# Uniform Steiner bundles and adjoint pairs of reflection functors for Kronecker representations

Daniel Bissinger, Kiel University

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# Motivation and overview, I

Let $\Bbbk = \overline{\Bbbk}$ field and $n \in \mathbb{N}$.

Interested in: Vector bundles (locally free sheaves) on $\mathbb{P}^n = \mathbb{P}^n(\Bbbk)$.

Important technique: Study behavior of Vbs along pullbacks.

## Examples

1. A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^n)$ splits (into line bundles) iff a restriction to $\mathbb{P}^2$ splits.
2. Let $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^n)$ and $\mathbb{P}^1 \cong L \subseteq \mathbb{P}^n$ a line, then

$$\mathcal{F}|_L \cong \bigoplus_{i \in \mathbb{Z}} a_i(L) \underbrace{\mathcal{O}_L(i).}_{\textit{Serre twist}} \quad \text{(Grothendieck, 1975)}$$

## Definition

$\mathcal{F}$ is uniform, provided $a_i(L) = a_i(L')$ for all $i \in \mathbb{Z}$, lines $L, L'$.
$\rightsquigarrow \mathcal{F}|_{\mathbb{P}^1} = \bigoplus_{i \in \mathbb{Z}} a_i \mathcal{O}_{\mathbb{P}^1}(i)$ (splitting type) & $\mathrm{supp}(\mathcal{F}) := \{i | a_i \neq 0\}$.

# Motivation and overview, II

## Definition (Dolgachev-Kapranov, 1993)

A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^n)$ is called Steiner bundle if there exist vector spaces $V_1, V_2$ and a short exact sequence $0 \to V_1 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^n}(-1) \to V_2 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F} \to 0$.

## Theorem (Marchesi-Miró-Roig, 2022)

*Let $k \in \mathbb{N}$, $\Bbbk = \mathbb{C}$. There exist uniform (and non-homogeneous) Steiner bundles $\mathcal{F}$ of k-type, i.e. $\mathcal{F}|_{\mathbb{P}^1} \cong \bigoplus_{i=0}^{k} a_i \mathcal{O}_{\mathbb{P}^1}(i)$ with $a_k \neq 0$.*

## Remark

1. *All known uniform Steiner bundles are of connected type, i.e. the support is an intervall in $\mathbb{Z}$.*
2. *All given examples contain 0 in the support.*

# Motivation and overview, III

**This talk:** Construct new uniform Steiner bundles using rep. theory of the Kronecker quiver

$$K_r := \quad 1 \;\overset{\gamma_1}{\underset{\gamma_r}{\rightrightarrows}}\; 2$$

with arrow space $A_r := \bigoplus_{i=1}^{r} \Bbbk\gamma_i$.

- $M \in \mathrm{rep}(K_r) \mathrel{\hat{=}} \psi_M \colon A_r \otimes_{\Bbbk} M_1 \to M_2; \gamma_i \otimes m \mapsto M(\gamma_i)(m)$.
- $\mathrm{rep}_{\mathrm{inj}}(K_r, 1) := \{M \mid \ker \psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1} = \{0\} \; \forall\, \mathfrak{v} \in \mathbb{P}(A_r)\}$ is the space 1-injective representations.

# Motivation and overview, IV

## Theorem (Jardim-Prata, 2015)

*There is a right exact functor* $\widetilde{\Theta}_r\colon \operatorname{rep}(K_r) \to \operatorname{Coh}(\mathbb{P}(A_r))$ *s.t. the following statements hold.*

1. $\widetilde{\Theta}_r(M)$ *is a Steiner bundle iff* $M \in \operatorname{rep}_{\mathrm{inj}}(K_r, 1)$.
2. $\widetilde{\Theta}_r\colon \operatorname{rep}_{\mathrm{inj}}(K_r, 1) \to \operatorname{StVect}(\mathbb{P}(A_r))$ *is an exact equivalence.*

## Theorem (Worch, 2012)

*There exist representations* $(X(\mathfrak{v}))_{\mathfrak{v}\in\mathbb{P}(A_r)}$ *s.t. for all* $M \in \operatorname{rep}(K_r)$

1. $\operatorname{rk}(\psi_M|_{\mathfrak{v}} \otimes_{\Bbbk} M_1) = \dim_{\Bbbk} M_1 - \dim_{\Bbbk} \operatorname{Hom}_{K_r}(X(\mathfrak{v}), M)$,
2. $M \in \operatorname{rep}_{\mathrm{inj}}(K_r, 1)$ *iff* $\operatorname{Hom}_{K_r}(X(\mathfrak{v}), M) = 0$ *f.a.* $\mathfrak{v} \in \mathbb{P}(A_r)$.

