Wild Kronecker algebras and Steiner bundles on Grassmannians

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Notation

- Let $\Bbbk$ be a field (of arbitrary characteristic).
- Throughout A denotes a finite dimensional $\Bbbk$-algebra,i.e. $A = (A, \iota)$ consists of a ring $A \neq 0$ and a ring homomorphism $\iota\colon \Bbbk \to A$ such that
  - (i) the $\Bbbk$-vector space A is finite dimensional, and
  - (ii) $\Bbbk \cong \operatorname{im}\iota \subseteq Z(A) := \{x \in A \mid \forall a \in A\colon xa = ax\}$.

For example:

1. $\iota\colon \Bbbk \to \mathrm{Mat}_n(\Bbbk); \lambda \mapsto \lambda \cdot I_n$
2. $\iota\colon \mathbb{R} \to \mathbb{H}; \lambda \mapsto \lambda \cdot 1 + 0 \cdot i + 0 \cdot j + 0 \cdot k$
3. $\iota\colon \Bbbk \to \Bbbk[X]/(f); \lambda \mapsto \lambda \cdot X^0 + (f)$

Representation Theory (of A)

Roughly: Represent $a \in A$ as a $\Bbbk$-linear map on vector spaces.

An A-module M consists of a f.d. $\Bbbk$-vector space and a map $A \times M \to M; (a, m) \mapsto a.m$ that satisfies the "vector space axioms": $1.m = m$, $(a_1 + a_2).m = a_1.m + a_2.m, \ldots$

An A-linear map $f : M \to N$ of A-modules is a $\Bbbk$-linear map s.t. $f(a.m) = a.f(m)$ for all $a \in A$ and $m \in M$.

Study $\operatorname{mod} A$ = category of A-modules (has submodules, direct sums,...).

Definition

$\{0\} \neq M \in \operatorname{mod} A$ is called indecomposable if $M \cong N \oplus L$ implies $N = \{0\}$ or $L = \{0\}$.

Consequences of KRS

Theorem (Krull-Remak-Schmidt, 1911-1928)

Let $M \in \operatorname{mod} A$. There exists a (up to isomorphism and order) unique decomposition $M = \bigoplus_{i=1}^{n} M_i$ into indecomposable modules $M_1, \ldots, M_n \in \operatorname{mod} A$.

$\rightsquigarrow$ Study $\operatorname{ind} A :=$ isoclasses of indec. modules (and morphisms).

Example

1. $A = \Bbbk$. Then $\operatorname{ind} A = \{\Bbbk\}$ and $\operatorname{Hom}_A(\Bbbk, \Bbbk) = \Bbbk \operatorname{id}_{\Bbbk}$ (LA I).
2. $A = \operatorname{Mat}_n(\Bbbk)$. Then $\operatorname{ind} A = \{\Bbbk^n\}$ and $\operatorname{Hom}_A(\Bbbk^n, \Bbbk^n) = \Bbbk \operatorname{id}_{\Bbbk^n}$.

$\rightsquigarrow \operatorname{mod} \Bbbk \cong \operatorname{mod} \operatorname{Mat}_n(\Bbbk)$.

3. $A = \Bbbk[X]/(X^n)$ and $x := X + (X^n)$.The indecomposable modules are

$$\{0\} \subsetneq (x^{n-1}) \subsetneq (x^{n-2}) \subsetneq \cdots \subsetneq (x) \subsetneq A \text{ (LA II)}.$$

A reminder - The Jordan canonical form

Let $A = \Bbbk[X]/(X^n)$, $x := X + (X^n)$ and $M \in \operatorname{mod} A$.

Observations:

- M determined by $x_M \colon M \to M; m \mapsto x.m$.
- x_M is nilpotent since $x^n = 0_A$.
- A subspace $M' \subseteq M$ is a submodule iff M' is x_M-invariant.
- $\Bbbk$-linear map $f \colon M \to N$ is A-linear iff $f \circ x_M = x_N \circ f$.
- $M \cong N$ iff there is $f \colon M \to M'$ isomorphism of vector spaces such that $f \circ x_M \circ f^{-1} = x_N$,i.e. $x_M \sim x_N$ are similar.
- M is indecomposable iff M allows no decomposition into x_M-invariant subspaces iff $x_M \sim J_{\dim_\Bbbk M}$ (Jordan block).

$\Longrightarrow$ $\leq n$ isoclasses $+ \{0\} \subsetneq (x^{n-1}) \subsetneq (x^{n-2}) \subsetneq \cdots \subsetneq (x) \subsetneq A$.

Semisimple algebras

Let $n \geq 2$: $\{0\} \subsetneq (x^{n-1}) \subsetneq (x^{n-2})$ is a proper submodule of (x^{n-2}).

Definition

A $\Bbbk$-algebra A is called semisimple if every indecomposable module has no proper submodules.

From now on $\Bbbk = \overline{\Bbbk}$.

