The Constant Jordan type property for generalized Kronecker quivers

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Notation

- Let k be an algebraically closed field.
- All k-vector spaces and modules are assumed to be finite-dimensional.
- All considered algebras Λ are associative.
- $\operatorname{mod}\Lambda$ denotes the category of left Λ-modules.
- G denotes a finite group and kG its group algebra.

Group algebras and wildness, I

Definition

Λ is called **semi-simple** if every indecomposable module is simple.

Let Λ be semi-simple.

- Thanks to the Theorem of Krull-Remak-Schmidt (1911-1928) the classification of Λ-modules reduces to the classification of all (finitely many) simple Λ-modules.
- The classification of the simple modules may be still very challenging but $\operatorname{mod}\Lambda$ is not rich in structure as $\operatorname{Hom}_\Lambda(S, T) = 0$ for all non-isomorphic indecomposable modules.

Semi-simplicity for group algebras may be characterized as follows:

Theorem (Maschke, 1899)

The algebra kG is semi-simple if and only if $\operatorname{char}(k) \nmid |G|$.

We are interested in the case where $\operatorname{mod} kG$ is rich in structure.

Group algebras and wildness, II

We assume from now on that $p := \mathrm{char}(k) > 0$ and $p \mid |G|$.

Theorem (Bondarenko-Drozd, 1977)

Assume that $p \neq 2$ and let P be a Sylow-p-subgroup of G that is not cyclic, then kG is a **wild** algebra.

Example

For $r \in \mathbb{N}$ we denote by $E_r \cong (\mathbb{Z}/(p))^r$ an elementary abelian p-group of rank r. Then kE_r is wild whenever $p \neq 2$ and $r \geq 2$.

Wildness: If Λ is a wild algebra, then the classification of $\mathrm{mod}\,\Lambda$ is as difficult as the classification of $\mathrm{mod}\,B$ for every other algebra B and therefore considered a hopless task.

Motivation

Goal: Better understanding of mod kG for kG wild.

Fruitful approaches:

1. Study suitable subcategories $\mathcal{C} \subseteq \operatorname{mod} kG$.
2. Study modules in mod kG via their restrictions to algebraic families of subalgebras of kG isomorphic to kE_r, $r \in \mathbb{N}$.
   - Chouinard, 1976: A kG-module M is projective if and only if $M|_{kE_r}$ is projective for every $E_r \subseteq G$, $\forall r \in \mathbb{N}$.
3. Attach reasonable invariants to modules in mod kG that are invariant under isomorphism classes.
   - Carlson, 1983: Rank varieties $V(E_r)_M$ for modules in mod kE_r.

This talk: Modules of constant Jordan type for mod kE_r.

- Introduced by Carlson-Friedlander-Pevtsova in 2008 for finite group (schemes) over k.
- This approach combines the other three approaches.

Modules of Constant Jordan type, I

Fix a k-complement $V \subseteq \mathrm{Rad}(kE_r)$ of $\mathrm{Rad}^2(kE_r)$ with basis $(x_1, \ldots, x_r)$. Given $M \in \operatorname{mod} kE_r$ and $x \in V$ we denote by x_M the nilpotent operator

$$x_M \colon M \to M, m \mapsto x.m.$$

We denote by

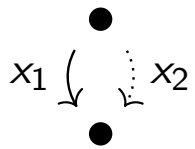
$$\mathrm{JT}(M, x) = [1]^{a_1}[2]^{a_2} \cdots [p]^{a_p}$$

the **Jordan type of** M **at** x, i.e. the Jordan canonical form of x_M has exactly $a_i \in \mathbb{N}_0$ blocks of size i for each $i \in \{1, \ldots, p\}$.

Modules of Constant Jordan type, II

Let $p \geq 2$, $E := (\mathbb{Z}/(p))^2$ with generating set $\{g_1, g_2\}$ and let $V := \langle \{x_i := g_i - 1 \mid 1 \leq i \leq 2\} \rangle$.

