Relative projective Kronecker representations and Steiner bundles on Grassmannians

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Motivation and overview, I

- $k = \overline{k}$ field, $\mathrm{char}(k) = p > 0$, $r \in \mathbb{N}$.
- $E_r \cong (\mathbb{Z}/(p))^r$ elementary abelian p-group of rank r.

Theorem (Friedlander-Pevtsova, 2008, simplified version)

There exists a functor $\widetilde{\Theta}\colon \operatorname{mod} kE_r \to \mathrm{Coh}(\mathbb{P}^{r-1})$ and a "large" subcategory $\mathrm{CR}(kE_r) \subseteq \operatorname{mod} kE_r$ such that for $M \in \operatorname{mod} kE_r$ the following statements are equivalent:

1. $M \in \mathrm{CR}(kE_r)$.
2. $\widetilde{\Theta}(M)$ is a vector bundle on $\mathbb{P}^{r-1}$.

"Restriction" to Loewy length ≤ 2 (Kronecker representations) yields:

$$\widetilde{\Theta}\colon \operatorname{rep}(K_r) \to \mathrm{Coh}(\mathbb{P}^{r-1})$$

(implicit in the works of Hulek, Drezet-Le Portier, Brambilla for $k = \mathbb{C}$).

Motivation and overview, II

There is a "large" subcategory $\mathrm{EKP}(K_r) \subseteq \mathrm{rep}(K_r)$ and an induced equivalence of categories

$$\mathrm{EKP}(K_r) \xrightarrow{\sim} \text{ Steiner bundles on } \mathbb{P}^{r-1}.$$

Generalization (Jardim-Prata, 2015, $\mathrm{char}(k) = 0$):

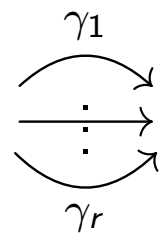
- X smooth proj. variety, $(\mathcal{E}, \mathcal{F})$ suitable pair of vector bundles with $r := \dim_{\Bbbk} \mathrm{Hom}(\mathcal{E}, \mathcal{F})$.
- $\exists\, \mathrm{rep}_{\mathrm{gi}}(K_r, X) \subseteq \mathrm{rep}(K_r)$ and $\widetilde{\Theta}_X \colon \mathrm{rep}(K_r) \to \mathrm{Coh}(X)$ that induces an equivalence ($\mathrm{rep}_{\mathrm{gi}}(K_r, X), \widetilde{\Theta}_X$ depend on $(\mathcal{E}, \mathcal{F})$)

$$\mathrm{rep}_{\mathrm{gi}}(K_r, X) \xrightarrow{\sim} \text{ cokernel bundles of type } (\mathcal{E}, \mathcal{F}).$$

This talk:

- $X = \mathrm{Gr}_d(k^r)$ and cokernel bundles of type $(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)})$.
- Give a detailed description and properties of $\mathrm{rep}_{\mathrm{gi}}(K_r, d) := \mathrm{rep}_{\mathrm{gi}}(K_r, \mathrm{Gr}_d(k^r))$.

Setup

- Let $\mathbb{k} = \overline{\mathbb{k}}$ field, arbitrary characteristic, $r \in \mathbb{N}_{\geq 2}$.
- $K_r :=$ 

Kronecker quiver, $A_r := \bigoplus_{i=1}^{r} \mathbb{k}\gamma_i$ arrow space , $\mathbb{k}K_r = \begin{pmatrix} \mathbb{k} & 0 \\ A_r & \mathbb{k} \end{pmatrix}$ Kronecker algebra.
- $\psi_M \colon A_r \otimes_{\mathbb{k}} M_1 \to M_2; a \otimes m \mapsto a.m \; \forall M \in \operatorname{mod} \mathbb{k}K_r \cong \operatorname{rep}(K_r)$.
- $\operatorname{EKP}(K_r) := \{M \mid \ker(\sum_{i=1}^{r} \alpha_i M(\gamma_i)) = \{0\} \; \forall \alpha \in \mathbb{k}^r - 0\}$

$$= \{M \mid \ker \psi_M|_{\mathfrak{v} \otimes_{\mathbb{k}} M_1} = \{0\} \; \forall \, \mathfrak{v} \in \operatorname{Gr}_1(A_r)\}.$$

Definition

Let $1 \leq d < r$. We say that $M \in \operatorname{rep}(K_r)$ is **relative d-projective**, provided $\ker \psi|_{\mathfrak{v} \otimes_{\mathbb{k}} M_1} = \{0\}$ for all $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$. We define $\operatorname{rep}_{\text{proj}}(K_r, d) := \{M \in \operatorname{rep}(K_r) \mid M \text{ relative } d\text{-projective}\}$.

