CONES GENERATED BY REGULAR MODULES, I

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1. Preliminaries

Let k be an algebraically closed field and Q a finite acyclic connected quiver without loops. We denote by $\operatorname{mod} kQ$ the category of finite dimensional (left) kQ-modules, where kQ is the path algebra of Q. Recall that kQ is a basic, connected and hereditary algebra, i.e. $\operatorname{Ext}^2_{kQ} = 0$. We denote the Auslander-Reiten translation of kQ by τ_{kQ}. Let $X, Y \in \operatorname{mod} kQ$ be finite dimensional kQ-modules. We say that X generates Y or Y is generated by X if there exists $n \in \mathbb{N}$ and an epimorphism $X^n \twoheadrightarrow Y$.

Definition. Let $M \in \operatorname{mod} kQ$ be indecomposable. We say that M is

(a) preprojective if there exists $n \in \mathbb{N}_0$ and a projective indecomposable module P such that $M \cong \tau_{kQ}^{-n} P$.
(b) preinjective if there exists $n \in \mathbb{N}_0$ and an injective indecomposable module I such that $M \cong \tau_{kQ}^{n} I$.
(c) regular if M is neither preprojective nor preinjective.

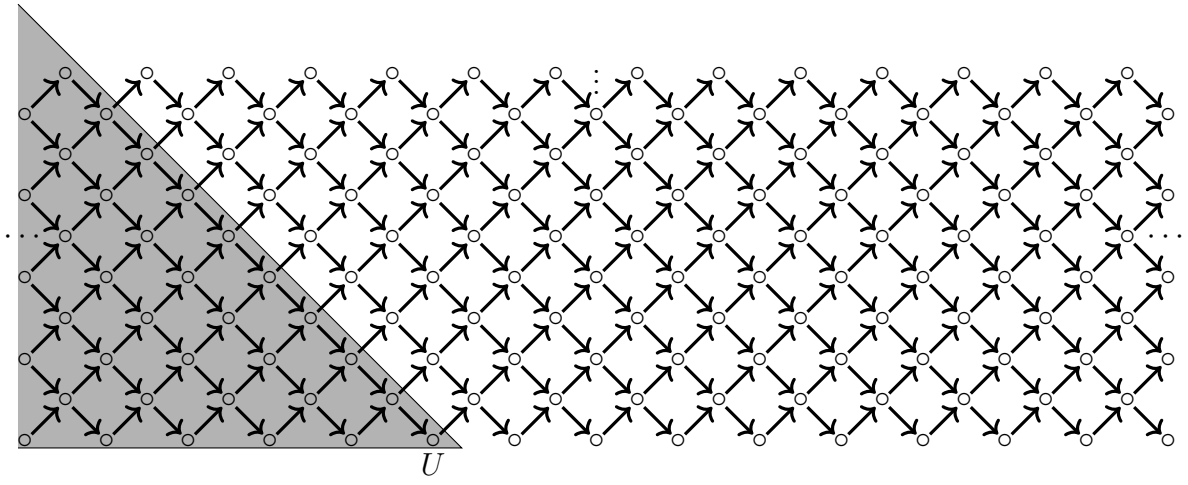
We call an arbitrary module $N \in \operatorname{mod} kQ$ preprojective (preinjective, regular) if all of its direct summands are preprojectve (preinjective, regular).

Definition. A quiver Q is called wild, provided the underlying graph $\overline{Q}$ is not a simply-laced Dynkin or Euclidean diagram.

In these talks we discuss the following result and outline its proof.

Theorem. [2, 8.3] *Let Q be a wild quiver, U, V be indecomposable regular modules with U quasi-simple. There exists $m(U,V) \in \mathbb{N}$ such that $\tau_{kQ}^{-m(U,V)} V$ generates all preinjective modules and all modules in $(\to U)$.*

Recall that every regular component $\mathcal{C}$ for a wild quiver algebra kQ is of type $\mathbb{Z}\mathbb{A}_\infty$. Such a component with quasi-simple module U is illustrated in the following figure. The cone $(\to U)$ of all predecessor of U in $\mathcal{C}$ is given by the grey shaded area.