- Consider the trivial module k and $x \in V$, then $x_k(k) = 0$ and therefore $\mathrm{JT}(k, x) = [1]^1[2]^0 \cdots [p]^0 = [1]$.
- Let M be the module given by the diagram



Let $x = \lambda_1 x_1 + \lambda_2 x_2 \in V$, then

$$\mathrm{JT}(M, x) = \begin{cases} [1]^2, \lambda_1 = -\lambda_2 \\ [1]^0[2]^1, \text{ else.} \end{cases}$$

Modules of Constant Jordan type, III

Definition

A module $M \in \operatorname{mod} kE_r$ has

- **constant Jordan type**, if $\mathrm{JT}(M, x) = \mathrm{JT}(M, y) \forall x, y \in V - 0$.
- the **equal images property**, if $\operatorname{im} x_M$ is independent of $x \neq 0$.
- the **equal kernels property**, if $\ker x_M$ is independent of $x \neq 0$.

$\rightsquigarrow$ full subcategories $\mathrm{CJT}(kE_r)$, $\mathrm{EIP}(kE_r)$ and $\mathrm{EKP}(kE_r)$.

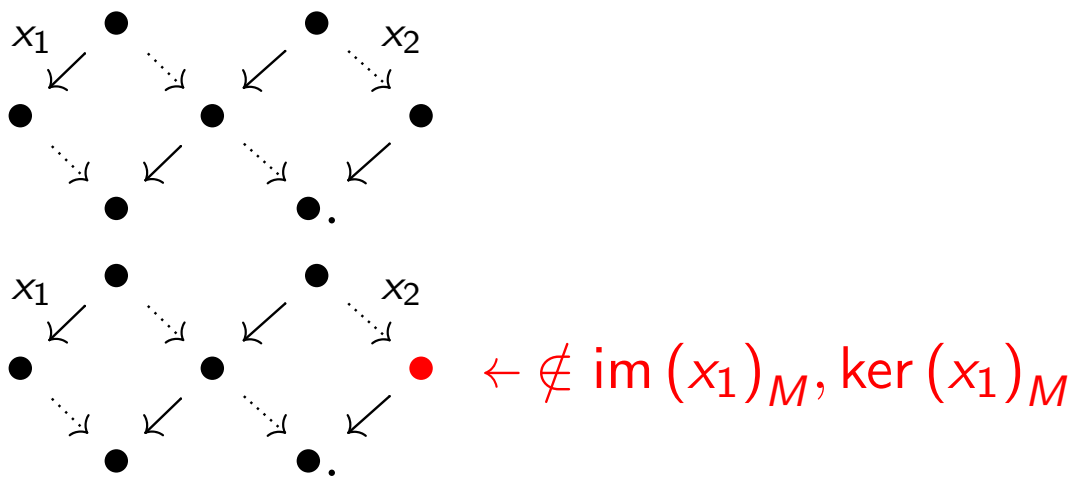
Notation: We say M has CJT $[1]^{a_1}[2]^{a_2}\cdots[p]^{a_p}$ if $M \in \mathrm{CJT}(kE_r)$ and $[1]^{a_1}[2]^{a_2}\cdots[p]^{a_p} = \mathrm{JT}(M, x)$ for some $x \in V - 0$.

Remark

The properties CJT, EKP, EIP do not depend on the complement V.

Examples

- The trivial module k has CJT $[1]^1[2]^0 \cdots [p]^0 = [1]$.
- The regular module kE_r has constant Jordan type $[p]^{p^{r-1}}$.
- Let $p \geq 3$, $E := (\mathbb{Z}/(p))^2$ with generating set $\{g_1, g_2\}$ and let $V := \langle \{x_i := g_i - 1 \mid 1 \leq i \leq 2\} \rangle$. Let M be the module given by the diagram



Note that $M \notin \mathrm{EKP}(kE_r) \cup \mathrm{EIP}(kE_r)$. Moreover, we have $M \in \mathrm{CJT}(kE_r)$ if and only if $p = 3$. Then M has CJT $[2]^2[3]$.

