On uniform and homogeneous Steiner bundles and reflection functors

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Motivation and overview, I

Classical viewpoint:

- A Steiner bundle $\mathcal{F}$ on $\mathbb{P}^n = \mathbb{P}^n(\mathbb{C})$ is defined by an exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n}^b \to \mathcal{F} \to 0.$$

- $f \mathrel{\hat{=}} b \times a$ matrix $A(f)$ with coefficients in $\mathbb{C}[X_0, \ldots, X_n]_1$.

- Proofs often rely on explicit matrix computations, involving algorithms or case-by-case considerations that are best understood via examples.

Motivation and overview, II

This talk:

1. Explain the connection between Steiner bundles on Grassmannians and Kronecker representations.

2. Present conceptual proofs for known results for Steiner bundles using tools from representation theory.

3. Compare uniform and homogeneous Steiner bundles.

Steiner bundles, I

Setup:

- $\Bbbk = \overline{\Bbbk}$ field, $\mathrm{Gr}_d(\Bbbk^n)$ Grassmannian, $1 \leq d < n$.
- $\mathrm{Vect}(\mathrm{Gr}_d(\Bbbk^n))$ category of vector bundles (locally free sheaves).

$$0 \to \underbrace{\mathcal{U}_{(n,d)}}_{\text{Universal subbundle}} \to \Bbbk^n \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(\Bbbk^n)} \to \underbrace{\mathcal{Q}_{(n,d)}}_{\text{Universal quotient bundle}} \to 0$$

For $\mathbb{P}^n = \mathrm{Gr}_1(\Bbbk^{n+1})$ Euler sequence:

$$0 \to \underbrace{\mathcal{O}_{\mathbb{P}^n}(-1)}_{\text{Serre twisting sheaf}} \to \Bbbk^{n+1} \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^n} \to \underbrace{\mathcal{T}_{\mathbb{P}^n}(-1)}_{\text{Shift of tangent bundle}} \to 0$$

Definition (Miró-Roig & Soares, Dolgachev-Kapranov)

$\mathcal{F} \in \mathrm{Vect}(\mathrm{Gr}_d(\Bbbk^n))$ is a Steiner bundle if there are $V_1, V_2 \in \mathrm{mod}\,\Bbbk$ and an exact sequence $0 \to V_1 \otimes_{\Bbbk} \mathcal{U}_{(n,d)} \to V_2 \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(\Bbbk^n)} \to \mathcal{F} \to 0$.

Steiner bundles, II

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in $\mathrm{Vect}(\mathrm{Gr}_d(\Bbbk^n))$.

1. $\mathcal{F}_1, \mathcal{F}_3 \in \mathrm{StVect}(\mathrm{Gr}_d(\Bbbk^n)) \implies \mathcal{F}_2 \in \mathrm{StVect}(\mathrm{Gr}_d(\Bbbk^n))$.
2. $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{StVect}(\mathrm{Gr}_d(\Bbbk^n)) \implies \mathcal{F}_3 \in \mathrm{StVect}(\mathrm{Gr}_d(\Bbbk^n))$.

Example (Steiner bundles on $\mathbb{P}^1$)

Let $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^1)$, then $\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} a_i \mathcal{O}_{\mathbb{P}^1}(i)$ (Grothendieck, 1957)

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \Bbbk^2 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^1} \to \underbrace{\mathcal{O}_{\mathbb{P}^1}(1)}_{\in \mathrm{StVect}(\mathbb{P}^1)} \to 0 \quad \text{Euler sequence}$$

$$\rightsquigarrow 0 \to \mathcal{O}_{\mathbb{P}^1} \to \Bbbk^2 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^1}(1) \to \underbrace{\mathcal{O}_{\mathbb{P}^1}(2)}_{\in \mathrm{StVect}(\mathbb{P}^1)} \to 0$$

$\rightsquigarrow \mathcal{F} \in \mathrm{Vect}(\mathbb{P}^1)$ is a Steiner bundle iff $\mathcal{F} \cong \bigoplus_{i \in \mathbb{N}_0} a_i \mathcal{O}_{\mathbb{P}^1}(i)$.

Steiner bundles, III

Theorem (Arrondo-Marchesi, 2012)

Let $(c, s) \in \mathbb{N}_0^2$. There exists a Steiner bundle $\mathcal{F} \in \mathrm{StVect}(\mathrm{Gr}_d(\Bbbk^n))$ with $c_1(\mathcal{F}) = c$ and $\mathrm{rk}(\mathcal{F}) = s$ iff $s \geq (n-d)\min\{d, c\}$.

Corollary

The rank of a non-trivial Steiner bundle on $\mathbb{P}^n$ is $\geq n$.

Connection to Kronecker representations, I

Theorem (Jardim-Prata, 2015, $\mathrm{char}(\Bbbk) = 0$)

Let X smooth proj. variety and $(\mathcal{E}, \mathcal{F})$ be a "suitable" pair of vector bundles. The category of cokernel bundles of type $(\mathcal{E}, \mathcal{F})$ is equivalent to a full subcategory $\mathcal{S}$ of representations of the Kronecker quiver K_r with $r := \dim_{\Bbbk} \mathrm{Hom}_X(\mathcal{E}, \mathcal{F})$ arrows.

One can show:

- The Theorem holds in arbitrary characteristic.
- Steiner bundles = Cokernel bundles of type $(\mathcal{U}_{(n,d)}, \mathcal{O}_{\mathrm{Gr}_d(\Bbbk^n)})$.
- $(\mathcal{U}_{(n,d)}, \mathcal{O}_{\mathrm{Gr}_d(\Bbbk^n)})$ is suitable.
- $\mathcal{S}$ can be described explicitly for $(\mathcal{U}_{(n,d)}, \mathcal{O}_{\mathrm{Gr}_d(\Bbbk^n)})$.

Connection to Kronecker representations, II

Let $1 \leq d < r$

$$K_r := \quad 1 \;\overset{\gamma_1}{\underset{\gamma_r}{\rightrightarrows}}\; 2$$

be the r-Kronecker quiver with arrow space $A_r := \bigoplus_{i=1}^{r} \Bbbk\gamma_i$.

- $M \in \mathrm{rep}(K_r) \mathrel{\hat{=}} \psi_M \colon A_r \otimes_{\Bbbk} M_1 \to M_2; \gamma_i \otimes m \mapsto M(\gamma_i)(m)$.
- $\mathrm{rep}_{\mathrm{inj}}(K_r, d) := \{M \mid \ker \psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1} = \{0\} \text{ for all } \mathfrak{v} \in \mathrm{Gr}_d(A_r)\}$ space of *d*-injective representations.