Steiner bundles, I

- $1 \leq d < r$, $\mathrm{Gr}_d(A_r)$ Grassmannian.
- $\mathrm{Vect}(\mathrm{Gr}_d(A_r))$ category of vector bundles on $\mathrm{Gr}_d(A_r)$.
- $\mathcal{O}_{\mathrm{Gr}_d(A_r)}$ structure sheaf, $\mathcal{U}_{(r,d)}$ universal subbundle of $\mathrm{Gr}_d(A_r)$.

Definition (Arrondo-Marchesi)

A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathrm{Gr}_d(A_r))$ is referred to as a **Steiner bundle** if there exist vector spaces V_1, V_2 and a short exact sequence $0 \to V_1 \otimes_{\Bbbk} \mathcal{U}_{(r,d)} \to V_2 \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(A_r)} \to \mathcal{F} \to 0$.

Remarks

1. *Steiner bundles are cokernel bundles of type* $(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(k^r)})$
2. *For* $d = 1$, *we have* $\mathrm{Gr}_d(A_r) \cong \mathbb{P}(A_r)$ *and* $\mathcal{U}_{(r,d)} \cong \mathcal{O}_{\mathbb{P}(A_r)}(-1)$.
3. $\mathrm{StVect}(\mathrm{Gr}_d(A_r))$ *is closed under extensions, direct sums and direct summands.*

Steiner bundles, II

Theorem (Jardim-Prata, 2015, $\mathrm{char}(\Bbbk) = 0$)

Let $1 \leq d < r$. There is a right exact functor $\widetilde{\Theta}_d \colon \mathrm{rep}(K_r) \to \mathrm{Coh}(\mathrm{Gr}_d(A_r))$ s.t. the following statements hold.

1. $\mathcal{F} \in \mathrm{Vect}(\mathrm{Gr}_d(A_r))$ *is a Steiner bundle iff there is* $M \in \mathrm{rep}_{\mathrm{gi}}(K_r, d)$ *such that* $\mathcal{F} \cong \widetilde{\Theta}_d(M)$.
2. $\widetilde{\Theta}_d \colon \mathrm{rep}_{\mathrm{gi}}(K_r, d) \to \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ *is an exact equivalence.*

Lemma

Let $1 \leq d < r$, then $\mathrm{rep}_{\mathrm{proj}}(K_r, d) = \mathrm{rep}_{\mathrm{gi}}(K_r, d)$.

Remarks

1. *The Theorem holds in arbitrary characteristic.*
2. *For $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$, we have* $\mathrm{rk}(\widetilde{\Theta}_d(M)) = \Delta_M(d) := \dim_{\Bbbk} M_2 - d \dim_{\Bbbk} M_1$.

Relative projective representations, I

A different point of view: Let $M \in \mathrm{rep}(K_r)$.

- $\mathrm{Inj}_{\Bbbk}(A_1, A_r) := \{\alpha \in \mathrm{Hom}_{\Bbbk}(A_1, A_r) \mid \ker \alpha = \{0\}\}$.
- For $\alpha, \beta \in \mathrm{Inj}_{\Bbbk}(A_1, A_r)$ with $\mathfrak{v} := \mathrm{im}\,\alpha = \mathrm{im}\,\beta$, we have $\alpha^*(M) \cong \beta^*(M) \in \mathrm{rep}(K_1)$ and set $M|_{\mathfrak{v}} := \alpha^*(M)$.
- We have $\mathrm{ind}(K_1) = \{0 \to \Bbbk, \Bbbk \xrightarrow{\mathrm{id}} \Bbbk, \Bbbk \to 0\}$.
- Write $M|_{\mathfrak{v}} = a(0 \to \Bbbk) \oplus b(\Bbbk \xrightarrow{\mathrm{id}} \Bbbk) \oplus c(\Bbbk \to 0)$, then $\mathrm{rk}(\psi_M|_{\mathfrak{v}} \otimes_{\Bbbk} M_1) = b$.

Hence

$$
\begin{aligned}
& M \in \mathrm{EKP}(K_r) \\
\Leftrightarrow\ & \mathrm{rk}(\psi_M|_{\mathfrak{v}} \otimes_{\Bbbk} M_1) = \dim_{\Bbbk} M_1 \forall\, \mathfrak{v} \in \mathrm{Gr}_1(A_r) \\
\Leftrightarrow\ & M|_{\mathfrak{v}} \cong \dim_{\Bbbk} M_1 (\Bbbk \xrightarrow{\mathrm{id}} \Bbbk) \oplus \Delta_M(1)(0 \to \Bbbk)\ \forall\, \mathfrak{v} \in \mathrm{Gr}_1(A_r) \\
\Leftrightarrow\ & M|_{\mathfrak{v}} \text{ projective } \forall\, \mathfrak{v} \in \mathrm{Gr}_1(A_r).
\end{aligned}
$$