Properties of CJT(kE_r)

- EKP(kE_r), EIP(kE_r), CJT(kE_r) are closed under direct sums and summands.
- EKP(kE_r), EIP(kE_r) $\subseteq$ CJT(kE_r).
- The category CJT(kE_r) is **wild** whenever kE_r is wild, i.e. $r \geq 2$ and $p \neq 2$ (Bondarenko-Lytvynchuk, 2012).
- Benson-Pevtsova, 2012: For $1 \leq i \leq p$ there are functors $\mathcal{F}_i \colon \operatorname{mod} kE_r \to \operatorname{Coh}(\mathbb{P}^{r-1})$ s.t. for $M \in \operatorname{mod} kE_r$ the following statements are equivalent:
  1. $M \in \mathrm{CJT}(kE_r)$ of constant Jordan type $[1]^{a_1}[2]^{a_2} \cdots [p]^{a_p}$.
  2. $\forall i \in \{1, \ldots, p\} : \mathcal{F}_i(M)$ alg. vector bundle on $\mathbb{P}^{r-1}$ of rank a_i.

The realization problem in CJT(kE_r)

Question: Which Jordan types are realized as the Jordan type of (indecomposable) modules in CJT(kE_r)?

There are several open problems related to this question:

Suslin's conjecture. Suppose that $M \in \operatorname{mod} kE_r$ has constant Jordan type $[1]^{a_1}[2]^{a_2}\cdots[p]^{a_p}$ such that $a_j \neq 0$ for some $j \in \{2, \ldots, p-1\}$, then $a_{j-1} + a_{j+1} \neq 0$.

This talk: Restrictions on Jordan types for modules in CJT(kE_r) $\cap \operatorname{mod}_2 kE_r$ (wild for $r \geq 3$), where $\operatorname{mod}_2 kE_r$ is the full subcategory of modules of Loewy length ≤ 2.

M has Loewy length $\leq 2 :\Longleftrightarrow \operatorname{Rad}^2(kE_r).M = 0$.

The realization problem in $\mathrm{CJT}(kE_r) \cap \mathrm{mod}_2\, kE_r$

Let $M \in \mathrm{mod}_2\, kE_r$ and assume that M has constant Jordan type.

- The CJT is of the form $[1]^c[2]^d$ with $c, d \in \mathbb{N}_0$.
- We have $d = 0$ if and only if M is semi-simple.
- If $d \neq 0$, then Westwick's Theorem implies that $c \geq r - 1$.

Theorem (Benson)

There exists a module $M \in \mathrm{mod}_2\, kE_r$ of constant Jordan type $[1]^c[2]^d$ if and only if $d = 0$ or $d \neq 0$ and $c \geq r - 1$.

Question: Is there an analogous result for M indecomposable?

$\rightsquigarrow$ We will investigate this question from a different viewpoint.

Kronecker quiver

For $r \in \mathbb{N}$ the quiver

$$\Gamma_r := \quad 1 \;\overset{\gamma_1}{\underset{\gamma_r}{\rightrightarrows}}\; 2$$

is called the (generalized) **Kronecker quiver** with r arrows.
Denote by $\operatorname{rep}(\Gamma_r)$ the category of f.d. Γ_r-representations:

- **Objects**: $M = (M_1, M_2, M(\gamma_1), \ldots, M(\gamma_r))$, where $M_1, M_2 \in \operatorname{mod} k$ and $M(\gamma_1), \ldots, M(\gamma_r) \colon M_1 \to M_2$ are k-linear maps.
- **Morphisms**: $f = (f_1, f_2) \colon M \to N$ where $f_i \colon M_i \to N_i$ are k-linear s.t. $\forall j \in \{1, \ldots, r\}$:

$$\begin{array}{ccc} M_1 & \xrightarrow{M(\gamma_j)} & M_2 \\ \downarrow{\scriptstyle f_1} & \circlearrowleft & \downarrow{\scriptstyle f_2} \\ N_1 & \xrightarrow{N(\gamma_j)} & N_2. \end{array}$$

$\mathfrak{F}\colon \operatorname{rep}(\Gamma_r) \to \operatorname{mod}_2 kE_r$

We have an exact, faithful, dense functor $\mathfrak{F}\colon \operatorname{rep}(\Gamma_r) \to \operatorname{mod}_2 kE_r$

$$\mathfrak{F}(M) := M_1 \oplus M_2 \text{ and } x_j.(m_1 + m_2) = M(\gamma_j)(m_1)$$

for all $m_1 + m_2 \in \mathfrak{F}(M)$, $1 \leq j \leq r$.