# Restrictions to $K_1$, I

- For $d < r$ let $\mathrm{Inj}_{\Bbbk}(A_d, A_r) := \{\beta \in \mathrm{Hom}_{\Bbbk}(A_d, A_r) \mid \ker \beta = 0\}$.
- For $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ and $M \in \mathrm{rep}(K_r)$, the representation

$$\psi_{\alpha^*(M)} \colon A_d \otimes_{\Bbbk} M_1 \overset{\alpha \otimes \mathrm{id}_{M_1}}{\to} A_r \otimes_{\Bbbk} M_1 \overset{\psi_M}{\to} M_2 \in \mathrm{rep}(K_d)$$

is called restriction of $M$ to $K_d$ (along $\alpha$).

## Example (Restriction to $K_1$)

Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_1, A_r)$, then

$$\alpha^*(M) \cong a_\alpha \underbrace{(\Bbbk \overset{\mathrm{id}}{\to} \Bbbk)}_{=P(1)} \oplus b_\alpha \underbrace{(0 \to \Bbbk)}_{=P(2)} \oplus c_\alpha (\Bbbk \to 0) \in \mathrm{rep}(K_1), \text{ and}$$

$$\begin{aligned} a_\alpha &= \mathrm{rk}(\psi_{\alpha^*(M)}) = \mathrm{rk}(\psi_M|_{\mathrm{im}\, \alpha \otimes_{\Bbbk} M_1}) \\ &\overset{Worch}{=} \dim_{\Bbbk} M_1 - \dim_{\Bbbk} \mathrm{Hom}_{K_r}(X(\mathrm{im}\, \alpha), M). \end{aligned}$$

# Restrictions to $K_1$, II

## Example (continued)

$$\alpha^*(M) \cong a_\alpha \underbrace{(\Bbbk \xrightarrow{\mathrm{id}} \Bbbk)}_{=P(1)} \oplus b_\alpha \underbrace{(0 \to \Bbbk)}_{=P(2)} \oplus c_\alpha (\Bbbk \to 0) \in \mathrm{rep}(K_1)$$

and $a_\alpha = \mathrm{rk}(\psi_{\alpha^*(M)}) = \dim_\Bbbk M_1 - \dim_\Bbbk \mathrm{Hom}_{K_r}(X(\mathrm{im}\,\alpha), M)$.

Since $a_\alpha + c_\alpha = \dim_\Bbbk M_1$, we conclude:

## Corollary

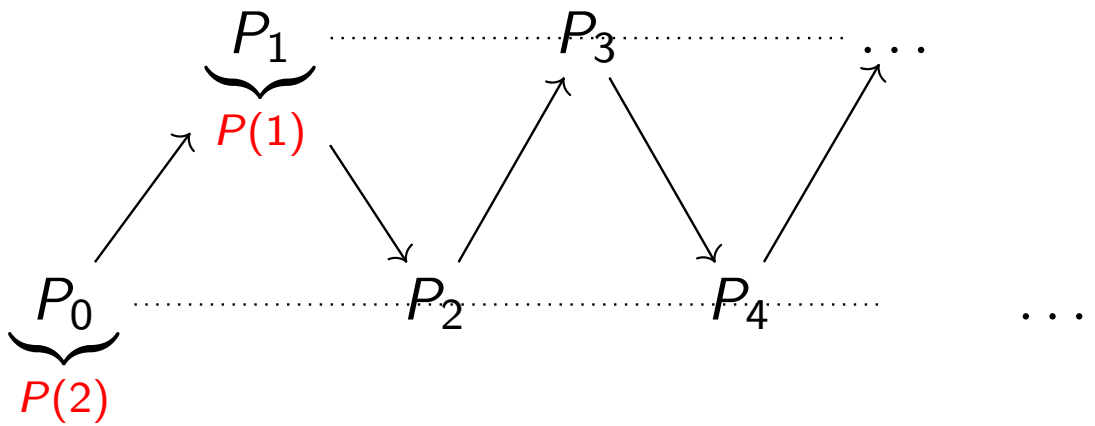
*Let $M \in \mathrm{rep}(K_r)$. The following statements are equivalent.*

1. $\mathrm{Hom}_{K_r}(X(\mathrm{im}\,\alpha), M) = 0$.
2. $\alpha^*(M) \in \mathrm{rep}(K_1)$ *is projective.*
3. $\alpha^*(M) \cong \dim_\Bbbk M_1 P(1) \oplus \underbrace{(\dim_\Bbbk M_2 - 1 \cdot \dim_\Bbbk M_1)}_{=:\Delta_M(1)} P(2)$.

# Restrictions to $K_2$, I

**Question**: Can we study restrictions to $K_2$ via $\mathrm{Hom}_{K_r}(-,-)$?

- Preprojective Auslander-Reiten component




with $P_n =: \; \Bbbk[X,Y]_{n-1} \;\underset{Y\cdot}{\overset{X\cdot}{\rightrightarrows}}\; \Bbbk[X,Y]_n \; \in \mathrm{rep}(K_2)$.