Theorem (Artin-Wedderburn, 1908)

Let A be semisimple, then

$$A \cong \mathrm{Mat}_{n_1}(\Bbbk) \times \mathrm{Mat}_{n_2}(\Bbbk) \times \cdots \times \mathrm{Mat}_{n_\ell}(\Bbbk) \text{ and } |\operatorname{ind} A| = \ell.$$

Group algebras and semisimplicity

Let G be a finite group, then $\Bbbk G := \bigoplus_{g \in G} \Bbbk g$ obtains the structure of a $\Bbbk$-algebra by extending the multiplication of G bilinearly to $\Bbbk G$.

Theorem (Maschke, 1899)

The algebra $\Bbbk G$ is semisimple iff $\operatorname{char}(\Bbbk) \nmid \operatorname{ord}(G)$.

Example

Let $p := \operatorname{char}(\Bbbk) > 0$ and $E_r := (\mathbb{Z}/(p))^r = \langle g_1, \ldots, g_r \rangle$ for $r \in \mathbb{N}$.

- $\Bbbk[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p) \xrightarrow{\sim} \Bbbk E_r; x_i \mapsto g_i - 1$.
- We know $\Bbbk E_1 \cong \Bbbk[X]/(X^p)$ and $\operatorname{ind} \Bbbk E_1 \cong \operatorname{ind} \Bbbk[X]/(X^p)$.
- $r \geq 2 \implies \operatorname{ind} \Bbbk E_r = ???$

Wild Representation Type

Definition

The $\Bbbk$-algebra A has wild representation type if for every $\Bbbk$-algebra B, there is a representation embedding $T\colon \operatorname{mod} B \hookrightarrow \operatorname{mod} A$ (exact, respects isomorphisms, takes indec. to indec.).

$\rightsquigarrow \operatorname{mod} B$ is contained in $\operatorname{mod} A$.

$\rightsquigarrow$ Classification of $\operatorname{ind} A$ is as difficult as classification of $\operatorname{ind} B$.

Group algebras and wildness

Assume that $p := \mathrm{char}(\Bbbk) > 0$ and $p \mid \mathrm{ord}(G)$.

Theorem (Bondarenko-Drozd, 1977)

Assume that $p \neq 2$ and let P be a Sylow-p-subgroup of G.
Then $\Bbbk G$ is of wild rep. type iff P is not cyclic.

Example

Let $E_r \cong (\mathbb{Z}/(p))^r$ be an elementary abelian p-group of rank r.
Then $\Bbbk E_r$ is of wild rep. type iff $p \neq 2$ and $r \geq 2$.

Dade (1978): "There are just too many modules over p-groups!"

Hartshorne (1978): "The problem [of finding VBs of rank $< \dim \mathbb{P}^{r-1}$] is intriguing because there is no particular reason why they should not exist, yet they have proven extremly difficult to find".

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Problem: too hard! Almost always wild.
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- *Study invariants* (cohomology, support varieties, local behaviour, coherent sheaves associated to representations)
- *Classify coarser structures* (thick subcategories)
- *Study special classes of representations* (e.g., modules of constant Jordan type)

Modules and vector bundles, I

Let $\Bbbk E_r = \Bbbk[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$ and $x \in V := \bigoplus_{i=1}^r \Bbbk x_i$.

- We have $x^p = 0$ (Freshman's dream: $(a+b)^p = a^p + b^p$).
- $M \in \operatorname{mod} \Bbbk E_r$, then $x_M \colon M \to M, m \mapsto x.m$ is nilpotent.
- $\operatorname{JT}(x_M) = [1]^{a_1}[2]^{a_2} \cdots [p]^{a_p}$ denotes Jordan type of M at x, i.e. the Jordan canonical form of x_M has exactly $a_i \in \mathbb{N}_0$ blocks of size i for $i \in \{1, \dots, p\}$.

Definition

A module $M \in \operatorname{mod} \Bbbk E_r$ has

- constant Jordan type, if $\operatorname{JT}(x_M) = \operatorname{JT}(y_M)\ \forall x, y \in V \setminus \{0\}$.
- constant rank, if $\operatorname{rk}(x_M) = \operatorname{rk}(y_M)\ \forall x, y \in V \setminus \{0\}$.

$\rightsquigarrow$ full subcategories $\operatorname{CJT}(\Bbbk E_r) \subseteq \operatorname{CR}(\Bbbk E_r) \subseteq \operatorname{mod} \Bbbk E_r$.

Modules and vector bundles, II

Theorem (Friedlander-Pevtsova, 2011)

There exists a functor $\widetilde{\Theta}\colon \operatorname{mod} \Bbbk E_r \to \operatorname{Coh}(\mathbb{P}^{r-1})$ *such that for* $M \in \operatorname{mod} \Bbbk E_r$ *the following statements are equivalent:*

1. $M \in \operatorname{CR}(\Bbbk E_r)$.
2. $\widetilde{\Theta}(M)$ *is a vector bundle on* $\mathbb{P}^{r-1}$.

Question: Is $\operatorname{CR}(\Bbbk E_r) \neq \varnothing$?