- $\mathfrak{F}$ almost reflects isomorphism classes on indecomposables: $\mathfrak{F}(S_1) = k = \mathfrak{F}(S_2)$, where $S_1 := k \xrightarrow{r} 0$, $S_2 := 0 \xrightarrow{r} k$.
- $\mathfrak{F}$ is far from being full ($r \geq 2$):

$$M := \quad k \overset{0}{\underset{\text{id}}{\rightrightarrows}} k$$

$$N := \quad k \overset{\text{id}}{\underset{0}{\rightrightarrows}} k.$$

$\Rightarrow \operatorname{Hom}(M, N) = 0$, but $\operatorname{Hom}_{kE_r}(\mathfrak{F}(M), \mathfrak{F}(N)) \neq 0$ as kE_r is local and $\mathfrak{F}(M), \mathfrak{F}(N) \neq 0$.

$\mathrm{EIP}(\Gamma_r), \mathrm{EKP}(\Gamma_r), \mathrm{CJT}(\Gamma_r)$

Definition

For $\alpha \in k^r$, $M \in \mathrm{rep}(\Gamma_r)$ let $M^\alpha := \sum_{i=1}^r \alpha_i M(\gamma_i) \colon M_1 \to M_2$.

- $\mathrm{EIP}(\Gamma_r) := \{M \in \mathrm{rep}(\Gamma_r) \mid \forall \alpha \in k^r - 0 : \operatorname{im} M^\alpha = M_2\}$.
- $\mathrm{EKP}(\Gamma_r) := \{M \in \mathrm{rep}(\Gamma_r) \mid \forall \alpha \in k^r - 0 : \ker M^\alpha = 0\}$.
- $\mathrm{CJT}(\Gamma_r) := \{M \in \mathrm{rep}(\Gamma_r) \mid \forall \alpha \in k^r - 0 : \mathrm{rk}(M^\alpha) \text{ is constant}\}$.

Let $\mathcal{X} \in \{\mathrm{CJT}, \mathrm{EIP}, \mathrm{EKP}\}$, then the essential image of

$$\mathfrak{F}_{|\mathcal{X}(\Gamma_r)} \colon \mathcal{X}(\Gamma_r) \to \mathrm{mod}_2\, kE_r$$

is $\mathcal{X}_2(kE_r) := \mathcal{X}(kE_r) \cap \mathrm{mod}_2\, kE_r$.

$\rightsquigarrow$ Study $\mathrm{CJT}_2(kE_r), \mathrm{EKP}_2(kE_r)$ with new tools in $\mathrm{rep}(\Gamma_r)$.

The path algebra of $1 \rightrightarrows 2$

The category $\mathrm{rep}(\Gamma_r)$ is in a natural way equivalent to the module category of the path algebra $k\Gamma_r$:

- Underlying k-space: $k\Gamma_r = k\varepsilon_1 \oplus k\varepsilon_2 \oplus \bigoplus_{i=1}^{r} k\gamma_i$.
- Multiplication: Given by concatenation of paths, where ε_i denotes the trivial path at the vertex i. For example:
  - $\gamma_i\gamma_j = 0$ for all $i, j \in \{1, \ldots, r\}$.
  - $\gamma_i\varepsilon_1 = \gamma_i$ and $\varepsilon_1\gamma_i = 0 \forall i \in \{1, \ldots, r\}$.

As a path algebra $k\Gamma_r$ is **hereditary**, i.e. submodules of projective modules are projective. The algebra $k\Gamma_r$ is wild if and only if $r \geq 3$.

$\rightsquigarrow$ $\mathrm{rep}(\Gamma_r)$ is an hereditary category with simple objects S_1, S_2 and $\mathrm{rep}(\Gamma_r)$ is a wild category iff $r \geq 3$.

Kac's Theorem

Let $q_{\Gamma_r} : \mathbb{Z}^2 \to \mathbb{Z}, (x, y) \mapsto x^2 + y^2 - rxy$ be the **Tits form of** Γ_r and let

$$\underline{\dim} : \ \mathrm{rep}(\Gamma_r) \to \mathbb{Z}^2, M \mapsto (\dim_k M_1, \dim_k M_2)$$

be the dimension vector function.