Relative projective representations, II

Corollary

$\mathrm{rep}_{\mathrm{proj}}(K_r, d) = \{M \mid M|_{\mathfrak{v}} \in \mathrm{rep}(K_d)$ *projective* $\forall\, \mathfrak{v} \in \mathrm{Gr}_d(A_r)\}$ *and for* $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ *we have*

$$M|_{\mathfrak{v}} \cong \dim_{\Bbbk} M_1 P(1) \oplus \Delta_M(d) P(2) \quad \forall\, \mathfrak{v} \in Gr_d(A_r).$$

Properties of $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$, I

Theorem

Let $1 \leq d < r$. There exists a familiy $(E(\mathfrak{v}))_{\mathfrak{v} \in \mathrm{Gr}_d(A_r)}$ of elementary representations such that

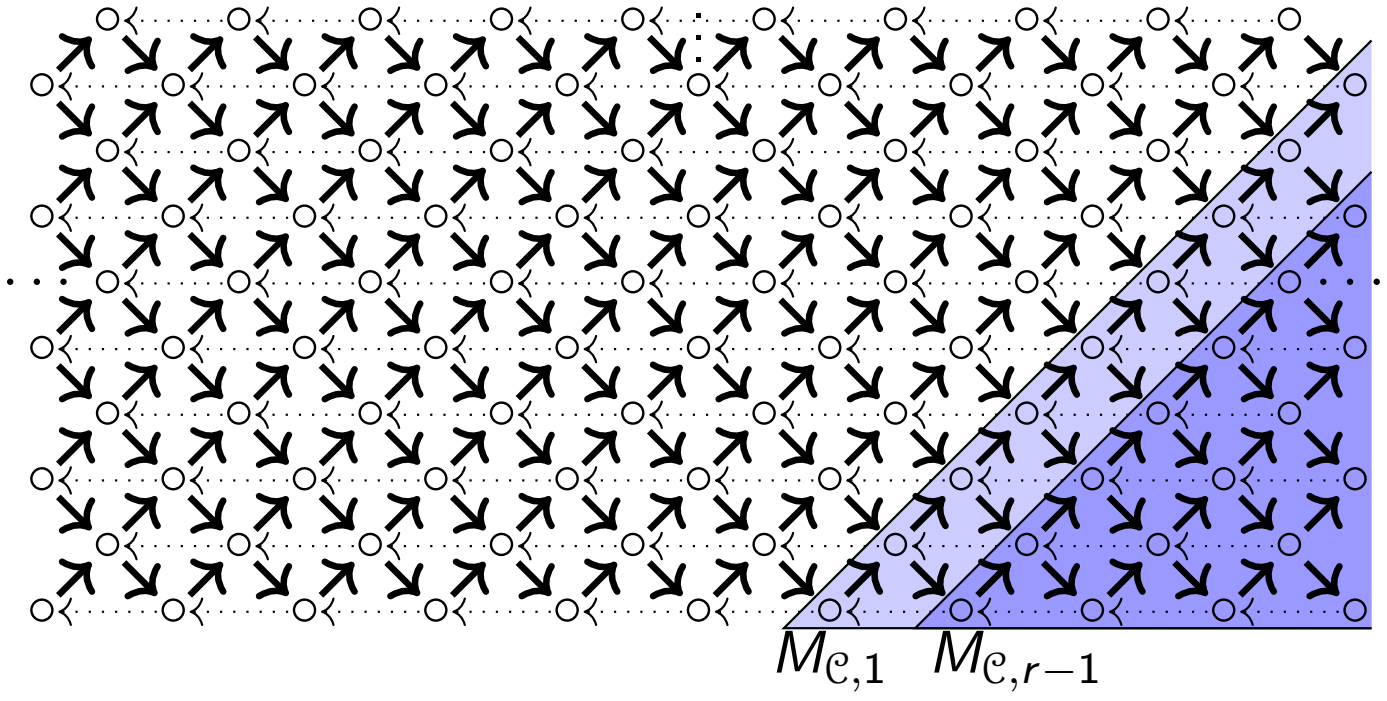
$$\mathrm{rep}_{\mathrm{proj}}(K_r, d) = \{M \mid \mathrm{Hom}(E(\mathfrak{v}), M) = 0 \ \forall\, \mathfrak{v} \in \mathrm{Gr}_d(A_r)\}.$$