Theorem (Bondarenko-Lytvynchuk, 2012)

The category $\operatorname{CJT}(\Bbbk E_r) \subseteq \operatorname{CR}(\Bbbk E_r)$ *is wild iff* $\Bbbk E_r$ *is wild.*

Modules and vector bundles, III

Similar developments for the Kronecker quiver

$$K_r = \quad 1 \;\; \overset{\gamma_1}{\underset{\gamma_r}{\overset{\gamma_2}{\underset{\vdots}{\rightrightarrows}}}} \;\; 2$$

Setting:

- $\mathrm{char}(\Bbbk) = 0$ or even $\Bbbk = \mathbb{C}$.
- Connection between VB on $\mathbb{P}^{r-1}$ and Kronecker representations first systematically used by Hulek (1981).
- First functorial approach due to Jardim-Prata (2015).
  - The result is formulated in very general terms.
  - Relation to coherent sheaves on smooth projective varieties.
  - Involved categories are abstractly described.

Modules and vector bundles, IV

Let $\Bbbk = \overline{\Bbbk}$ of arbitrary characteristic and $1 \leq d < r$.

Theorem (B.-Farnsteiner, 2022)

There exists a right exact functor $\widetilde{\Theta}_d \colon \operatorname{rep}(K_r) \to \operatorname{Coh}(\operatorname{Gr}_d(\Bbbk^r))$ *s.t. for* $M \in \operatorname{rep}(K_r)$ *the following statements are equivalent:*

1. $M \in \operatorname{CR}(K_r, d) \subseteq \operatorname{rep}(K_r)$.
2. $\widetilde{\Theta}_d(M)$ *is a vector bundle on* $\operatorname{Gr}_d(\Bbbk^r)$.

Restriction to $\operatorname{rep}_{\mathrm{inj}}(K_r, d) \subseteq \operatorname{CR}(K_r, d)$ *yields an exact equivalence* $\widetilde{\Theta}_d \colon \operatorname{rep}_{\mathrm{inj}}(K_r, d) \to \operatorname{StVect}(\operatorname{Gr}_d(\Bbbk^r))$.

How can we use this?

- Both categories are interesting in their own right and well-studied ($d = 1$: $\mathrm{Gr}_d(\Bbbk^r) = \mathbb{P}^{r-1}$).
- The methods are very different (non-exhaustive list):

$\mathrm{rep}(K_r)$	$\mathrm{StVect}(\mathrm{Gr}_d(\Bbbk^r))$
Studied for $\Bbbk = \overline{\Bbbk}$	Usually $\Bbbk = \mathbb{C}$
$\Bbbk K_r$ non-commutative	Realm of commutative algebra
Auslander-Reiten theory,...	-
Hereditary category	-
-	Stability
-	Homogeneity and equivariance
-	Uniformity
-	Powerful tools from alg. geometry

- $\mathrm{char}(\Bbbk) = p$: $\mathrm{rep}(K_r) \mathrel{\hat{=}} \Bbbk E_r$-modules of Loewy length ≤ 2.

Kronecker representations, I

Recall:

$$K_r = \quad 1 \;\overset{\gamma_1}{\underset{\gamma_r}{\overset{\gamma_2}{\vdots}}}\; 2$$

We call $A_r := \bigoplus_{i=1}^{r} \Bbbk\gamma_i$ the arrow space.

- A representation $M \in \mathrm{rep}(K_r)$ is a triple (M_1, M_2, ψ_M) consisting of vector spaces M_1, M_2 and a $\Bbbk$-linear map
$$\psi_M \colon A_r \otimes_{\Bbbk} M_1 \to M_2.$$
- A morphism $f : M \to N$ of representations is a pair (f_1, f_2) of $\Bbbk$-linear maps $f_i \colon M_i \to N_i$ such that
$$\begin{array}{ccc} A_r \otimes_{\Bbbk} M_1 & \xrightarrow{\psi_M} & M_2 \\ {\scriptstyle \mathrm{id}_{A_r} \otimes f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_r \otimes_{\Bbbk} N_1 & \xrightarrow{\psi_N} & N_2 \end{array}$$
commutes.

Kronecker representations, II

Example

Let $M, N \in \mathrm{rep}(K_1)$.

- $A_1 \otimes_{\Bbbk} M_1 \cong M_1 \implies M \mathrel{\hat{=}} \psi_M \colon M_1 \to M_2$ $\Bbbk$-linear.
- $M \cong N \Leftrightarrow \exists f_i \colon M_i \to N_i$ $\Bbbk$-isomorphisms : $\psi_N \circ f_1 = f_2 \circ \psi_M$
  $\Leftrightarrow \psi_N = f_2 \circ \psi_M \circ f_1^{-1}$
  $\overset{\text{LA I}}{\Leftrightarrow} \mathrm{rk}(\psi_N) = \mathrm{rk}(\psi_M)$ and $\dim_{\Bbbk} M_i = \dim_{\Bbbk} N_i$.
- Let $s := \mathrm{rk}(\psi_M)$ and $m_i := \dim_{\Bbbk} M_i$, then
  $\mathrm{rk}(s(\Bbbk \overset{\mathrm{id}_{\Bbbk}}{\to} \Bbbk) \oplus (m_2 - s)(0 \to \Bbbk) \oplus (m_1 - s)(\Bbbk \to 0)) = s$.
- $\mathrm{ind}(K_1) = \{\Bbbk \overset{\mathrm{id}_{\Bbbk}}{\to} \Bbbk, 0 \to \Bbbk, \Bbbk \to 0\} = \{P_1(1), P_2(1), \Bbbk \to 0\}$.