Theorem (Kac, 1982), Γ_r-version.

Let $\delta \in \mathbb{N}_0^2 - 0$.

1. There exists an indecomposable representation with dimension vector δ if and only if $q_{\Gamma_r}(\delta) \leq 1$. (**positive roots**)
2. If $q_{\Gamma_r}(\delta) < 1$, there exist infinitely many p.w. non-isomorphic indecomposable representations with dimension vector δ. (**positive imaginary roots**)

The classical cases

1. Let $r = 1$. There are exactly (up to isomorphism) three indecomposable representations $S_1 = k \to 0$, $S_2 = 0 \to k$ and $k \xrightarrow{\mathrm{id}} k$. We have
   - $\mathrm{EIP}(\Gamma_1) = \mathrm{add}[S_1 \oplus k \xrightarrow{\mathrm{id}} k]$, $\mathrm{EKP}(\Gamma_1) = \mathrm{add}[S_2 \oplus k \xrightarrow{\mathrm{id}} k]$, and
   - $\mathrm{CJT}(\Gamma_1) = \mathrm{rep}(\Gamma_1)$.
2. Let $r = 2$ (Classification originally by Kronecker)
   - We have $q_{\Gamma_2}(a, b) = (a - b)^2 \geq 0$.
   - There exists an indecomposable representation M with $\underline{\dim}\, M = (a, b) \in \mathbb{N}_0^2 - 0$ iff $|a - b| \in \{0, 1\}$.
   - For $M \in \mathrm{rep}(\Gamma_2)$ indecomposable one has

$$M \in \mathrm{CJT}(\Gamma_r) \Leftrightarrow |\dim_k M_1 - \dim_k M_2| = 1.$$

First observations

Initial question: Let $X \in \mathrm{CJT}_2(kE_r)$ be indecomposable of CJT $[1]^c[2]^d$.What can we say about (c,d)?

Example

Assume that $X \in \mathrm{EKP}_2(kE_r) \subseteq \mathrm{CJT}_2(kE_r)$.

- There is $M \in \mathrm{EKP}(\Gamma_r)$ indecomposable s.t. $\mathfrak{F}(M) \cong X$.
- We have $d = \mathrm{rk}(M^\alpha)\ \forall \alpha \in k^r - 0 \overset{M \in \mathrm{EKP}(\Gamma_r)}{\Longrightarrow} d = \dim_k M_1$.
- $c = \dim_k X - 2d = \dim_k M_2 - \dim_k M_1$.

$\Longrightarrow (d, d+c) = \underline{\dim}\, M \overset{\text{Kac}}{\Longrightarrow} q_{\Gamma_r}(d, d+c) \leq 1$. Moreover, Westwick's Theorem implies $(c,d) = (1,0)$ or $d \neq 0$ and $c \geq r-1$.

Summary: $q_{\Gamma_r}(d, d+c) \leq 1$ and $(c,d) = (1,0)$ or $c \geq r-1$.

Restrictions on Jordan types in $\mathrm{CJT}_2(kE_r)$

From now on we assume that $r \geq 3$.

Theorem (2019)

Let $X \in \mathrm{CJT}_2(kE_r)$ be indecomposable of CJT $[1]^c[2]^d$, then $q_{\Gamma_r}(d, d+c) \leq 1$ and either $(c,d) = (1,0)$ or $c \geq r-1$.

Sketch of proof: Let $M \in \mathrm{CJT}(\Gamma_r)$ be indecomposable s.t. $\mathfrak{F}(M) \cong X$. By Kac's Theorem, we have $q_{\Gamma_r}(\underline{\dim}\, M) \leq 1$.
We consider 3 cases:

1. $M \in \mathrm{EKP}(\Gamma_r) \cup \mathrm{EIP}(\Gamma_r)$. ✓
2. $q_{\Gamma_r}(\underline{\dim}\, M) = 1$, then $M \in \mathrm{EKP}(\Gamma_r) \cup \mathrm{EIP}(\Gamma_r)$. ✓
3. $q_{\Gamma_r}(\underline{\dim}\, M) < 1$ and $M \notin \mathrm{EKP}(\Gamma_r) \cup \mathrm{EIP}(\Gamma_r)$.
   The main tool in this case is the **generic Jordan type**.