Let $T \in \operatorname{mod} kQ$ be a module. We define

$$\operatorname{Gen}(T) = \{M \in \operatorname{mod} kQ \mid M \text{ is generated by } T\}.$$

Clearly $\operatorname{Gen}(T)$ is closed under images and direct sums. The picture above suggests that $\operatorname{Gen}(T)$ is also closed under extensions. However, this is in general not the case. In other words, $\operatorname{Gen}(T)$ is not always a torsion class as the following example shows. Assume that $|Q_0| \neq 1$. Since Q is connected, we find an arrow $i \to j \in Q_1$ with $i \neq j$. Hence $0 \neq \operatorname{Ext}^1_{kQ}(S(i), S(j))$ where $S(i), S(j)$ are the simple modules corresponding to the vertices i and j. Let $0 \to S(j) \to E \to S(i) \to 0$ be a non-split exact sequence. The module E is indecomposable and not semisimple, but

$$\operatorname{Gen}(S(i) \oplus S(j)) = \{S(i)^a \oplus S(j)^b \mid a, b \in \mathbb{N}_0\}.$$

Hence $E \notin \operatorname{Gen}(S(i) \oplus S(j))$.
When we consider kQ as the regular module, we get $\operatorname{Gen}(kQ) = \operatorname{mod} kQ$. Hence $\operatorname{Gen}(kQ)$ is closed under extensions. These examples also show that the "size" of $\operatorname{Gen}(T)$ depends on T.
In the next section we study a property for modules $T \in \operatorname{mod} kQ$ that implies that $\operatorname{Gen}(T)$ is closed under extensions.

2. $\operatorname{Gen}(T)$ and Tilting modules

Most of the following proofs may be found in [1].

Lemma 1. *Let $T, M \in \operatorname{mod} kQ$, then $M \in \operatorname{Gen}(T)$ if and only if the kQ-linear map $\varepsilon_M \colon T \otimes_\Gamma \operatorname{Hom}_{kQ}(T, M) \to M$ given by $t \otimes f \to f(t)$ is surjective, where $\Gamma := \operatorname{End}_{kQ}(T)^{op}$.*

Proof. We assume first that M is generated by T. We fix a k-basis of $\{f_1, \dots, f_d\}$ of $\operatorname{Hom}_{kQ}(T, M)$. We define $f := [f_1, \dots, f_d] \colon T^d \to M$. Since M is generated by T, we find $l \in \mathbb{N}$ and an epimorphism $g \colon T^l \to M$. For $i \in \{1, \dots, l\}$ we denote by $\iota_i \colon T \to T^l$ the canonical embedding into the i-th component and define $g_i := g \circ \iota_i$. For $i \in \{1, \dots, l\}$ we find $\lambda_{ij} \in k$ such that $g_i = \sum_{j=1}^d f_j \lambda_{ij}$. Let $\lambda \colon T^l \to T^d$ be the k-linear map given by the matrix $(\lambda_{ij}) \in \operatorname{Mat}_{l \times d}(k)$, i.e. $\lambda(x)_i = \sum_{j=1}^l \lambda_{ji} x_j$ for all $i \in \{1, \dots, d\}$. Then $g = f \circ \lambda$. Since g is surjective, we conlude that f is surjective.
By definition of f the map $f_* := \operatorname{Hom}_{kQ}(T, f)$ is surjective. Hence we get a short exact sequence of Γ-modules

$$0 \to \operatorname{Hom}_{kQ}(T, \ker f) \to \operatorname{Hom}_{kQ}(T, T^d) \xrightarrow{f_*} \operatorname{Hom}_{kQ}(T, M) \to 0.$$

There results a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & T \otimes_\Gamma \operatorname{Hom}_{kQ}(T, \ker f) & \longrightarrow & T \otimes_\Gamma \operatorname{Hom}_{kQ}(T, T^d) & \xrightarrow{\mathrm{id} \otimes f_*} & T \otimes_\Gamma \operatorname{Hom}_{kQ}(T, M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon_{\ker f}} & & \downarrow{\scriptstyle \varepsilon_{T^d}} & & \downarrow{\scriptstyle \varepsilon_M} & & \\
0 & \longrightarrow & \ker f & \longrightarrow & T^d & \xrightarrow{f} & M & \longrightarrow & 0.
\end{array}$$