In general, the representations

$$P_1(r) := \psi_{P(1)} \colon A_r \otimes_{\Bbbk} \Bbbk \to \Bbbk^r; \gamma_i \otimes 1 \mapsto e_i,$$

$$P_2(r) := \psi_{P(2)} \colon A_r \otimes_{\Bbbk} \{0\} \to \Bbbk; \gamma_i \otimes 0 \mapsto 0$$

in $\mathrm{rep}(K_r)$ are indecomposable and are called PIMs.

Kronecker representations, III

Let $1 \leq d < r$, denote by $\mathrm{Gr}_d(A_r)$ the Grassmannian

$$\mathrm{Gr}_d(A_r) := \{\mathfrak{v} \subseteq A_r \mid \dim_{\Bbbk} \mathfrak{v} = d\}.$$

Recall:

$$\begin{array}{ccc} \mathrm{CR}(K_r, d) & \xrightarrow{\widetilde{\Theta}_d} & \mathrm{Vect}(\mathrm{Gr}_d(A_r)) \\ \uparrow & & \uparrow \\ \mathrm{rep}_{\mathrm{inj}}(K_r, d) & \underset{\widetilde{\Theta}_d}{\xrightarrow{\cong}} & \mathrm{StVect}(\mathrm{Gr}_d(A_r)). \end{array}$$

Definition

- $\mathrm{CR}(K_r, d) := \{M \mid \mathrm{rk}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) \text{ ind. of } \mathfrak{v} \in \mathrm{Gr}_d(A_r)\}$.
- $\mathrm{rep}_{\mathrm{inj}}(K_r, d) := \{M \mid \ker(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = \{0\}\ \forall\, \mathfrak{v} \in \mathrm{Gr}_d(A_r)\}$.

Theorem (B. 2017)

The category $\mathrm{rep}_{\mathrm{inj}}(K_r, d)$ *is wild iff* $r \geq 3$.

Restrictions to K_d, I

Let $M = (M_1, M_2, \psi_M) \in \mathsf{rep}(K_r)$.

- We define $\Delta_M(d) := \dim_{\Bbbk} M_2 - d \dim_{\Bbbk} M_1$.
- For $\alpha \in \mathsf{Inj}_{\Bbbk}(A_d, A_r) := \{\beta \in \mathsf{Hom}_{\Bbbk}(A_d, A_r) \mid \ker \beta = 0\}$ let

$$\psi_{\alpha^*(M)} \colon A_d \otimes_{\Bbbk} M_1 \overset{\alpha \otimes \mathsf{id}_{M_1}}{\to} A_r \otimes_{\Bbbk} M_1 \overset{\psi_M}{\to} M_2,$$

and call $\alpha^*(M) := (M_1, M_2, \psi_{\alpha^*(M)}) \in \mathsf{rep}(K_d)$ restriction of M to K_d (along α).

Proposition (B.-Farnsteiner, 2023)

Let $1 \leq d < r$. The following statements are equivalent:

1. $M \in \mathsf{rep}_{\mathsf{inj}}(K_r, d)$.
2. $\alpha^*(M) \cong \dim_{\Bbbk} M_1 P_1(d) \oplus \Delta_M(d) P_2(d)$ $\forall \alpha \in \mathsf{Inj}_{\Bbbk}(A_d, A_r)$.

Restrictions to K_d, II

Sketch of proof for $d = 1$.

Let $M \in \mathsf{rep}_{\mathsf{inj}}(K_r, 1)$ and $\alpha \in \mathsf{Inj}_{\Bbbk}(A_1, A_r)$. By KRS there are $a_\alpha, b_\alpha, c_\alpha \in \mathbb{N}_0$ s.t.

$$\alpha^*(M) \cong a_\alpha \underbrace{(\Bbbk \xrightarrow{\mathsf{id}} \Bbbk)}_{=P_1(1)} \oplus b_\alpha \underbrace{(0 \to \Bbbk)}_{=P_2(1)} \oplus c_\alpha (\Bbbk \to 0).$$

We have

- $\operatorname{im}\alpha \in \mathsf{Gr}_1(A_r)$,
- $\operatorname{im}\psi_{\alpha^*(M)} = \operatorname{im}(\psi_M \circ (\alpha \otimes \mathsf{id}_{M_1})) = \operatorname{im}(\psi_M|_{\operatorname{im}\alpha \otimes_{\Bbbk} M_1})$, and
- $\dim_{\Bbbk} M_1 = \mathsf{rk}(\psi_M|_{\operatorname{im}\alpha \otimes_{\Bbbk} M_1}) = \mathsf{rk}(\psi_{\alpha^*(M)}) = a_\alpha$.

$$\begin{aligned} \implies (a_\alpha, b_\alpha, c_\alpha) &= (\dim_{\Bbbk} M_1, \dim_{\Bbbk} M_2 - \dim_{\Bbbk} M_1, 0) \\ &= (\dim_{\Bbbk} M_1, \Delta_M(1), 0). \end{aligned}$$

What are algebraic vector bundles?