The generic Jordan Type

Definition

For $M \in \mathrm{rep}(\Gamma_r)$ we define

$$\mathrm{MaxRk}(M) := \{[\alpha] \in \mathbb{P}^{r-1} \mid \mathrm{rk}(M^\alpha) \text{ is maximal}\} \subseteq \mathbb{P}^{r-1}.$$

For $[\alpha] \in \mathrm{MaxRk}(M)$ the number $d_M := \mathrm{rk}(M^\alpha)$ is called the **generic rank** or **maximal rank**. We set $c_M := \dim_k M_1 + \dim_k M_2 - 2d_M$ and call the pair (c_M, d_M) the **generic Jordan type of** M.

Examples:

- $k \underset{\mathrm{id}}{\overset{0}{\rightrightarrows}} k$ has generic Jordan type $(0, 1)$.
- Let $M \in \mathrm{CJT}(\Gamma_r)$, then $\mathfrak{F}(M)$ has constant Jordan type $[1]^{c_M}[2]^{d_M}$.

Filtering representations

Let $0 \to A \to M \to B \to 0$ be a short exact sequence, then

$$(*) \quad d_M \geq d_A + d_B \text{ and } c_M \leq c_A + c_B.$$

Idea: For $M \in \mathrm{rep}(\Gamma_r)$ consider filtrations

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

and use $(*)$ to get information about the (generic) Jordan type of representations.

Example (of an unuseful filtration)

Let $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a Jordan-Hölder filtration. Since S_1, S_2 have generic Jordan type $(1, 0)$, iterated application of $(*)$ yields $d_M \geq 0$ and $c_M \leq n = \dim_k M_1 + \dim_k M_2$.

Regular representations

Recall: We are interested in the Jordan type of indecomposable representations M with $q_{\Gamma_r}(\underline{\dim}\, M) < 1$.

Definition

An indec. representation M is called **regular** if $q_{\Gamma_r}(\underline{\dim}\, M) < 1$.
An arbitrary representation N is called **regular** if every indec. direct summand of M is regular.

- The simple representations S_1, S_2 are not regular.
- Regular representations are closed under extensions, direct sums and direct summands.
- Regular representations are not closed under subrepresentations and factors.

Elementary representations, I

Definition

Let $r \geq 3$. A non-zero regular representation $E \in \operatorname{rep}(\Gamma_r)$ is called **elementary**, provided there is no short exact sequence $0 \to A \to E \to B \to 0$ with A, B regular and non-zero.

- Each non-zero regular representation of minimal dimension is elementary and elementary representations are indecomposable.
- Each regular representation M has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ s.t. all M_i/M_{i-1} are elementary.
- S_1 and S_2 are not regular and therefore not elementary. E elementary $\implies d_E \neq 0$.

$\rightsquigarrow$ Study the generic Jordan type of elementary representations.

Elementary representations, II

Remarks:

- There are many elementary representations: Let E be elementary and $\Phi := \begin{pmatrix} r^2-1 & -r \\ r & -1 \end{pmatrix}$ the **Coxeter-Transformation**. For each $n \in \mathbb{Z}$, there is an indecomposable representation with dimension vector $\Phi^n \underline{\dim}\, E$. Moreover, $\Phi^n \underline{\dim}\, E \neq \Phi^m \underline{\dim}\, E$ for all $m \neq n$.
- If E is elementary and M is indecomposable with $\underline{\dim}\, M = \underline{\dim}\, E$ then M is not necessarily elementary.
- For $r \geq 4$ the dimension vectors of elementary representations are not classified. Classification for $r = 3$ is due to Ringel.
- One can show: If E is elementary and $E \notin \mathrm{EKP}(\Gamma_r) \cup \mathrm{EIP}(\Gamma_r)$ then $\dim_k E_1, \dim_k E_2 \leq r-1$.