- $\mathrm{rep}_{\mathrm{inj}}(K_2,1) = \mathrm{add}(P_n \mid n \in \mathbb{N}_0)$.
- $\widetilde{\Theta}_2(P_n) = \mathcal{O}_{\mathbb{P}(A_2)}(n)$ f.a. $n \in \mathbb{N}_0$.

# Restrictions to $K_2$, II

$M \in \mathsf{rep}_{\mathsf{inj}}(K_r, 1)$, $\alpha \in \mathsf{Inj}_{\Bbbk}(A_2, A_r)$, then $\alpha^*(M) \in \mathsf{rep}_{\mathsf{inj}}(K_2, 1)$ and

$$\alpha^*(M) \cong \bigoplus_{n \in \mathbb{N}_0} a_n(\alpha) P_n.$$

For $L := \mathbb{P}(\operatorname{im} \alpha) \subseteq \mathbb{P}(A_r)$ we obtain

$$\widetilde{\Theta}_r(M)|_L \cong \widetilde{\Theta}_2(\alpha^*(M)) \cong \bigoplus_{n \in \mathbb{N}_0} a_n(\alpha) \widetilde{\Theta}_2(P_n) \cong \bigoplus_{n \in \mathbb{N}_0} a_n(\alpha) \mathcal{O}_{\mathbb{P}(A_2)}(n).$$

**Proposition (B.-Farnsteiner, 2022)**

*There exist representations* $(X(\mathfrak{v}))_{\mathfrak{v} \in \mathsf{Gr}_2(A_r)}$ *s.t. for all* $M \in \mathsf{rep}(K_r)$

1. $\operatorname{rk}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = 2 \cdot \dim_{\Bbbk} M_1 - \dim_{\Bbbk} \operatorname{Hom}_{K_r}(X(\mathfrak{v}), M)$, *and*
2. $\alpha^*(M) \in \mathsf{rep}(K_2)$ *is projective iff* $\ker \psi_M|_{\operatorname{im} \alpha \otimes_{\Bbbk} M_1} = \{0\}$.

# Restrictions to $K_2$, III

## Corollary

*Let* $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2) := \{M \mid \ker \psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1} = \{0\}\ \forall\, \mathfrak{v} \in \mathrm{Gr}_2(A_r)\}$ *be 2-injective with* $\dim_{\Bbbk} M_1 \neq 0$. *Then* $\widetilde{\Theta}_r(M)$ *is a uniform Steiner bundle of* 1*-type.*

## Proof.

We have $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2) \subseteq \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$. Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$, then

$$\alpha^*(M) \cong \underbrace{(\dim_{\Bbbk} M_2 - 2 \cdot \dim_{\Bbbk} M_1)}_{=:\Delta_M(2)} P_0 \oplus \dim_{\Bbbk} M_1 P_1$$

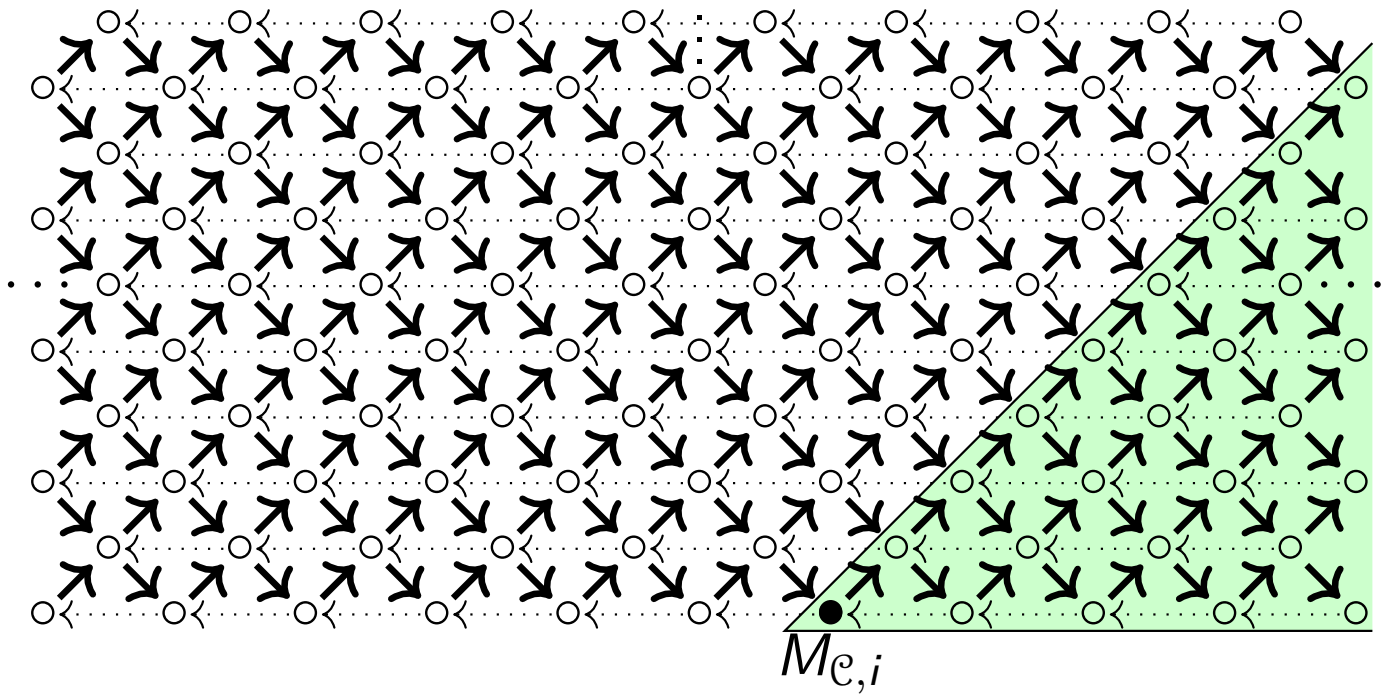
and $\widetilde{\Theta}_r(M)|_{\mathbb{P}(\mathrm{im}\,\alpha)} \cong \Delta_M(2)\mathcal{O}_{\mathbb{P}(A_2)} \oplus \dim_{\Bbbk} M_1 \mathcal{O}_{\mathbb{P}(A_2)}(1)$. □