Theorem

There are (at least) two equivalent ways to think about algebraic vector bundles on varieties.

Today: Study a variety via its module category.

Definition

Let $\mathbb{A}^r := \Bbbk^r$ and $R := \Bbbk[X_1, \ldots, X_r]$.Given $F \subseteq R$ we define

$$Z(F) := \{x \in \mathbb{A}^r \mid f(x) = 0 \ \forall \ f \in F\}.$$

We have

1. $Z(\{0_R\}) = \mathbb{A}^r$, $Z(R) = \varnothing$,
2. $Z(F) \cup Z(G) = Z((F) \cap (G))$, and
3. $\bigcap_{i \in I} Z(F_i) = Z(\bigcup_{i \in I} F_i)$.

$\rightsquigarrow$ topology on $\mathbb{A}^r$ called Zariski topology.

The Zariski topology

$\mathbb{A}^r$ is irreducible, i.e. the following equivalent definitions hold.

1. $\mathbb{A}^r = Z(F_1) \cup Z(F_2) \implies Z(F_1) = \mathbb{A}^r$ or $Z(F_2) = \mathbb{A}^r$.
2. $\overline{U} = \mathbb{A}^r$ for $\varnothing \neq U \subseteq \mathbb{A}^r$ open.
3. $\bigcap_{i=1}^n U_i \neq \varnothing$ for $U_1, \ldots, U_n \subseteq \mathbb{A}^r$ open non-empty.

We have $\chi_{AB} = \chi_{BA}$ for all $A, B \in \mathrm{Mat}_n(\Bbbk) \cong \mathbb{A}^{n^2}$

- Let $\det := \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^n X_{i\sigma(i)} \in \Bbbk[X_{ij} \mid 1 \leq i, j \leq n]$. Then $\mathrm{GL}_n(\Bbbk) = \mathrm{Mat}_n(\Bbbk) \setminus Z(\det)$ is open.
- There are $c_\ell \in \Bbbk[X_{ij} \mid 1 \leq i, j \leq n]$ for $\ell \in \{0, \ldots, n\}$ s.t. $\chi_D = \sum_{i=0}^n c_i(D) t^i \in \Bbbk[t]$ f.a. $D \in \mathrm{Mat}_n(\Bbbk)$.
- Fix $B \in \mathrm{Mat}_n(\Bbbk)$,then $X := \{A \in \mathrm{Mat}_n(\Bbbk) \mid c_\ell(AB) - c_\ell(BA) = 0 \ \forall \ell\}$ is closed.
- Let $A \in \mathrm{GL}_n(\Bbbk)$, then $AB = A(BA)A^{-1}$ and $\chi_{AB} = \chi_{A(BA)A^{-1}} = \chi_{BA}$. Therefore $\mathrm{GL}_n(\Bbbk) \subseteq X \subseteq \mathrm{Mat}_n(\Bbbk)$.

The sheaf of regular functions

Given $U \subseteq \mathbb{A}^r$ open, set

$$\mathcal{O}_{\mathbb{A}^r}(U) := \{r \in \mathrm{Map}(U, \Bbbk) \mid \text{ locally } r = \frac{f}{g}, f, g \in R\}.$$

This renders $(\mathbb{A}^r, \mathcal{O}_{\mathbb{A}^r})$ a space with functions:

Definition

A space with functions $(X, \mathcal{O}_X)$ consists of a top. space X and $\Bbbk$-algebras $\mathcal{O}_X(U) \subseteq \mathrm{Map}(U, \Bbbk)$ for all $U \subseteq X$ open, s.t.

(i) For $V \subseteq U$ open and $r \in \mathcal{O}_X(U)$ we have $r|_V \in \mathcal{O}_X(V)$.

(ii) Given $U = \bigcup_{i \in I} U_i$ and $r_i \in \mathcal{O}_X(U_i)$ such that for all $i, j \in I$

$$r_i|_{U_i \cap U_j} = r_j|_{U_i \cap U_j},$$

the map $r \colon U \to \Bbbk; x \in U_i \mapsto r_i(x)$ is in $\mathcal{O}_X(U)$.

Modules for spaces with functions

$(X, \mathcal{O}_X)$ spaces with functions, $\operatorname{Mod} \mathcal{O}_X$ category of $\mathcal{O}_X$-modules (sheaf version of A-modules) and $\mathcal{F} \in \operatorname{Mod} \mathcal{O}_X$.