Proposition (B.-Farnsteiner, 2022)

There is a right exact functor $\widetilde{\Theta}_{d,r} \colon \mathrm{rep}(K_r) \to \mathrm{Coh}(\mathrm{Gr}_d(A_r))$ *s.t.*

1. $\widetilde{\Theta}_{d,r}(M)$ *is a Steiner bundle iff* $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, d)$.
2. $\widetilde{\Theta}_{d,r} \colon \mathrm{rep}_{\mathrm{inj}}(K_r, d) \to \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ *is an exact equivalence.*

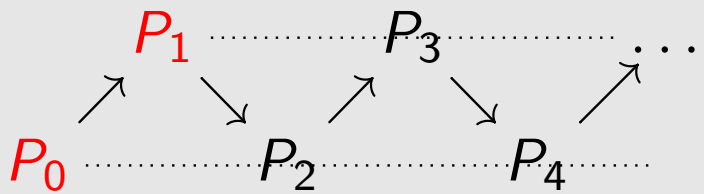
Connection to Kronecker representations, III

Let $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, d)$, $\widetilde{\Theta}_{d,r}\colon \mathrm{rep}_{\mathrm{inj}}(K_r, d) \to \mathrm{StVect}(\mathrm{Gr}_d(A_r))$

- $\mathrm{rk}(\widetilde{\Theta}_{d,r}(M)) = \dim_{\Bbbk} M_2 - d \dim_{\Bbbk} M_1$
- $c_1(\widetilde{\Theta}_{d,r}(M)) = \dim_{\Bbbk} M_1$.

Example (Steiner bundles on $\mathbb{P}^1 \cong \mathbb{P}(A_2)$)

Preprojective Auslander-Reiten component of K_2



$\underline{\dim}\, P_n = (n, n+1)$, $P_n \cong \Bbbk[X,Y]_{n-1} \overset{X\cdot}{\underset{Y\cdot}{\rightrightarrows}} \Bbbk[X,Y]_n \in \mathrm{rep}_{\mathrm{inj}}(K_2, 1)$.

$\rightsquigarrow \widetilde{\Theta}_{1,2}(P_n) = \mathcal{O}_{\mathbb{P}(A_2)}(n)$ and $\mathrm{rep}_{\mathrm{inj}}(K_2, 1) = \mathrm{add}(P_n \mid n \in \mathbb{N}_0)$.

Steiner bundles of minimal rank, I

Theorem (Arrondo-Marchesi, 2012)

Let $(c,s)\in\mathbb{N}_0^2$. *There exists a Steiner bundle* $\mathcal{F}\in \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ *with* $c_1(\mathcal{F})=c$ *and* $\mathrm{rk}(\mathcal{F})=s$ *iff* $s\geq (r-d)\min\{d,c\}$.

Representation-theoretic approach:
Given $(x,y)\in\mathbb{N}_0^2$, consider the variety of representations

$$\mathcal{V}(K_r,\Bbbk^x,\Bbbk^y):=\mathrm{Hom}_{\Bbbk}(A_r\otimes_{\Bbbk}\Bbbk^x,\Bbbk^y).$$

Application of $\widetilde{\Theta}_{d,r}$ yields:
We find $\mathcal{F}\in\mathrm{StVect}(\mathrm{Gr}_d(A_r))$ with $\mathrm{rk}(\mathcal{F})=s$ and $c_1(\mathcal{F})=c$ iff

$$\mathrm{rep}_{\mathrm{inj}}(K_r,d)\cap\mathcal{V}(K_r,\Bbbk^c,\Bbbk^{s+dc})\neq\varnothing.$$

Goal

$\mathrm{rep}_{\mathrm{inj}}(K_r,d)\cap\mathcal{V}(K_r,\Bbbk^x,\Bbbk^y)\neq\varnothing$ iff $y-dx\geq(r-d)\min\{d,x\}$.

Steiner bundles of minimal rank, II

Let $e = (e_1, e_2), f = (f_1, f_2) \in \mathbb{N}_0^2$, then

$$\mathcal{V}_{e,f}(K_r) := \{M \in \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2}) \mid \exists N \leq M \colon \underline{\dim}\, N = e\}$$

is closed.

Theorem (Reineke, 2024)

Let $q_r(f) \leq 0$ and $e \leq f$. The following statements are equivalent.

1. $\mathcal{V}_{e,f}(K_r) \neq \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2})$.
2. $\langle e, f - e\rangle_r < 0$. *(Euler form)*

Corollary

Let $e \leq f$ and $q_r(e) \leq 1$. The following statements are equivalent.

1. $\mathcal{V}_{e,f}(K_r) \neq \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2})$.
2. $\langle e, f - e\rangle_r < 0$.

Steiner bundles of minimal rank, III

Let $M \in \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2})$ such that $e := (1, r-1) \leq f$.

$$\begin{aligned} M \notin \mathrm{rep}_{\mathrm{inj}}(K_r, 1) \Rightarrow \quad & a \in A_r \setminus \{0\} \colon \ker(\psi_M|_{\Bbbk a \otimes_{\Bbbk} M_1}) \neq \{0\} \\ \Rightarrow \quad & \exists m \in M_1, \underline{\dim}\langle m \rangle = (1, \ell) \leq (1, r-1) \\ \Rightarrow \quad & N := \langle m \rangle \oplus S(2)^{r-1-\ell} \leq M, \underline{\dim}\, N = e \\ \Rightarrow \quad & M \in \mathcal{V}_{e,f}(K_r). \end{aligned}$$

Corollary

1. $X := \mathrm{rep}_{\mathrm{inj}}(K_r, 1) \cap \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2}) = \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2}) \setminus \mathcal{V}_{e,f}(K_r)$.
2. $X \neq \varnothing \Leftrightarrow \langle (1, r-1), (f_1 - 1, f_2 - r + 1) \rangle_r < 0 \Leftrightarrow f_2 - f_1 \geq r - 1$.

Steiner bundles of minimal rank, III

Let $M \in \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2})$ such that $e := (1, r-1) \leq f$.

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Corollary

1. $\operatorname{rep}_{\mathrm{inj}}(K_r, d) \cap \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2})$ *is open.*
2. $\operatorname{rep}_{\mathrm{inj}}(K_r, d) \cap \mathcal{V}(K_r, \Bbbk^{f_1}, \Bbbk^{f_2}) \neq \varnothing \Leftrightarrow f_2 - d \cdot f_1 \geq (r-d) \min\{d, f_1\}$.

Steiner bundles arrange nicely in AR components, I

We have the following general results.