The generic Jordan type of regular representations, II

Proposition

Let E be an elementary representation, then $q_{\Gamma_r}(d_E, d_E + c_E) < 1$.

Corollary

Let M be a regular representation, then $q_{\Gamma_r}(d_M, d_M + c_M) < 1$.

Sketch of proof: Consider a filtration
$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ with elementary filtrations factors
$E_i := M_i/M_{i-1}$.

We have $q_{\Gamma_r}(d_{E_i}, d_{E_i} + c_{E_i}) < 1 \Leftrightarrow d_{E_i} + c_{E_i} < d_{E_i} L_r$,
where $L_r := \frac{r+\sqrt{r^2-4}}{2} \notin \mathbb{Q}$.
$\overset{(*)}{\Longrightarrow}$ $c_M + d_M < d_M L_r \Leftrightarrow q_{\Gamma_r}(d_M, d_M + c_M) < 1$.

The realization problem in $\mathrm{CJT}(kE_r) \cap \mathrm{mod}_2\, kE_r$

Theorem (2019)

1. Let $M \in \mathrm{CJT}_2(kE_r)$ be indecomposable of CJT $[1]^c[2]^d$, then $q_{\Gamma_r}(d, d+c) \leq 1$ and either $(c,d) = (1,0)$ or $c \geq r-1$.
2. For each $(c,d) \in \mathbb{N}_0^2 - 0$ such that $q_{\Gamma_r}(d, d+c) \leq 1$ and $c \geq r-1$, there exists an indecomposable module in $\mathrm{EKP}_2(kE_r)$ (and $\mathrm{EIP}_2(kE_r)$) of constant Jordan type $[1]^c[2]^d$.

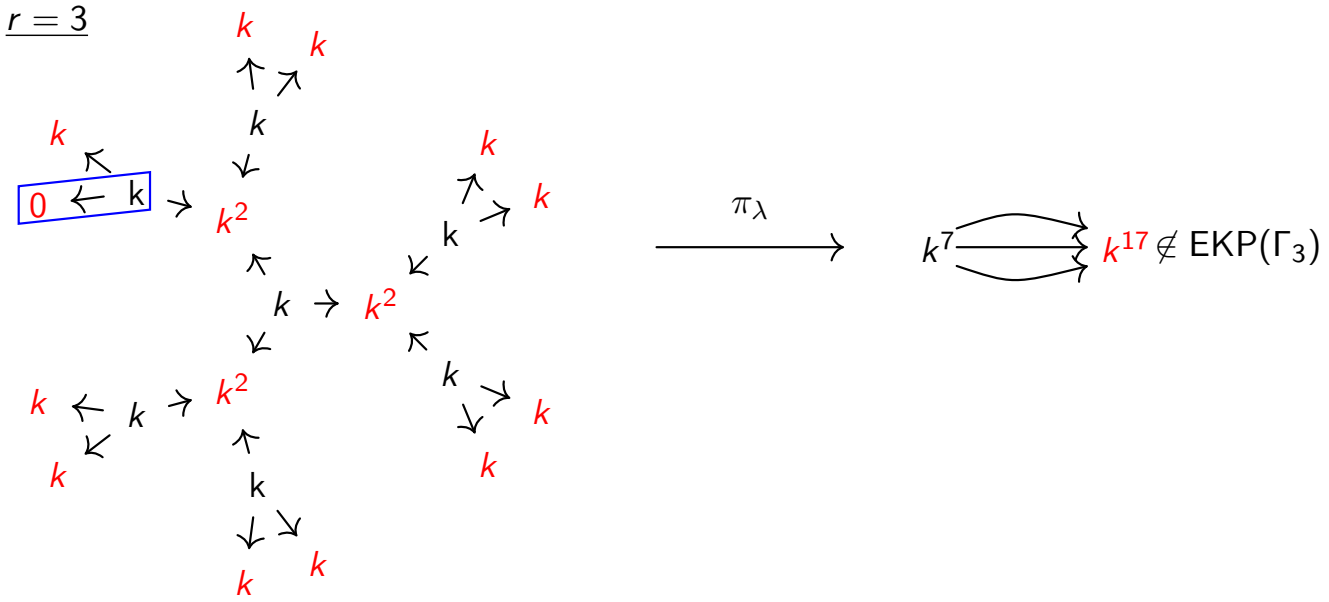
Remark

The second statement is equivalent to the existence of an indecomposable representation $M \in \mathrm{EKP}(\Gamma_r)$ for each element $0 \neq (a,b) \in \mathbb{N}_0^2$ such that $r-1 \leq b-a$ and $q_{\Gamma_r}(a,b) \leq 1$.