Corollary

1. $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ *is a τ^{-1}-invariant torsion-free class.*
2. $\mathrm{rep}_{\mathrm{proj}}(K_r, r-1) \subsetneq \mathrm{rep}_{\mathrm{proj}}(K_r, r-2) \subsetneq \cdots \subsetneq \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$.
3. $\mathrm{rep}_{\mathrm{proj}}(K_r, r-1)$ *contains all preprojective representations.*
4. *For $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$, we have $\tau^{-1}(M) \in \mathrm{rep}_{\mathrm{proj}}(K_r, r-1)$.*

Properties of $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$, II

Let $r \geq 3$. For each regular component $\mathcal{C}$ and $1 \leq d < r$ of the AR quiver there is a uniquely determined quasi-simple representation $M_{\mathcal{C},d}$ such that $\mathrm{rep}_{\mathrm{proj}}(K_r, d) \cap \mathcal{C} = (M_{\mathcal{C},d} \to)$.



$\rightsquigarrow$ Steiner bundles arrange nicely in AR-components.

Uniform vs homogeneous vector bundles

Theorem (Grothendieck)

Let $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^1)$. There exist uniquely determined $(a_i)_{i\in\mathbb{Z}}$ such that $\mathcal{F} \cong \bigoplus_{i\in\mathbb{Z}} a_i \mathcal{O}_{\mathbb{P}^1}(i)$.

Let $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}(A_r))$ and $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$, then

$$\mathcal{F}|_{\mathbb{P}(\mathfrak{v})} \cong \bigoplus_{i\in\mathbb{Z}} a_i(\mathfrak{v}) \mathcal{O}_{\mathbb{P}(\mathfrak{v})}(i).$$

Definition

1. $\mathcal{F}$ is called **uniform**, provided $(a_i(\mathfrak{v})) = (a_i(\mathfrak{v}'))$ for all $\mathfrak{v}, \mathfrak{v}' \in \mathrm{Gr}_2(A_r)$.
2. $\mathcal{F}$ is called **homogeneous**, provided $g^*.\mathcal{F} \cong \mathcal{F}$ for all $g \in \mathrm{GL}(A_r)$. $[(g^*.\mathcal{F})(U) := \mathcal{F}(g^{-1}.U)$ for all $U \subseteq \mathbb{P}(A_r)]$

- $\mathcal{F}$ homogeneous $\implies$ $\mathcal{F}$ uniform.
- $\mathcal{F}$ uniform $\not\Longrightarrow$ $\mathcal{F}$ homogeneous (Elencwaig, 1979).

Homogeneous representations

$\mathrm{GL}(A_r)$ acts on $\operatorname{mod} \Bbbk K_r$ via autoequivalences.

Definition

$M \in \operatorname{mod} \Bbbk K_r$ is called **homogeneous**, provided $M^{(g)} \cong M$ for all $g \in \mathrm{GL}(A_r)$.

Lemma

1. *Let $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1) = \mathrm{EKP}(K_r)$. Then M is homogeneous iff $\widetilde{\Theta}(M)$ is homogeneous.* $(\widetilde{\Theta} := \widetilde{\Theta}_1)$
2. *An AR-component $\mathcal{C}$ contains a homogeneous representation iff all representations in $\mathcal{C}$ are homogeneous.*

Remark

We don't know many components that contain homogeneous representations.

Uniform vs homogeneous Steiner bundles

Corollary

Let $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 2)$, *then* $\widetilde{\Theta}(M) \in \mathrm{StVect}(\mathbb{P}(A_r))$ *is uniform.*

Proof

Let $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$. We have $M|_{\mathfrak{v}} \cong \dim_{\Bbbk} M_1 P(1) \oplus \Delta_M(2)P(2)$ and conclude

$$\begin{aligned}\widetilde{\Theta}(M)|_{\mathbb{P}(\mathfrak{v})} &\cong \widetilde{\Theta}(M|_{\mathfrak{v}}) \cong \dim_{\Bbbk} M_1 \widetilde{\Theta}(P(1)) \oplus \Delta_M(2)\widetilde{\Theta}(P(2)) \\ &\cong \dim_{\Bbbk} M_1 \mathcal{O}_{\mathbb{P}(\mathfrak{v})}(1) \oplus \Delta_M(2)\mathcal{O}_{\mathbb{P}(\mathfrak{v})}.\end{aligned}$$

Theorem

Let $\mathcal{C}$ *be an AR-component that contains a non-homogeneous representation. Let* $N \in (M_{\mathcal{C},2} \to)$, *then* $\widetilde{\Theta}(N)$ *is a uniform but not homogeneous bundle.*

Example: Consider $M \in \mathrm{rep}(K_r)$ with $M_j = \Bbbk$, $M(\gamma_i) = \delta_{1i}\,\mathrm{id}_{\Bbbk}$.

Thank you for your attention!