}$ and $i \in \{1, 2\}$. We get

$$\dim_k \operatorname{Hom}_{k\Gamma_r}(E_{l+1}, \tau^{-1}_{k\Gamma_r}I_t) = \dim_k \operatorname{Hom}_{k\Gamma_r}(\tau^{-a+1}_{k\Gamma_r}E_{l+1}, I(i)) = \dim_k(\tau^{-a+1}_{k\Gamma_r}E_{l+1})_i \neq 0$$

since every regular $k\Gamma_r$-module U satisfies $\underline{\dim}\, U \in \mathbb{N} \times \mathbb{N}$. Consider a non-zero morphism $f\colon E_{l+1} \to \tau^{-1}_{k\Gamma_r}I_t = I_{t-2}$. Since $t \geq 4$, we have $I_{t-2} \ncong I_1$. Since every proper factor of E_{l+1} is isomorphic to I_1, we conclude that f is injective.
By [1, VIII.2.5] every indecomposable direct summand of $\operatorname{coker} f$ is a proper successor of I_{t-2} in the preinjective component, i.e. of the form I_t for some $s < t - 2$. We apply τ_{kQ} and get a short exact sequence

$$0 \to \tau_{kQ}E_{l+1} \to I_t \to \tau_{kQ} \operatorname{coker} f \to 0$$

where every indecomposable direct summand of $\tau_{kQ}\operatorname{coker} f$ is of the form I_s for some $s < t$. Now the inductive hypothesis yields $\tau_{kQ}\operatorname{coker} f \in \operatorname{Gen}(\mathcal{E})$. We apply Lemma 2 to the short exact sequence $0 \to E_{l+1} \to I_{t-2} \to \operatorname{coker} f \to 0$ to conclude that $I_t \in \operatorname{Gen}(\mathcal{E})$. □

Proposition 4. *Let U be a regular indecomposable module. There exists $m(U) \in \mathbb{N}$ such that $\tau^m_{k\Gamma_r}U \in \operatorname{Gen}(\mathcal{E}) \cap \ker \operatorname{Ext}^1_{k\Gamma_r}(\mathcal{E}, -)$ for all $m \geq m(U)$.*

Proof. Since U possesses a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_k = U$ such that U_i/U_{i-1} is elementary for all $i \in \{1, \ldots, k\}$ and $\operatorname{Gen}(\mathcal{E}) \cap \ker \operatorname{Ext}^1_{k\Gamma_r}(\mathcal{E}, -)$ is closed under extensions, it sufficies to prove the statement for U elementary.
By [2, Lemma 5] there exists $c(E_{l+1}, U) \in \mathbb{N}$ such that $\operatorname{Hom}_{kQ}(E_{l+1}, \tau^m_{k\Gamma_r}U) \neq 0$ for all $m \geq c(E_{l+1}, U)$. Moreover, we find $b(E_{l+1}, \mathcal{E}) \in \mathbb{N}$ such that $0 = \operatorname{Hom}_{k\Gamma_r}(E_{l+1}, \tau^{-m}_{k\Gamma_r}, \mathcal{E})$ for all $m \geq b(E_{l+1}, \mathcal{E})$. We set $m(U) := b(E_{l+1}, \mathcal{E}) + c(E_{l+1}, U) + 1$. Let $m \geq m(U)$, then $m - 1 \geq c(E_{l+1}, U)$. We fix $f\colon E_{l+1} \to \tau^{m-1}_{k\Gamma_r}U$ non-zero. Since every proper factor of E_{l+1} is injective and U is regular, we get a short exact sequence

$$0 \to E_{l+1} \xrightarrow{f} \tau^{m-1}_{k\Gamma_r}U \to \operatorname{coker} f \to 0.$$

Since E_{l+1} is regular and $\tau^{m-1}_{k\Gamma_r}U$ is elementary it follows from [8] that $\operatorname{coker} f$ is preinjective. Now it follows from Lemma 2 and Lemma 3 that $\tau^m_{k\Gamma_r}U = \tau_{k\Gamma_r}(\tau^{m-1}_{k\Gamma_r}U) \in \operatorname{Gen}(\mathcal{E})$.
We write $m = b + c$ with $b \geq b(E_{l+1}, \mathcal{E}) + 1$ and $c \geq c(E_{l+1}, U)$. Assume there is $i \in \{1, \ldots, l\}$ such that

$$\begin{aligned} 0 \neq \dim_k \operatorname{Ext}^1_{k\Gamma_r}(E_i, \tau^m_{k\Gamma_r}U) &= \dim_k \operatorname{Hom}_{k\Gamma_r}(U, \tau^{-m+1}_{k\Gamma_r}E_i) \\ &= \dim_k \operatorname{Hom}_{k\Gamma_r}(\tau^c_{k\Gamma_r}U, \tau^{-b+1}_{k\Gamma_r}E_i). \end{aligned}$$

Let $h\colon \tau^c_{k\Gamma_r}U \to \tau^{-b+1}_{k\Gamma_r}E_i$ be non-zero. Since $c \geq c(E_{l+1}, U)$, we find a non-zero morphism $g\colon E_{l+1} \to \tau^c_{k\Gamma_r}U$. Since $\tau^c_{k\Gamma_r}U$ is elementary, we conclude that $0 \neq h \circ g \in \operatorname{Hom}_{k\Gamma_r}(E_{l+1}, \tau^{-b+1}_{k\Gamma_r}E_i)$. But $\operatorname{Hom}_{k\Gamma_r}(E_{l+1}, \tau^{-(b-1)}_{k\Gamma_r}E_i) = 0$ by the choice of $b \geq b(E_{l+1}, \mathcal{E}) + 1$, a contradiction.
Therefore $\tau^m_{k\Gamma_r}U \in \ker \operatorname{Ext}^1_{k\Gamma_r}(\mathcal{E}, -)$ for all $m \geq m(U)$. □

Theorem 5. *Let $r \geq 3$ and $U, V \in \operatorname{mod} k\Gamma_r$ be indecomposable regular modules with U quasi-simple. There exists $m(U, V) \in \mathbb{N}$ such that $\tau^{-m(U,V)}_{k\Gamma_r}V$ generates all preinjective modules and all modules in $(\to U)$.*

Proof. Let $m(U)$ be such that $\tau^m_{\Gamma_r}U$ is in $\operatorname{Gen}(\mathcal{E}) \cap \ker \operatorname{Ext}^1_{k\Gamma_r}(\mathcal{E}, -)$ for all $m \geq m(U)$. Therefore $\tau^m_{k\Gamma_r}U \in \operatorname{Gen}(\tau^{-m(U)}_{k\Gamma_r}\mathcal{E}) \cap \operatorname{Ext}^1_{k\Gamma_r}(\tau^{-m(U)}_{k\Gamma_r}\mathcal{E}, -)$ for all $m \geq 0$. We conclude $(\to U) \subseteq \operatorname{Gen}(\tau^{-m(U)}_{k\Gamma_r}\mathcal{E})$. By [2, Lemma 5] we find $m(U, V) := a(\tau^{-m(U)}_{k\Gamma_r}\mathcal{E}, V)$ such that $\tau^{-m(U)}_{k\Gamma_r}\mathcal{E}$ is generated by $\tau^{-l}_{k\Gamma_r}V$ for all $l \geq a(\tau^{-m(U)}_{k\Gamma_r}\mathcal{E}, V)$. □

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