# Steiner bundles arrange nicely in AR components

For $r \geq 3$ and $\mathcal{C}$ regular AR component there are $M_{\mathcal{C},1}, M_{\mathcal{C},2} \in \mathcal{C}$ quasi-simple s.t.

$$\mathcal{C} \cap \mathrm{rep}_{\mathrm{inj}}(K_r, i) = (M_{\mathcal{C},i} \to) \quad (\text{cone of successors}).$$

**Illustration:**

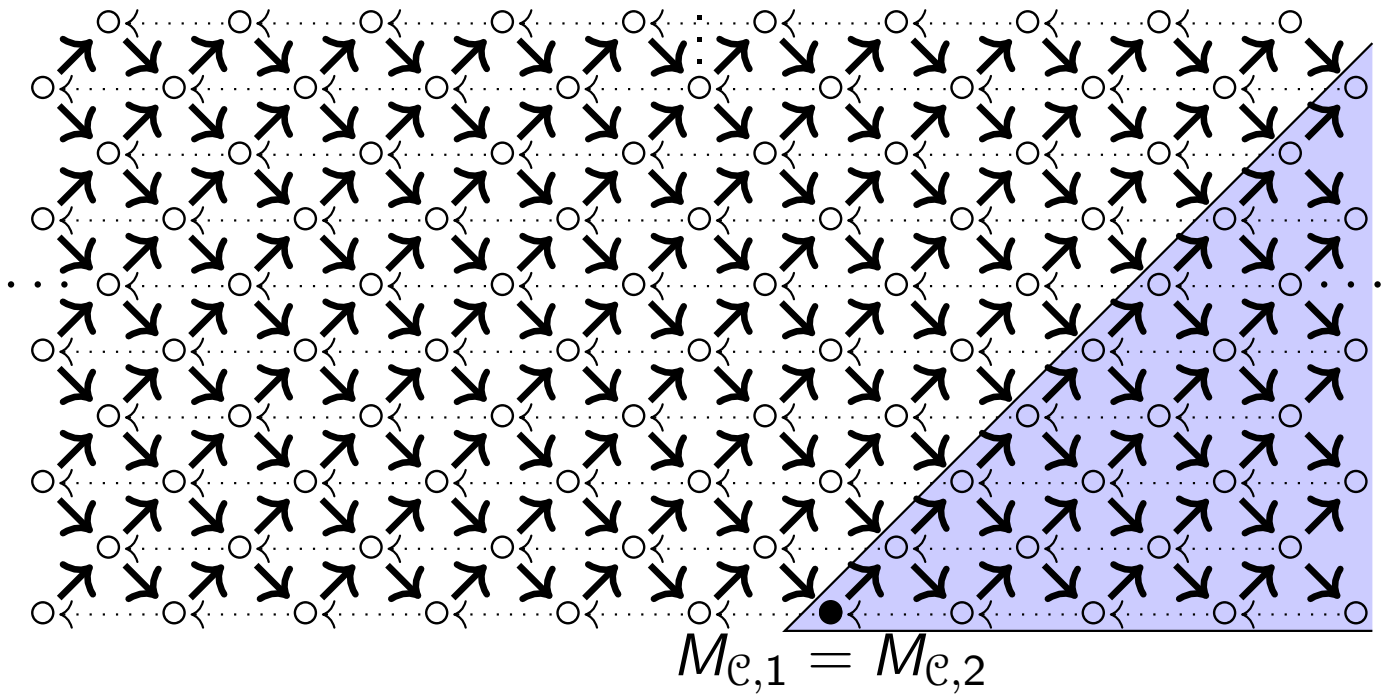


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**Case 1:**

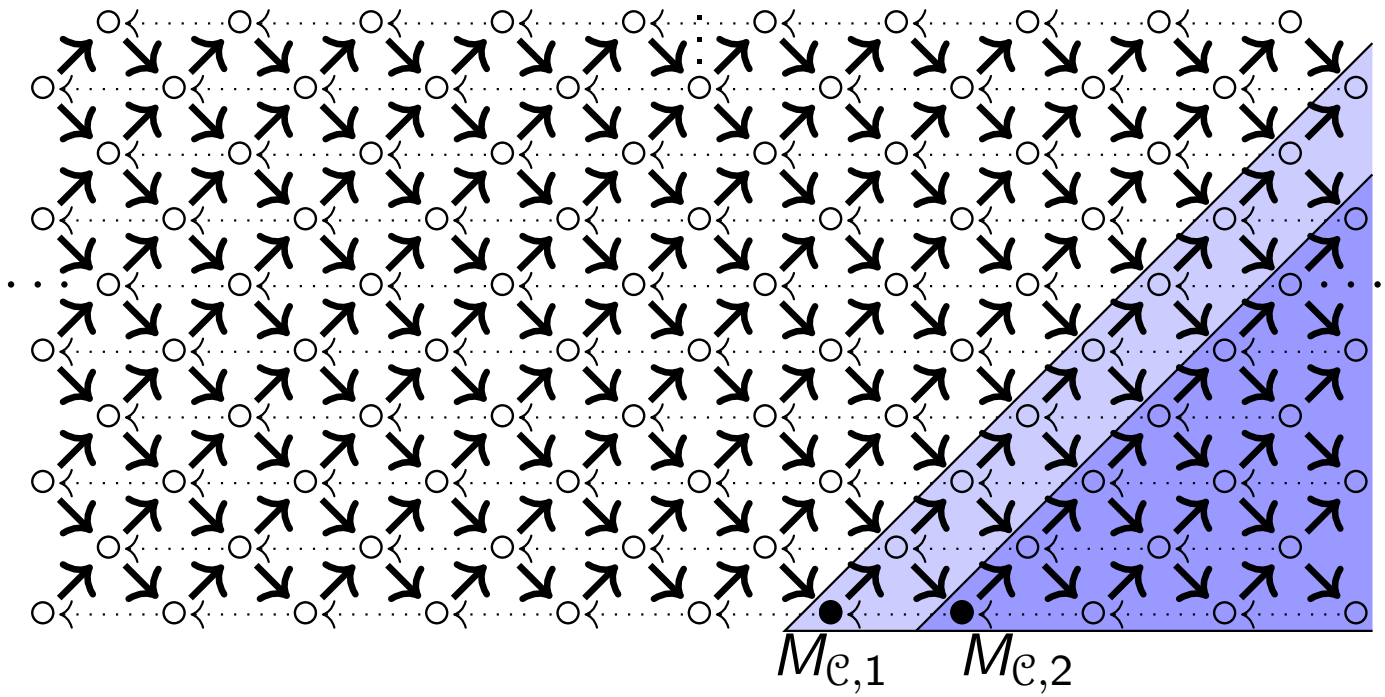


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**Case 2:**



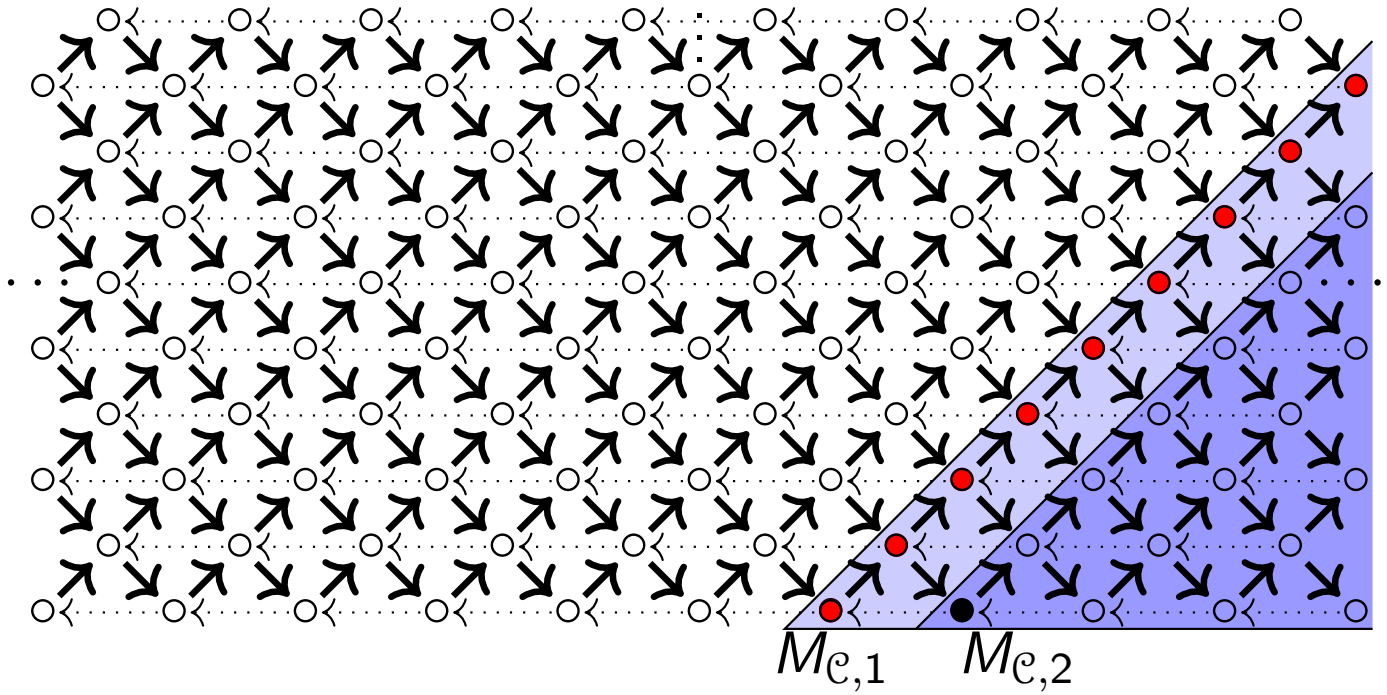
# Steiner bundles arrange nicely in AR components

For $r \geq 3$ and $\mathcal{C}$ regular AR component there are $M_{\mathcal{C},1}, M_{\mathcal{C},2} \in \mathcal{C}$ quasi-simple s.t.

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**Case 2:**

Application of $\widetilde{\Theta}_r$: interesting bundles