The universal covering

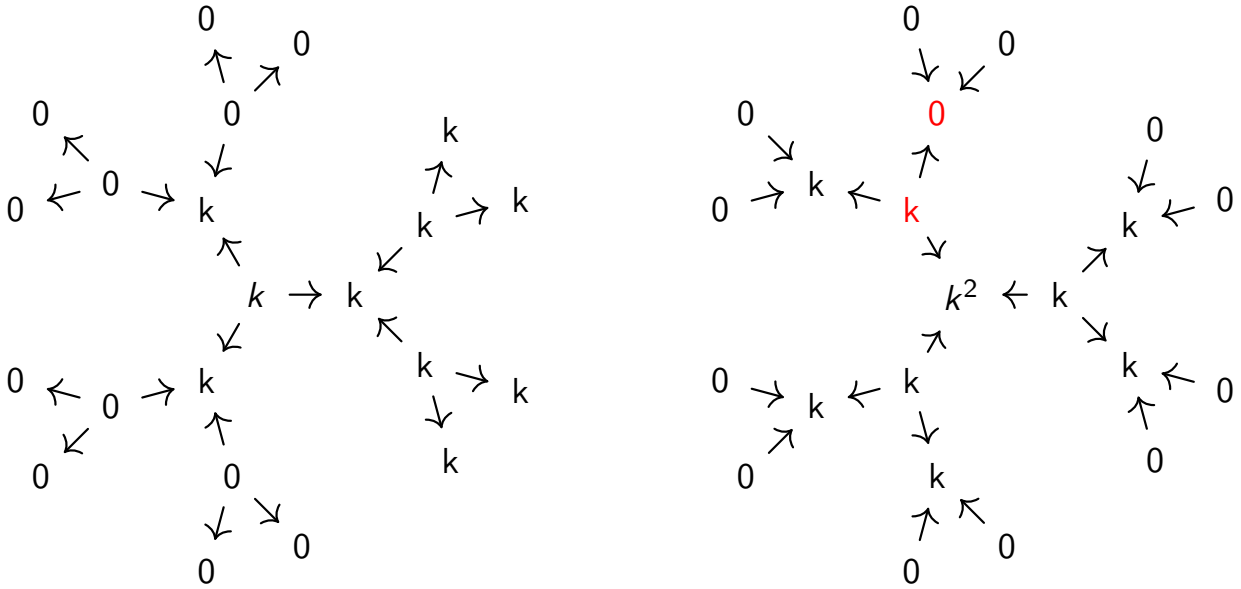
A powerful tool, that is used in our proof, for constructing indecomposable representations (with certains properties) is the universal covering $\pi\colon C_r \to \Gamma_r$.

- C_r is the r-regular (infinite) tree with bipartite orientation.
- There exists an exact functor $\pi_\lambda\colon \operatorname{rep}(C_r) \to \operatorname{rep}(\Gamma_r)$ that preserves and reflects indecomposables.
- For $M \in \operatorname{rep}(C_r)$ the push-down $\pi_\lambda(M)$ satisfies $\pi_\lambda(M) \in \operatorname{EKP}(\Gamma_r)$ iff $M(\gamma)$ is injective for all arrows $\gamma \in C_r$.



Examples

Consider the following two indecomposable representations $M, N \in \mathrm{rep}(\Gamma_3)$ given by the following diagrams:



We have $\underline{\dim}\,\pi_\lambda(M) = (3,7) = \underline{\dim}\,\pi_\lambda(N)$, $\pi_\lambda(M) \in \mathrm{EKP}(\Gamma_3)$ and $\pi_\lambda(N) \notin \mathrm{EKP}(\Gamma_3)$ as $N(k \to 0)\colon k \to 0$ is not injective.