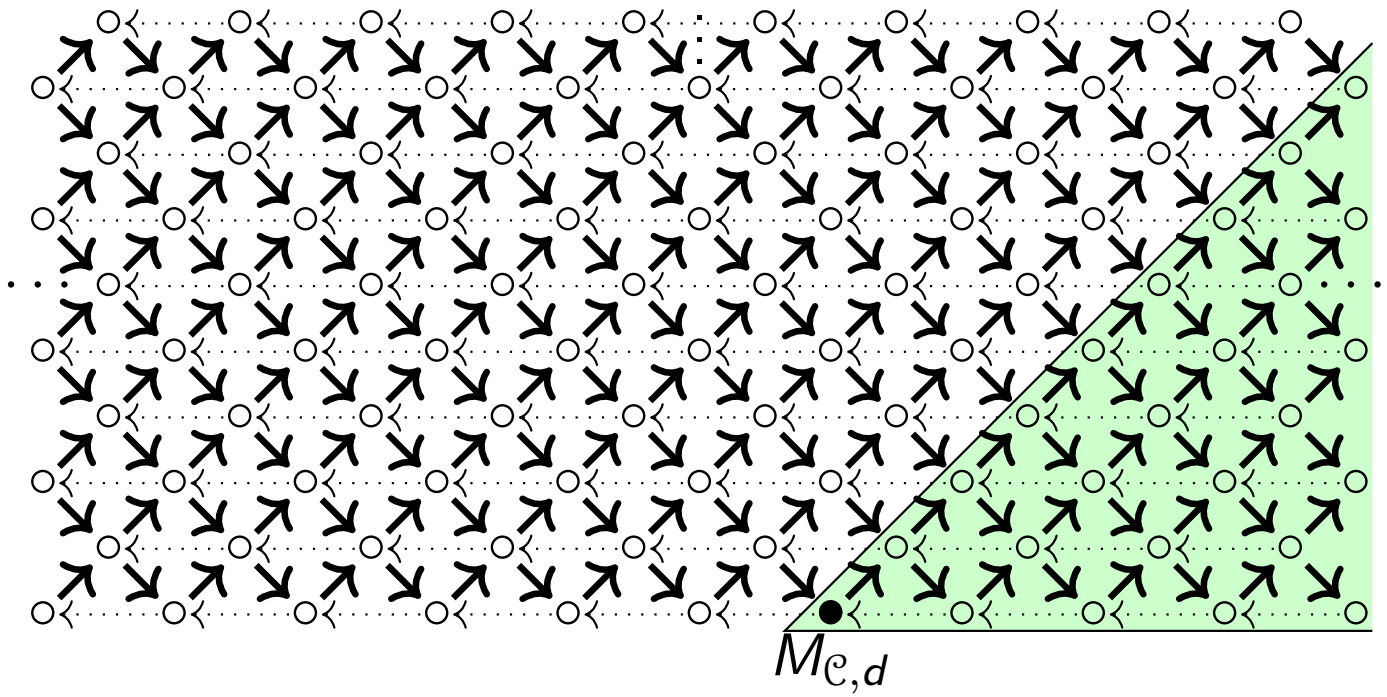
- $\mathrm{rep}_{\mathrm{inj}}(K_r, r-1) \subseteq \mathrm{rep}_{\mathrm{inj}}(K_r, r-2) \subseteq \cdots \subseteq \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$.
- $\mathrm{rep}_{\mathrm{inj}}(K_r, d)$ torsion free class.
- $\mathcal{P} \subseteq \mathrm{rep}_{\mathrm{inj}}(K_r, d)$.
- $\mathrm{rep}_{\mathrm{inj}}(K_r, d)$ is closed under
  1. $\sigma_{K_r}^{-1}\colon\ \mathrm{rep}(K_r) \to \mathrm{rep}(K_r)$ (reflection functor)
  2. $\tau_{K_r}^{-1} \cong \sigma_{K_r}^{-2}$ (AR translation).
- $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, r-1)$ iff $\sigma_{K_r}(M) \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$.

Steiner bundles arrange nicely in AR components, II

For $r \geq 3$ and $\mathcal{C}$ regular AR component there are $M_{\mathcal{C},d} \in \mathcal{C}$ quasi-simple ($1 \leq d < r$) s.t.

$$\mathcal{C} \cap \mathrm{rep}_{\mathrm{inj}}(K_r, d) = (M_{\mathcal{C},d} \to) \quad \text{(cone of successors)}.$$

Illustration:

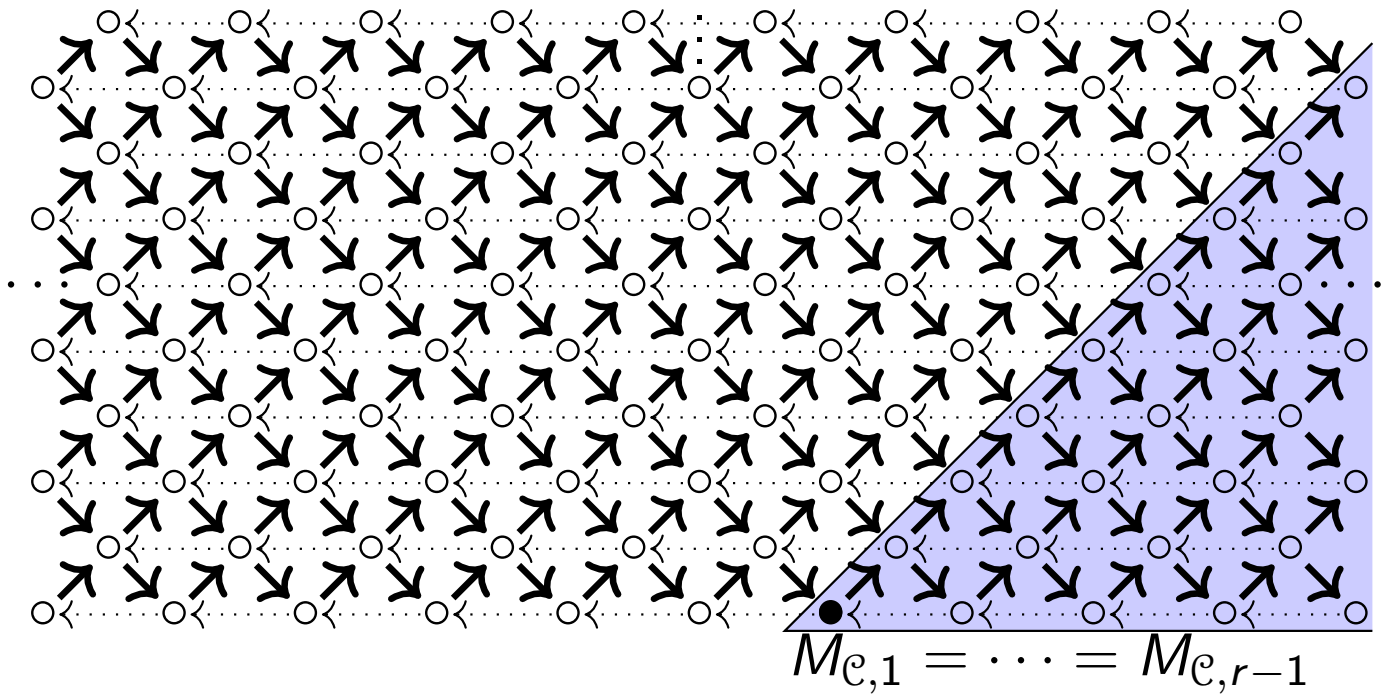


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Case 1:

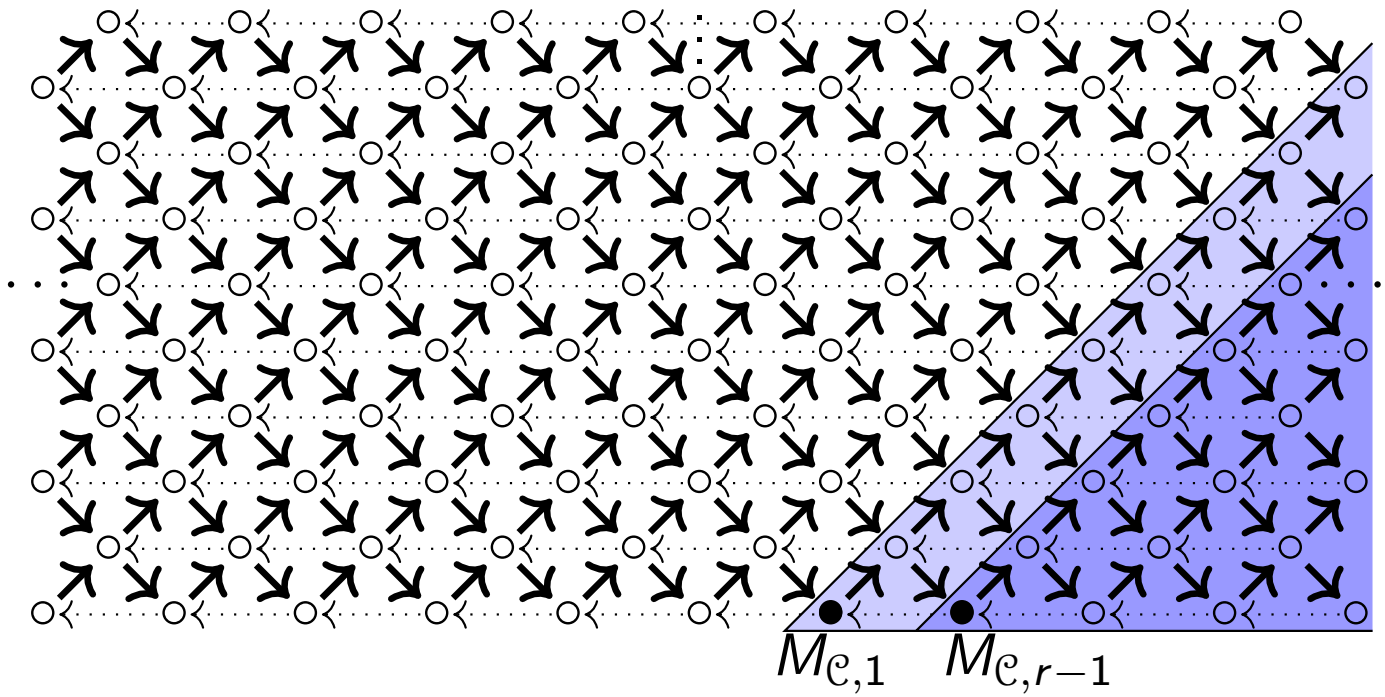


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Case 2:



Uniform vs homogeneous, I

Let $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^n)$ and $\varphi\colon \mathbb{P}^d \hookrightarrow \mathbb{P}^n$ be a linear embedding, then

$$\varphi^*(\mathcal{F}) \in \mathrm{Vect}(\mathbb{P}^d) \quad (\textit{restriction/pullback}).$$

Definition

$\mathcal{F}$ is called

1. homogeneous, if $\varphi^*(\mathcal{F}) \cong \mathcal{F}$ for all $\varphi\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^n \in \mathrm{Aut}(\mathbb{P}^n)$.
2. uniform, if $\varphi^*(\mathcal{F}) \cong \psi^*(\mathcal{F})$ for all $\varphi, \psi\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$.

- $\mathcal{F}$ uniform $\Leftrightarrow \exists (a_i)_{i\in\mathbb{Z}}$ s.t. $\varphi^*(\mathcal{F}) \cong \bigoplus_{i\in\mathbb{Z}} a_i \mathcal{O}_{\mathbb{P}^1}(i)$ for all φ.
- homogeneous $\Rightarrow$ uniform; only on $\mathbb{P}^1$ "iff" holds (Elencwajg).
- Conjecture: For $\mathrm{char}(\Bbbk) = 0$ every uniform bundle of rank $< 2n$ on $\mathbb{P}^n$ is homogeneous (known for $n \in \{1, 2, 3\}$).

Uniform vs homogeneous, II

Question: Are uniform Steiner bundles homogeneous?

Theorem (Marchesi & Miró-Roig, 2021)

Let $\mathrm{char}(\Bbbk) = 0$ *and* $c \geq 4$. *There exists* $\mathcal{E}_{c,n} \in \mathrm{StVect}(\mathbb{P}^n)$ *with*

- $c_1(\mathcal{F}) = c$ *and* $\mathrm{rk}(\mathcal{F}) = c + 2(n-1)$,
- *not homogeneous,*
- *but uniform of 1-type, i.e.* $\mathcal{F}|_{\mathbb{P}^1} = a_0\mathcal{O}_{\mathbb{P}^1} \oplus a_1\mathcal{O}_{\mathbb{P}^1}(1)$ *with* $a_1 \neq 0$.

Main step: Construct $\mathcal{E}_{c,2} \in \mathrm{StVect}(\mathbb{P}^2)$ of rank $c+2$.

Homogeneous and uniform representations, I

Let $\widetilde{\Theta}_r := \widetilde{\Theta}_{1,r}\colon \mathsf{rep}_{\mathsf{inj}}(K_r, 1) \to \mathsf{StVect}(\mathbb{P}(A_r))$ and $M \in \mathsf{rep}(K_r)$.