# Restriction, inflation and reflection, I

- Consider $\iota\colon A_2 \hookrightarrow A_r; \gamma_i \mapsto \gamma_i$ and let

$$\mathrm{res}\colon \mathrm{rep}(K_r) \to \mathrm{rep}(K_2); \psi_M \mapsto \psi_{\iota^*(M)} \text{ (restriction)}.$$

- Let

$$\mathrm{inf}\colon \mathrm{rep}(K_2) \to \mathrm{rep}(K_r) \text{ (inflation)}$$

$$\psi_X \mapsto \psi_{\mathrm{inf}(X)}(\gamma_i \otimes x) = \begin{cases} \psi_X(\gamma_i \otimes x), & i \in \{1,2\} \\ 0, & \textit{else}. \end{cases}$$

- Let

$$\sigma_{K_d}\colon \mathrm{rep}(K_d) \to \mathrm{rep}(K_d) \text{ (reflection)}$$

$$\psi_{\sigma_{K_d}}\colon A_d \otimes_{\Bbbk} \ker \psi_M \to M_1; \gamma_i \otimes (\gamma_j \otimes m) \mapsto \delta_{ij} \cdot m.$$

**Proposition (B. 2024)**

*The functor* $\sigma_{K_r}^{-1} \circ \mathrm{inf}\colon \mathrm{rep}(K_2) \to \mathrm{rep}(K_r)$ *is left adjoint to* $\sigma_{K_2} \circ \mathrm{res}\colon \mathrm{rep}(K_r) \to \mathrm{rep}(K_2)$.