Note that ε_{T^d} is surjective since $\varepsilon_{T^d}(\sum_{i=1}^d (t_i \otimes \bar{\iota}_i)) = (t_1, \dots, t_d)$ for all $t_i \in T$, where $\bar{\iota}_i \colon T \to T^d$ is the embedding into the i-th coordinate. We conclude that ε_M is surjective.
Now we assume that ε_M is surjective. Since $\operatorname{Hom}_{kQ}(T, M)$ is a finitely generated Γ-module, we find $l \in \mathbb{N}$ and an epimorphism $\Gamma^l \to \operatorname{Hom}_{kQ}(T, M)$. Right exactness of $T \otimes_\Gamma -$ yields an epimorphism

$$f \colon T^l \xrightarrow{\sim} T \otimes_\Gamma \Gamma^l \to T \otimes_\Gamma \operatorname{Hom}_{kQ}(T, M)$$

Hence $\varepsilon_M \circ f$ is surjective and $M \in \operatorname{Gen}(T)$. □

Corollary 2. *Let $T \in \operatorname{mod} kQ$, then $\operatorname{Gen}(T) \cap \ker \operatorname{Ext}^1_{kQ}(T, -)$ is closed under extensions.*

Proof. Consider a short exact sequene $0 \to N \to E \to M \to 0$ with $N, M \in \mathrm{Gen}(T) \cap \ker \mathrm{Ext}^1_{kQ}(T, -)$. Clearly $E \in \ker \mathrm{Ext}^1_{kQ}(T, -)$. We also have a short exact sequence

$$0 \to \mathrm{Hom}_{kQ}(T, N) \to \mathrm{Hom}_{kQ}(T, E) \to \mathrm{Hom}_{kQ}(T, M) \to 0$$

and the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & T \otimes_\Gamma \mathrm{Hom}_{kQ}(T, N) & \longrightarrow & T \otimes_\Gamma \mathrm{Hom}_{kQ}(T, E) & \longrightarrow & T \otimes_\Gamma \mathrm{Hom}_{kQ}(T, M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varepsilon_N} & & \downarrow{\scriptstyle \varepsilon_E} & & \downarrow{\scriptstyle \varepsilon_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Now we use the suitable Four Lemma and Lemma 1 to conclude that $E \in \mathrm{Gen}(T)$. □

Corollary 3. *Let $T \in \mathrm{mod}\, kQ$ be a kQ-module such that $\mathrm{Ext}^1_{kQ}(T, T) = 0$, then $\mathrm{Gen}(T) \subseteq \ker \mathrm{Ext}^1_{kQ}(T, -)$ and $\mathrm{Gen}(T)$ is closed under extensions.*

Proof. Let $M \in \mathrm{Gen}(T)$, then we find a short exact sequence $0 \to \ker f \to T^d \xrightarrow{f} M \to 0$. Since kQ is hereditary, there results an exact sequence

$$0 = \mathrm{Ext}^1_{kQ}(T, T^d) \to \mathrm{Ext}^1_{kQ}(T, M) \to \mathrm{Ext}^2_{kQ}(T, \ker f) = 0.$$

Hence $\mathrm{Gen}(T) \subseteq \ker \mathrm{Ext}^1_{kQ}(T, -)$ and $\mathrm{Gen}(T) = \mathrm{Gen}(T) \cap \ker \mathrm{Ext}^1_{kQ}(T, -)$ is closed under extensions by Corollary 2. □

Definition. A module $T \in \mathrm{mod}\, kQ$ is called a (multiplicity free) tilting module of kQ, provided the following statements hold.

(1) $\mathrm{Ext}^1_{kQ}(T, T) = 0$, and
(2) $T = \bigoplus_{i=1}^n T_i$, with $n = |Q_0|$, $T_1, \ldots, T_n$ pairwise non-isomorphic indecomposable kQ-modules.

We say that T is a (multiplicity free) partial tilting module, provided (1) is satisfied and T decomposes as $T_1 \oplus \cdots \oplus T_l$ with pairwise non-isomorphic indecomposable modules $T_1, \ldots, T_l$.