- For $\alpha \in \mathsf{Inj}_{\Bbbk}(A_d, A_r) := \{\beta \in \mathsf{Hom}_{\Bbbk}(A_d, A_r) \mid \ker \beta = 0\}$

$$\psi_{\alpha^*(M)}\colon A_d \otimes_{\Bbbk} M_1 \overset{\alpha\otimes \mathsf{id}_{M_1}}{\to} A_r \otimes_{\Bbbk} M_1 \overset{\psi_M}{\to} M_2 \in \mathsf{rep}(K_d)$$

  is called restriction of M to K_d (along α).
- $\widetilde{\Theta}_d(\alpha^*(M)) \cong \hat{\alpha}^*(\widetilde{\Theta}_r(M))$ for $\hat{\alpha}\colon \mathbb{P}(A_d) \hookrightarrow \mathbb{P}(A_r)$; $\Bbbk x \mapsto \Bbbk\alpha(x)$.

Corollary

Let $M \in \mathsf{rep}_{\mathsf{inj}}(K_r, 1)$, then $\widetilde{\Theta}_r(M) \in \mathsf{StVect}(\mathbb{P}(A_r))$.

1. *$\widetilde{\Theta}_r(M)$ is uniform iff $\alpha^*(M) \cong \beta^*(M)$ for all $\alpha, \beta \in \mathsf{Inj}_{\Bbbk}(A_2, A_r)$.*
2. *$\widetilde{\Theta}_r(M)$ is homogeneous iff $M \cong g^*(M)$ for all $g \in \mathsf{GL}(A_r)$.*

$\rightsquigarrow$ homogeneous + uniform representations.

Homogeneous and uniform representations, II

Lemma

For $X \in \mathrm{rep}(K_2)$ are equivalent:

1. $\psi_X \colon A_2 \otimes_{\Bbbk} X_1 \to X_2$ *is injective.*
2. X *is projective, i.e.* $X \cong \underbrace{(\dim_{\Bbbk} X_2 - 2\dim_{\Bbbk} X_1)}_{=:\Delta_X(2)} P_0 \oplus (\dim_{\Bbbk} X_1) P_1$

Corollary (B.-Farnsteiner, 2022)

For $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$ with $M_1 \neq 0$ are equivalent:

1. $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$ $(\mathrm{rk}(\psi_{\alpha^*(M)}) = \mathrm{rk}(\psi_M|_{\mathrm{im}\,\alpha \otimes_{\Bbbk} M_1}))$.
2. $\forall \alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r) \colon \alpha^*(M) \cong \Delta_M(2) P_0 \oplus (\dim_{\Bbbk} M_1) P_1$.
3. $\widetilde{\Theta}_r(M)$ *is uniform of* 1*-type.*

Observe : $\hat{\alpha}^*(\widetilde{\Theta}_r(M)) \cong \widetilde{\Theta}_2(\bigoplus_{n \in \mathbb{N}_0} a_n(\alpha) P_n) \cong \bigoplus_{n \in \mathbb{N}_0} a_n(\alpha) \mathcal{O}_{\mathbb{P}(A_2)}(n)$.

Homogeneous and uniform representations, III

Corollary (Marchesi & Miró-Roig)

Let $c \geq 4$ *and* $\mathrm{char}(\Bbbk) = 0$. *There exists* $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}^2)$ *with*

1. $c_1(\mathcal{F}) = c$ *and* $\mathrm{rk}(\mathcal{F}) = c + 2$
2. *uniform of* 1*-type,*

Every such bundle is not homogeneous.

Sketch of proof.

- We have $M \in \mathrm{rep}_{\mathrm{inj}}(K_3, 2) \cap \mathcal{V}(K_r, \Bbbk^c, \Bbbk^{2c+2}) \neq \varnothing$.
- $\mathcal{F} := \widetilde{\Theta}_3(M)$ with $M \in \mathrm{rep}_{\mathrm{inj}}(K_3, 2) = \mathrm{rep}_{\mathrm{inj}}(K_3, r-1)$.
- $N := \sigma_{K_3}(M) \in \mathrm{rep}_{\mathrm{inj}}(K_3, 1)$ and $\underline{\dim}\, N = (c-2, c)$.
- $\mathcal{G} := \widetilde{\Theta}_3(N) \in \mathrm{StVect}(\mathbb{P}(A_3))$ of rank 2 and $c_1(\mathcal{G}) = c - 2$.

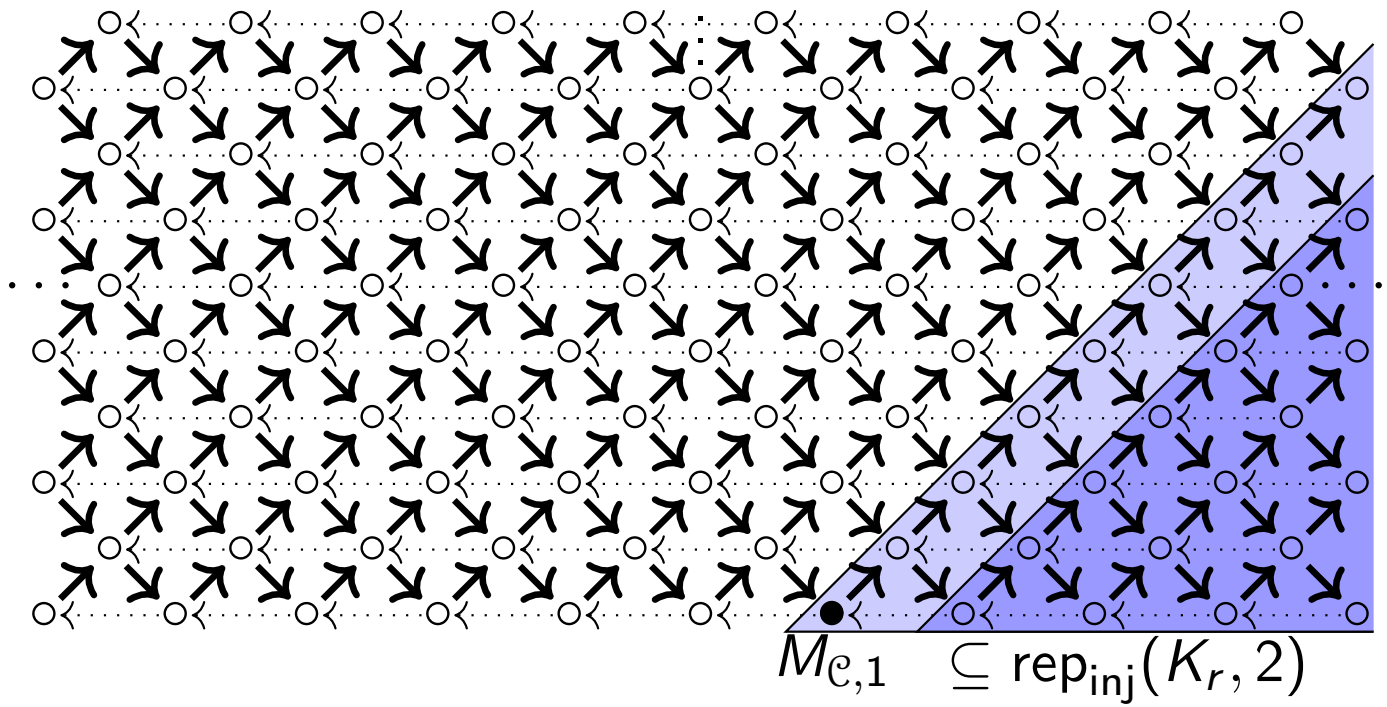
$\mathcal{F}$ homogeneous $\Rightarrow$ $\mathcal{G}$ homogeneous $\overset{\text{Van de Ven}}{\Rightarrow}$ $\mathcal{G} \cong \mathcal{T}_{\mathbb{P}(A_3)}(-1)$.
Hence $c - 2 = c_1(\mathcal{T}_{\mathbb{P}(A_3)}(-1)) = 1$. ↯ □