# Restriction, inflation and reflection, II

## Facts

1. $\sigma_{K_2}^{-1}(P_n) = P_{n+1}$ for all $n \in \mathbb{N}_0$.
2. We have $\dim_{\Bbbk} \mathrm{Hom}_{K_2}(P_n, P_m) = \max\{m - n + 1, 0\}$.

## Application

Let $n \in \mathbb{N}$ and $X_n := (\sigma_{K_r}^{-1} \circ \inf)(P_n) \in \mathrm{rep}(K_r)$.
Let $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$, then $\mathrm{res}(M) = \bigoplus_{n \geq 0} a_n P_n \in \mathrm{rep}_{\mathrm{inj}}(K_2, 1)$.

- $$\dim_{\Bbbk} \mathrm{Hom}_{K_r}(X_n, M) \overset{\textit{Prop.}}{=} \dim_{\Bbbk} \mathrm{Hom}_{K_2}(P_n, (\sigma_{K_2} \circ \mathrm{res})(M))$$
  $$\overset{1.}{=} \dim_{\Bbbk} \mathrm{Hom}_{K_2}(P_{n+1}, \mathrm{res}(M))$$
  $$\overset{2.}{=} \sum_{i \geq n+1} a_i (i - n).$$
- Similarly, $\dim_{\Bbbk} \mathrm{Ext}^1_{K_r}(X_n, M) = \sum_{i=0}^{n-1} a_i (n - i)$.