- $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module for $U \subseteq X$ open.
- $\mathcal{F}$ is called free (of rank $n \in \mathbb{N}$), if $\mathcal{F} \cong (\mathcal{O}_X)^n$.
- $\mathcal{F}$ is called locally free or vector bundle, if there is an open cover $X = \bigcup_{i \in I} X_i$ s.t. $\mathcal{F}|_{X_i}$ is a free $\mathcal{O}_{X_i}$-module for all $i \in I$.
- $\operatorname{Vect}(X) = \operatorname{Vect}(\mathcal{O}_X)$.

$$\underbrace{\operatorname{Vect}(X) = \operatorname{Vect}(\mathcal{O}_X)}_{\text{not abelian}} \subseteq \operatorname{Coh}(X) = \operatorname{Coh}(\mathcal{O}_X) \subseteq \underbrace{\operatorname{Mod}(X) = \operatorname{Mod}(\mathcal{O}_X)}_{\text{abelian category}}.$$

Serre's conjecture

Theorem (Serre (FAC), 1955)

Let $R = \Bbbk[X_1, \ldots, X_r]$.

1. *There exists a functor* $\tilde{\cdot}\colon \operatorname{mod} R \to \operatorname{Mod} \mathcal{O}_{\mathbb{A}^r}; M \mapsto \tilde{M}$ *that is an exact equivalence onto its image* $\operatorname{Coh}(\mathcal{O}_{\mathbb{A}^r}) = \operatorname{Coh}(\mathbb{A}^r)$.
2. $M \in \operatorname{mod} R$ *is free iff* $\widetilde{M}$ *is free.*
3. $M \in \operatorname{mod} R$ *is projective iff* $\widetilde{M}$ *is locally free (= VB).*

Conjecture (Serre)

Serre's Problem: Are locally free $\mathcal{O}_{\mathbb{A}^r}$-modules free?

Theorem (Quillen-Suslin, 1975)

Yes.

Vector bundles on $\mathbb{P}^{r-1}$

Let $\mathbb{P}^{r-1} := \{\Bbbk v \mid v \in \Bbbk^r \setminus \{0\}\}$ and $R = \Bbbk[X_1, \ldots, X_r]$.

- Zariski-Topology: The closed sets of $\mathbb{P}^{r-1}$ are $Z_+(F) := \{x \in \mathbb{P}^{r-1} \mid f(x) = 0 \ \forall\ f \in F\}$ for $F \subseteq R$.
- Sheaf of regular functions: Given $U \subseteq \mathbb{P}^r$ open, set

$$\mathcal{O}_{\mathbb{P}^{r-1}}(U) := \{r \in \mathrm{Map}(U, \Bbbk) \mid \text{ locally } r = \frac{f}{g}, \deg(f) = \deg(g)\}.$$

Theorem (Atiyah, 1956)

The category of vector bundles $\mathrm{Vect}(\mathbb{P}^{r-1})$ *is a KRS category.*

Theorem (Grothendieck, 1957)

Let $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^1)$. *There exist uniquely determined* $(a_i)_{i\in\mathbb{Z}}$ *such that* $\mathcal{F} \cong \bigoplus_{i\in\mathbb{Z}} a_i \mathcal{O}_{\mathbb{P}^1}(i)$.

Way less is known for $\mathbb{P}^{r-1}$ for $r \geq 3$ and $\mathrm{Gr}_d(A_r)$.

Steiner bundles on Grassmannians

The projective variety $(\mathrm{Gr}_d(A_r), \mathcal{O}_{\mathrm{Gr}_d(A_r)})$, called Grassmannian is obtained via structure transport along the Plücker embedding

$$\mathrm{pl}_{A_r}\colon\ \mathrm{Gr}_d(A_r) \hookrightarrow \mathbb{P}(\bigwedge^d A_r); \langle a_1, \ldots, a_d\rangle_{\Bbbk} \mapsto \Bbbk(a_1 \wedge a_2 \wedge \ldots \wedge a_d).$$

Definition (Arrondo-Marchesi, 2015)

A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathrm{Gr}_d(A_r))$ is called Steiner bundle if there exist vector spaces V_1, V_2 and a short exact sequence

$$0 \to V_1 \otimes_{\Bbbk} \mathcal{U}_{(r,d)} \to V_2 \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(A_r)} \to \mathcal{F} \to 0,$$

where $\mathcal{U}_{(r,d)}$ denotes the universal subbundle of $\mathrm{Gr}_d(A_r)$.

Remarks

For $d = 1$, we have $\mathrm{Gr}_d(A_r) \cong \mathbb{P}^{r-1}$ and $\mathcal{U}_{(r,d)} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(-1)$.

Application: Uniform bundles, I

Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$ and $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}(A_r))$.

- $\hat{\alpha}\colon \mathbb{P}(A_2) \hookrightarrow \mathbb{P}(A_r); \Bbbk x \mapsto \Bbbk\alpha(x)$,
- $\hat{\alpha}^*(\mathcal{F}) \in \mathrm{Vect}(\mathbb{P}(A_2))$ such that $\hat{\alpha}^*(\mathcal{F}) \cong \hat{\beta}^*(\mathcal{F})$ if $\operatorname{im}\alpha = \operatorname{im}\beta$.