Homogeneous vs uniform in AR components, I

Proposition

Let $r \geq 3$, $\mathcal{C}$ be a regular component.

1. $X \in (\tau_{K_r}^{-1}(M_{\mathcal{C},1}) \to)$ *is uniform.*
2. $M_{\mathcal{C},1}$ *uniform iff all* $E \in (M_{\mathcal{C},1} \to)$ *are uniform.*

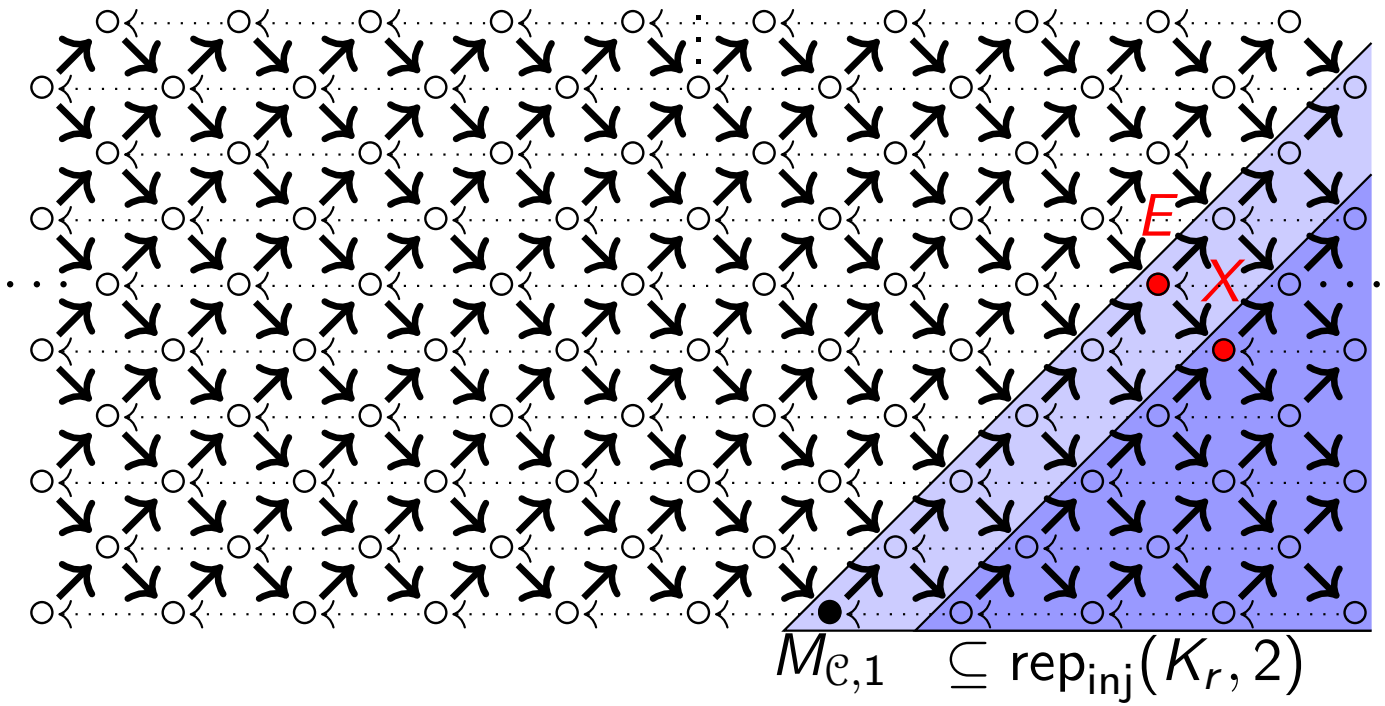


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2. $M_{\mathcal{C},1}$ *uniform iff all* $E \in (M_{\mathcal{C},1} \to)$ *are uniform.*



$$\Longrightarrow \quad 0 \to M_{\mathcal{C},1} \to E \to X \to 0.$$

Homogeneous vs uniform in AR components, II

Proof.

Let $\alpha \in \mathsf{Inj}_{\Bbbk}(A_2, A_r)$, then $0 \to \alpha^*(M_{\mathcal{C},1}) \to \alpha^*(E) \to \alpha^*(X) \to 0$. Since $X \in \mathsf{rep}_{\mathsf{inj}}(K_r, 2)$, $\alpha^*(X)$ is projective and $\alpha^*(E) \cong \alpha^*(M_{\mathcal{C},1}) \oplus \Delta_X(2)P_0 \oplus \dim_{\Bbbk} X_1 P_1$. □

By contrast, we have:

Proposition

$\mathcal{C}$ contains a homogeneous representation iff all representations in $\mathcal{C}$ are homogeneous.

Example: $\Bbbk \overset{\overset{\mathsf{id}}{\curvearrowright}}{\underset{0}{\rightarrow}} \Bbbk \ \curvearrowleft 0 \in \mathcal{C}$ contains no homogeneous representations.