# Restriction, inflation and reflection, III

**Theorem (B. 2024)**

*Let $n \in \mathbb{N}$. There exist representations $(X_n(\mathfrak{v}))_{\mathfrak{v} \in \mathrm{Gr}_2(A_r)}$ s.t. for all $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$ and $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$*

1. $\mathrm{Hom}_{K_r}(X_n(\mathrm{im}\,\alpha), M) = 0$ *iff* $\alpha^*(M) \in \mathrm{add}(P_0, \ldots, P_n)$, *and*
2. $\mathrm{Ext}^1_{K_r}(X_n(\mathrm{im}\,\alpha), M) = 0$ *iff* $\alpha^*(M) \in \mathrm{add}(P_i \mid i \geq n)$.

Using Reineke's result (2024) on general representations we obtain:

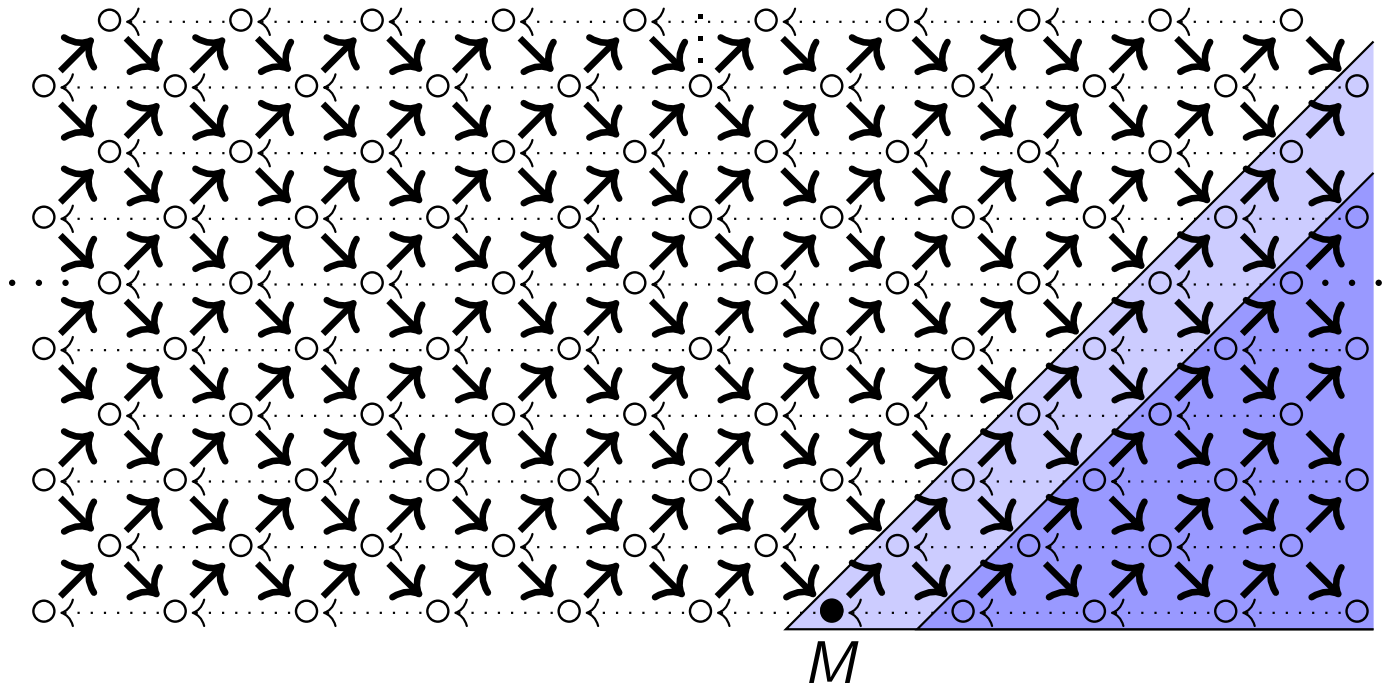
**Corollary**

*Let $n \in \mathbb{N}_0$. There are vector spaces $M_1, M_2$, $a_n, b_n \in \mathbb{N}$ and a non-empty open set $\mathcal{O} \subseteq \mathrm{Hom}_{\Bbbk}(A_r \otimes_{\Bbbk} M_1, M_2)$ such that every $\psi_M \in \mathcal{O}$ satisfies*

1. $M := (M_1, M_2, \psi_M)$ *is a brick in* $\mathrm{rep}_{\mathrm{inj}}(K_r, 1)$, *and*
2. $\alpha^*(M) \cong a_n P_n \oplus b_n P_{n+1}$ *for all* $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$.