Definition

Let $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$ and $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$ s.t. $\operatorname{im}\alpha = \mathfrak{v}$. The bundle

$$\mathcal{F}|_{\mathbb{P}(\mathfrak{v})} := \hat{\alpha}^*(\mathcal{F}) \in \mathrm{Vect}(\mathbb{P}(A_2)) \cong \mathrm{Vect}(\mathbb{P}^1)$$

is called restriction of $\mathcal{F}$ to $\mathfrak{v}$.

$$\mathcal{F}|_{\mathbb{P}(\mathfrak{v})} \cong \bigoplus_{i\in\mathbb{Z}} a_i(\mathfrak{v})\mathcal{O}_{\mathbb{P}(\mathfrak{v})}(i) \in \mathrm{Vect}(\mathbb{P}(\mathfrak{v})) \text{ (Grothendieck)}$$

$\rightsquigarrow$ Consider $(a_i(\mathfrak{v}))_{\mathfrak{v}\in\mathrm{Gr}_2(A_r)}$ as an analogue of the Jordan type.

Application: Uniform bundles, II

Proposition

Let $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ *and* $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$ *s.t.* $\widetilde{\Theta}_1(M) \cong \mathcal{F}$. *The following statements are equivalent.*

1. $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2) \subseteq \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$.
2. $\mathcal{F}|_{\mathbb{P}(\mathfrak{v})} \cong \dim_{\Bbbk} M_1 \mathcal{O}_{\mathbb{P}^1}(1) \oplus \Delta_M(2)\mathcal{O}_{\mathbb{P}^1}$ *f.a.* $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$.

Sketch of proof.

„$\Rightarrow$“: Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$ and $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$. We have

$$\alpha^*(M) \cong \dim_{\Bbbk} M_1 P_1(2) \oplus \Delta_M(2) P_2(2) \text{ (PIMs)}.$$

Let $\mathfrak{v} = \mathrm{im}\,\alpha \in \mathrm{Gr}_2(A_r)$, then

$$\begin{aligned}\mathcal{F}|_{\mathbb{P}(\mathfrak{v})} &\cong \hat{\alpha}^*(\mathcal{F}) \cong \hat{\alpha}^*(\widetilde{\Theta}_1(M)) \cong \widetilde{\Theta}_1(\alpha^*(M)) \\ &\cong \dim_{\Bbbk} M_1 \widetilde{\Theta}_1(P_1(2)) \oplus \Delta_M(2)\widetilde{\Theta}_1(P_2(2)) \\ &\cong \dim_{\Bbbk} M_1 \mathcal{O}_{\mathbb{P}^1}(1) \oplus \Delta_M(2)\mathcal{O}_{\mathbb{P}^1}.\end{aligned}$$

□

Application: Uniform bundles, III

$\implies \mathcal{F} \cong \widetilde{\Theta}_1(M)$ for $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$ is uniform of 1-type.

Theorem (Marchesi, Miro-Roig, 2021)

Let $c \in \mathbb{N}_{\geq 4}$. There exists a Steiner bundle $\mathcal{F}_c \in \mathrm{StVect}(\mathbb{P}^2(\mathbb{C}))$ with $c_1(\mathcal{F}) = c$ and $\mathrm{rk}(\mathcal{F}) = c + 2$ that is uniform of 1-type and not homogeneous.

Theorem (B. 2023)

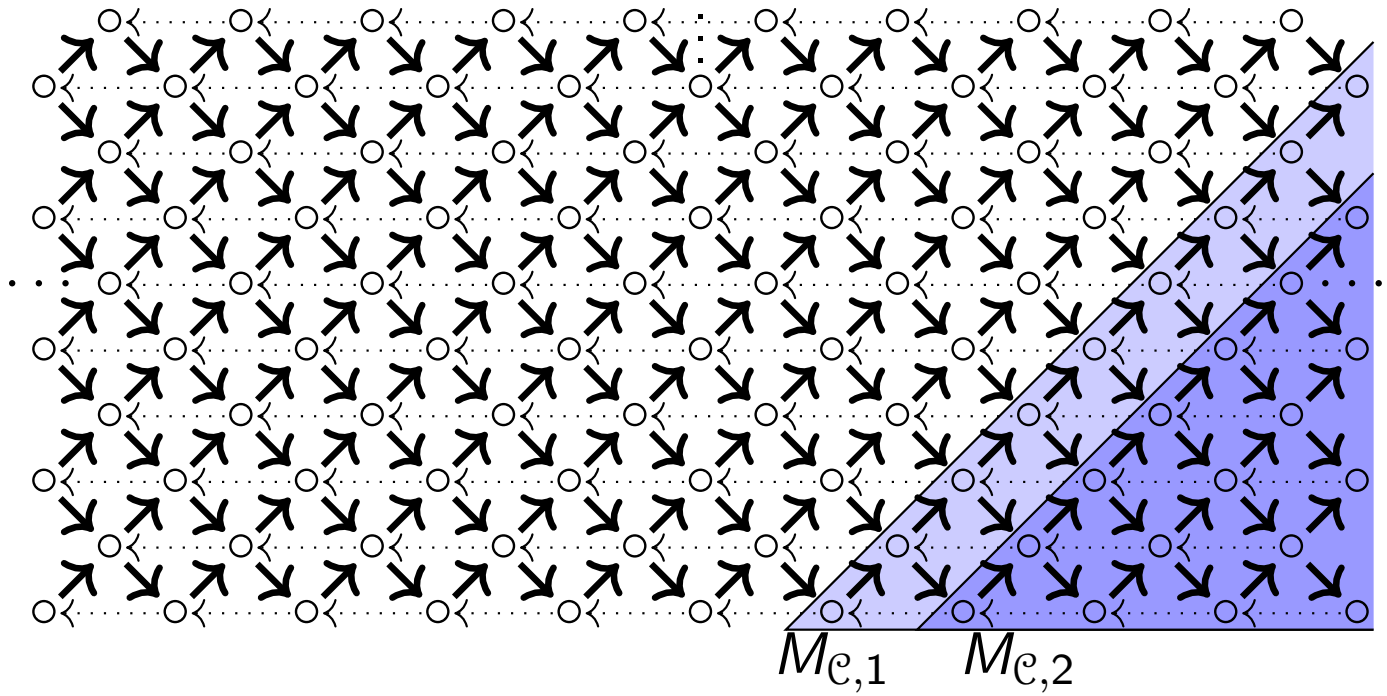
Let $c \in \mathbb{N}_{\geq 4}$. A general Steiner bundle $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}^2(\mathbb{C}))$ with $c_1(\mathcal{F}) = c$ and $\mathrm{rk}(\mathcal{F}) = c + 2$ is uniform of 1-type and not homogeneous.