Uniform Steiner bundles of k-type

Theorem (Marchesi & Miró-Roig, 2021)

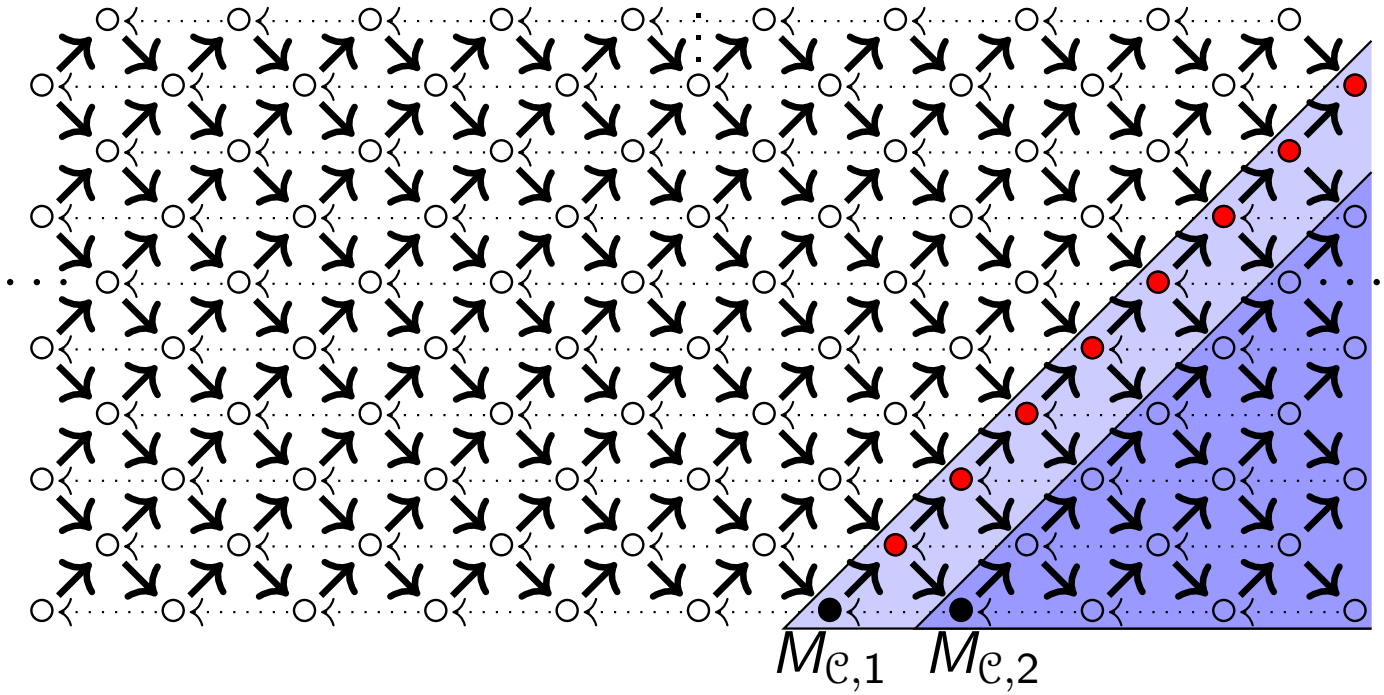
Let $k \in \mathbb{N}$, $\Bbbk = \mathbb{C}$. There exist uniform (and non-homogeneous) Steiner bundles $\mathcal{F}$ of k-type, i.e. $\mathcal{F}|_{\mathbb{P}^1} \cong \bigoplus_{i=0}^{k} a_i \mathcal{O}_{\mathbb{P}^1}(i)$ with $a_k \neq 0$.

All constructed uniform Steiner bundles $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}^n)$ are of connected type, i.e. for $\mathcal{F}$ uniform of k-type we have

$$\{i \in \mathbb{N}_0 \mid a_i \neq 0\} = \{0, 1, \ldots, k\}.$$

Where to look for k-splitting type with $k \geq 2$?

For $r \geq 3$ and $\mathcal{C}$ regular AR component such that $M_{\mathcal{C},1} \neq M_{\mathcal{C},2}$:



$\widetilde{\Theta}_r(M_{\mathcal{C},1})$ determines uniformity and k-type of $\widetilde{\Theta}_r(\bullet)$.

Restriction, inflation and reflection, I

Lemma (Happel-Unger, 1990)

There exists $X \in \operatorname{rep}(K_r)$ *such that for all* $M \in \operatorname{rep}(K_r)$

$$\begin{aligned}\operatorname{rk}(M(\gamma_1)) &= \dim_{\Bbbk} M_1 - \dim_{\Bbbk} \operatorname{Hom}_{K_r}(X, M) \\ &= \dim_{\Bbbk} M_2 - \operatorname{Ext}^1_{K_r}(X, M).\end{aligned}$$

Example (Restriction to K_1)

Let $\iota\colon A_1 \to A_r; \gamma_1 \mapsto \gamma_1 \in \operatorname{Inj}_{\Bbbk}(A_1, A_r)$, then

$$\iota^*(M) \cong a(\Bbbk \xrightarrow{\mathrm{id}} \Bbbk) \oplus b(0 \to \Bbbk) \oplus c(\Bbbk \to 0) \in \operatorname{rep}(K_1), \text{ and}$$

$$\begin{aligned}a &= \operatorname{rk}(\psi_{\iota^*(M)}) = \operatorname{rk}(\psi_M|_{\Bbbk\gamma_1 \otimes_{\Bbbk} M_1}) = \operatorname{rk}(M(\gamma_1)) \\ &= \dim_{\Bbbk} M_1 - \dim_{\Bbbk} \operatorname{Hom}_{K_r}(X, M) \\ &= \dim_{\Bbbk} M_2 - \dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(X, M).\end{aligned}$$

Question: Can this be generalized?

Restriction, inflation and reflection, II

- Consider $\iota\colon A_2 \hookrightarrow A_r; \gamma_i \mapsto \gamma_i$ and let

$$\mathrm{res}\colon \mathrm{rep}(K_r) \to \mathrm{rep}(K_2); \psi_M \mapsto \psi_{\iota^*(M)} \text{ (restriction)}.$$

- Let

$$\mathrm{inf}\colon \mathrm{rep}(K_2) \to \mathrm{rep}(K_r) \text{ (inflation)}$$

$$\psi_X \mapsto \psi_{\mathrm{inf}(X)}(\gamma_i \otimes x) = \begin{cases} \psi_X(\gamma_i \otimes x), & i \in \{1,2\} \\ 0, & \textit{else}. \end{cases}$$

- Let

$$\sigma_{K_d}\colon \mathrm{rep}(K_d) \to \mathrm{rep}(K_d) \text{ (reflection)}$$

$$\psi_{\sigma_{K_d}}\colon A_d \otimes_{\Bbbk} \ker \psi_M \to M_1; \gamma_i \otimes (\gamma_j \otimes m) \mapsto \delta_{ij} \cdot m.$$

Proposition (B. 2024)

The functor $\sigma_{K_r}^{-1} \circ \mathrm{inf}\colon \mathrm{rep}(K_2) \to \mathrm{rep}(K_r)$ *is left adjoint to* $\sigma_{K_2} \circ \mathrm{res}\colon \mathrm{rep}(K_r) \to \mathrm{rep}(K_2)$.