We omit the proof of the following result.

Theorem 4. *For a partial tilting module $T \in \mathrm{mod}\, kQ$ the following statements are equivalent:*

(*a*) *T is a tilting module.*
(*b*) $\ker \mathrm{Ext}^1_{kQ}(T, -) = \mathrm{Gen}(T)$.

3. Wild quiver with $n \geq 3$ simple modules

The proof of the following Lemma may be found in [2] and [3].

Lemma 5. *Let Q be a wild quiver and let X and Y be non-zero regular kQ-modules. The following statements hold:*

(*a*) *There exists $a(X, Y) \in \mathbb{N}$ such that $X \in \mathrm{Gen}(\tau_{kQ}^{-m} Y)$ for all $m \geq a(X, Y)$.*
(*b*) *There exists $b(X, Y) \in \mathbb{N}$ such that $\mathrm{Hom}_{kQ}(X, \tau_{kQ}^{-m} Y) = 0$ for all $m \geq b(X, Y)$.*
(*c*) *There exists $c(X, Y) \in \mathbb{N}$ such that $\mathrm{Hom}_{kQ}(X, \tau_{kQ}^{m} Y) \neq 0$ for all $m \geq c(X, Y)$.*

Theorem 6. *Let Q be a wild quiver with $n := |Q_0| \geq 3$. Let U, V be indecomposable regular modules with U quasi-simple. There exists $m(U, V) \in \mathbb{N}$ such that $\tau_{kQ}^{-m(U,V)} V$ generates all preinjective modules and all modules in $(\to U)$.*

Proof. Since $n \geq 3$, there exists a tilting module $T \in \mathrm{mod}\, kQ$ such that T is regular (see [3]). Since kQ is hereditary, τ_{kQ} induces an auto-equivalence on the full subcategory of regular

modules. Therefore $\tau_{kQ}^{-m}T$ is a regular tilting module for each $m \in \mathbb{N}$. Consider $b(U,T)$ as in Lemma 5. The Auslander-Reiten formula yields

$$0 = \operatorname{Hom}_{kQ}(U, \tau_{kQ}^{-m}T) \cong \operatorname{Ext}^1_{kQ}(\tau_{kQ}^{-m-1}T, U) \cong \operatorname{Ext}^1_{kQ}(\tau_{kQ}^{-b(U,T)-1}T, \tau_{kQ}^{-b(U,T)+m}U)$$

for all $m \geq b(U,T)$. In other words, the tilting module $S := \tau_{kQ}^{-b(U,T)-1}T$ satisfies $0 = \operatorname{Ext}^1_{kQ}(S, \tau_{kQ}^{l}U)$ for all $l \geq 0$. We apply Theorem 4 and conclude that $\tau_{kQ}^{l}U \in \operatorname{Gen}(S)$ for all $l \geq 0$. Since $\operatorname{Gen}(S)$ is closed under extensions, we conclude that $(\to U) \subseteq \operatorname{Gen}(S)$. By Lemma 5 we find $a(S,V) \in \mathbb{N}$ such that $S \in \operatorname{Gen}(\tau_{kQ}^{-a(S,V)}V)$. Hence $(\to U) \subseteq \operatorname{Gen}(\tau_{kQ}^{-a(S,V)}V)$. We set $m(U,V) := a(S,V)$. To finish the proof it sufficies to show that $\operatorname{Ext}^1_{kQ}(S,X) = 0$ for each preinjective indecomposable module X. The result is trivial if X is injective. Otherwise the Auslander-Reiten formula yields $\operatorname{Ext}^1_{kQ}(S,X) \cong \operatorname{Hom}_{kQ}(\tau_{kQ}^{-1}X, S)$. Since $\operatorname{Hom}_{kQ}(Y,R) = 0$ for all modules $Y, R \in \operatorname{mod} kQ$ with Y preinjective and R regular the claim follows. □

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, I: Techniques of Representation Theory*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2006.

[2] O. Kerner, *More Representations of Wild Quivers*, Contemporary Mathematics **607** (2014), 35-66.

[3] ———, *Representations of Wild Quivers*, Representation theory of algebras and related topics, CMS Conf. Proc. **19** (1996), 65-107.