Thank you for your attention!

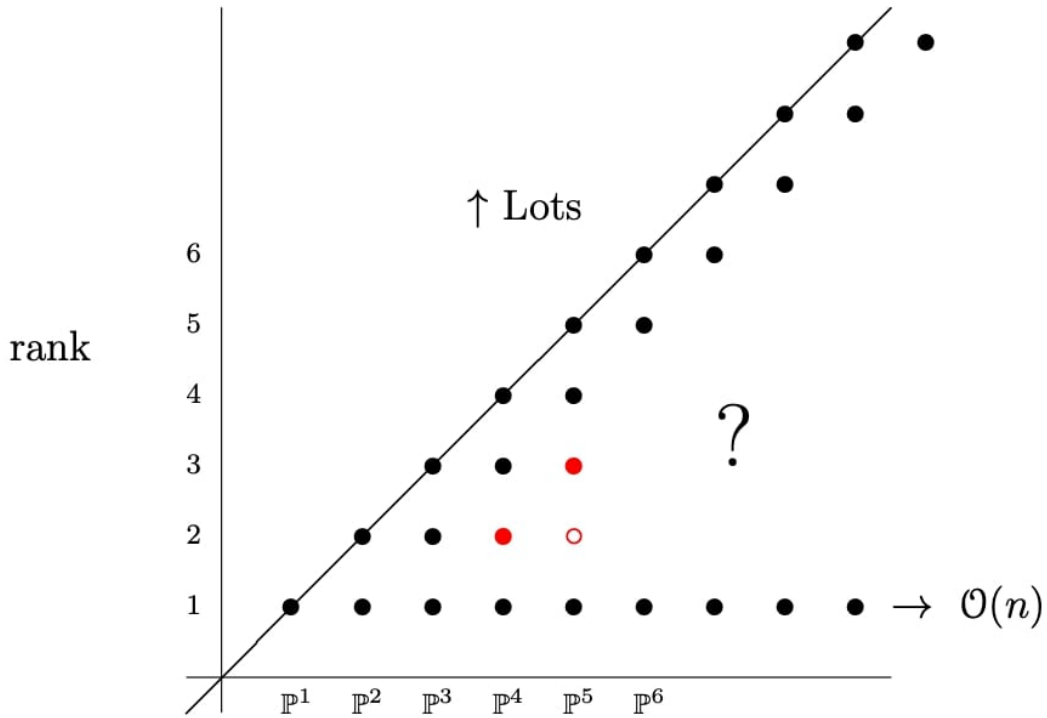
Steiner bundles arrange nicely in AR-components

Let $r \geq 3$. For each regular component $\mathcal{C}$ and $1 \leq d < r$ of the AR quiver there is a uniquely determined quasi-simple representation $M_{\mathcal{C},d}$ such that

$$\mathcal{C} \cap \mathrm{rep}_{\mathrm{inj}}(K_r, d) = (M_{\mathcal{C},d} \to) \quad \text{(cone of successors)}$$



Bundles of small rank



REMARK 6.3.3. There are no known restrictions on the possible ranks of indecomposable bundles on $\mathbb{P}^{r-1}$ for $r \geq 3$. For example, it is not known whether there exists an indecomposable vector bundle of rank 2 on $\mathbb{P}^5$ in characteristic not equal to 2, or on $\mathbb{P}^{r-1}$ $(r \geq 7)$ in any characteristic.