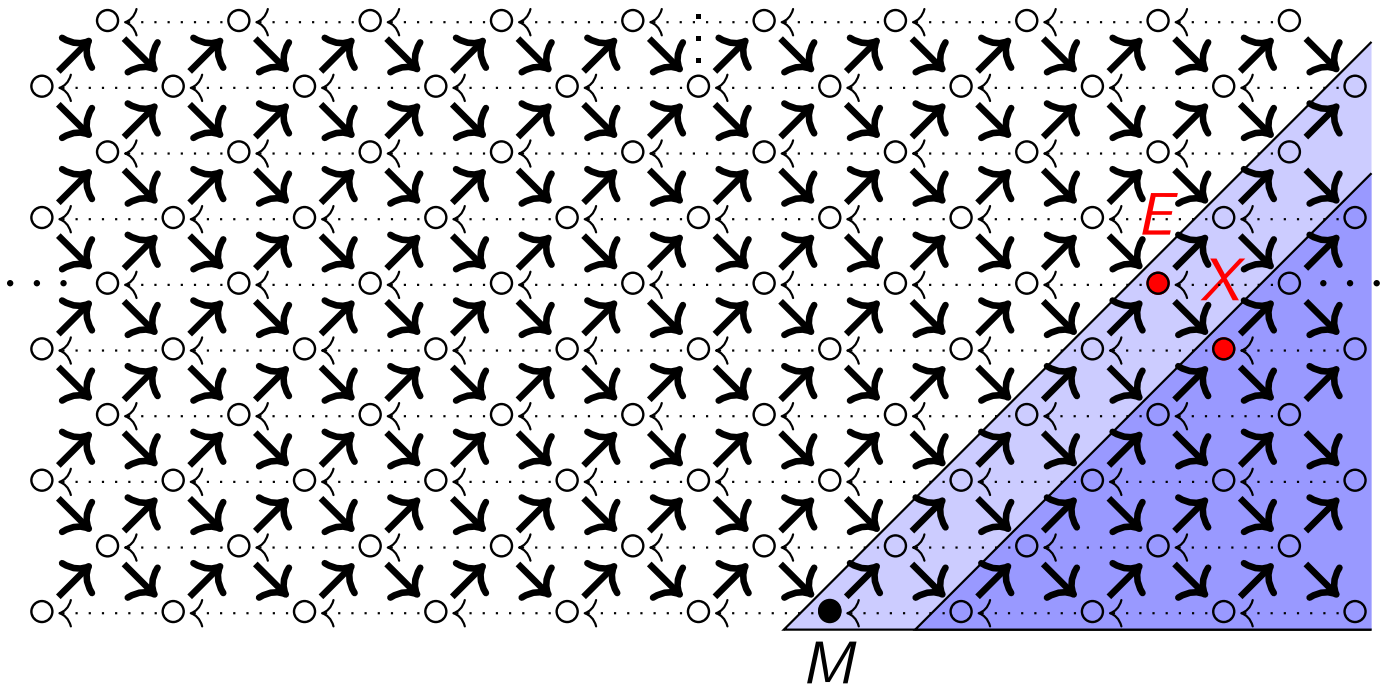
# Application, I

Let $n \in \mathbb{N}$ and $M = (M_1, M_2, \psi_M)$ with the mentioned properties. Then $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1) \setminus \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$ is regular and quasi-simple.



# Application, I

Let $n \in \mathbb{N}$ and $M = (M_1, M_2, \psi_M)$ with the mentioned properties. Then $M \in \mathrm{rep}_{\mathrm{inj}}(K_1, 1) \setminus \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$ is regular and quasi-simple.




$$\Longrightarrow \quad 0 \to M \to E \to X \to 0.$$

# Application, II

Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$, then

$$0 \to \alpha^*(M) \to \alpha^*(E) \to \alpha^*(X) \to 0.$$

Since $X \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$, $\alpha^*(X)$ is projective and

$$\alpha^*(E) \cong a_n P_n \oplus b_n P_{n+1} \oplus \Delta_X(2) P_0 \oplus \dim_{\Bbbk} X_1 P_1$$

**Corollary**

*Let $n \in \mathbb{N}$. There exist $c_0, c_1, c_n, c_{n+1} \in \mathbb{N}$ and an indecomposable and uniform Steiner bundle $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ such that*

$$\mathcal{F}|_{\mathbb{P}(A_2)} \cong c_0 \mathcal{O}_{\mathbb{P}(A_2)} \oplus c_1 \mathcal{O}_{\mathbb{P}(A_2)}(1) \oplus c_n \mathcal{O}_{\mathbb{P}(A_2)}(n) \oplus c_{n+1} \mathcal{O}_{\mathbb{P}(A_2)}(n+1).$$

$\rightsquigarrow$ New indecomposable uniform Steiner bundles with disconnected splitting types (having arbitrarily wide gaps).

**Thank you for your attention!**