Restriction, inflation and reflection, III

Facts

1. $\sigma_{K_2}^{-1}(P_n) = P_{n+1}$ for all $n \in \mathbb{N}_0$.
2. We have $\dim_{\Bbbk} \mathrm{Hom}_{K_2}(P_n, P_m) = \max\{m - n + 1, 0\}$.

Application

Let $n \in \mathbb{N}$ and $X_n := (\sigma_{K_r}^{-1} \circ \inf)(P_n) \in \mathrm{rep}(K_r)$.
Let $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$, then $\mathrm{res}(M) = \bigoplus_{n \geq 0} a_n P_n \in \mathrm{rep}_{\mathrm{inj}}(K_2, 1)$.

- $$\dim_{\Bbbk} \mathrm{Hom}_{K_r}(X_n, M) \overset{\text{Prop.}}{=} \dim_{\Bbbk} \mathrm{Hom}_{K_2}(P_n, (\sigma_{K_2} \circ \mathrm{res})(M))$$
  $$\overset{1.}{=} \dim_{\Bbbk} \mathrm{Hom}_{K_2}(P_{n+1}, \mathrm{res}(M))$$
  $$\overset{2.}{=} \sum_{i \geq n+1} a_i (i - n).$$
- Similarly, $\dim_{\Bbbk} \mathrm{Ext}^1_{K_r}(X_n, M) = \sum_{i=0}^{n-1} a_i (n - i)$.

Restriction, inflation and reflection, IV

Theorem (B. 2024)

Let $n \in \mathbb{N}$. There exist representations $(X_n(\mathfrak{v}))_{\mathfrak{v} \in \mathrm{Gr}_2(A_r)}$ s.t. for all $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$ and $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$

1. $\mathrm{Hom}_{K_r}(X_n(\mathrm{im}\,\alpha), M) = 0$ *iff* $\alpha^*(M) \in \mathrm{add}(P_0, \ldots, P_n)$, *and*
2. $\mathrm{Ext}^1_{K_r}(X_n(\mathrm{im}\,\alpha), M) = 0$ *iff* $\alpha^*(M) \in \mathrm{add}(P_i \mid i \geq n)$.

Using Reineke's result on general subrepresentations we obtain:

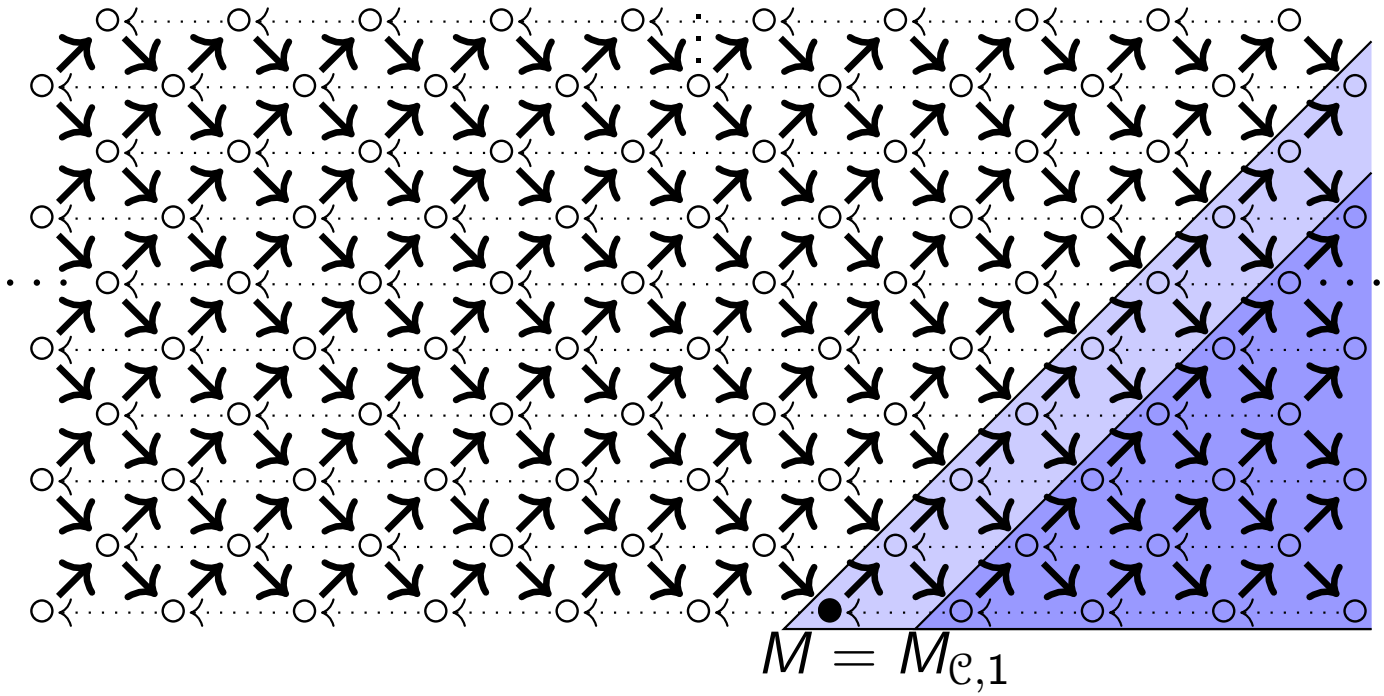
Corollary

Let $n \in \mathbb{N}_0$. There are $M_1, M_2 \in \mathrm{mod}\,\Bbbk$, $a_n, b_n \in \mathbb{N}$, and $\varnothing \neq \mathcal{O} \subseteq \mathrm{Hom}_{\Bbbk}(A_r \otimes_{\Bbbk} M_1, M_2)$ open s.t. every $\psi_M \in \mathcal{O}$ satisfies

1. $M := (M_1, M_2, \psi_M) \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1)$ *is a brick,*
2. $\alpha^*(M) \cong a_n P_n \oplus b_n P_{n+1}$ *for all* $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$, *and*
3. M *is not homogeneous.*

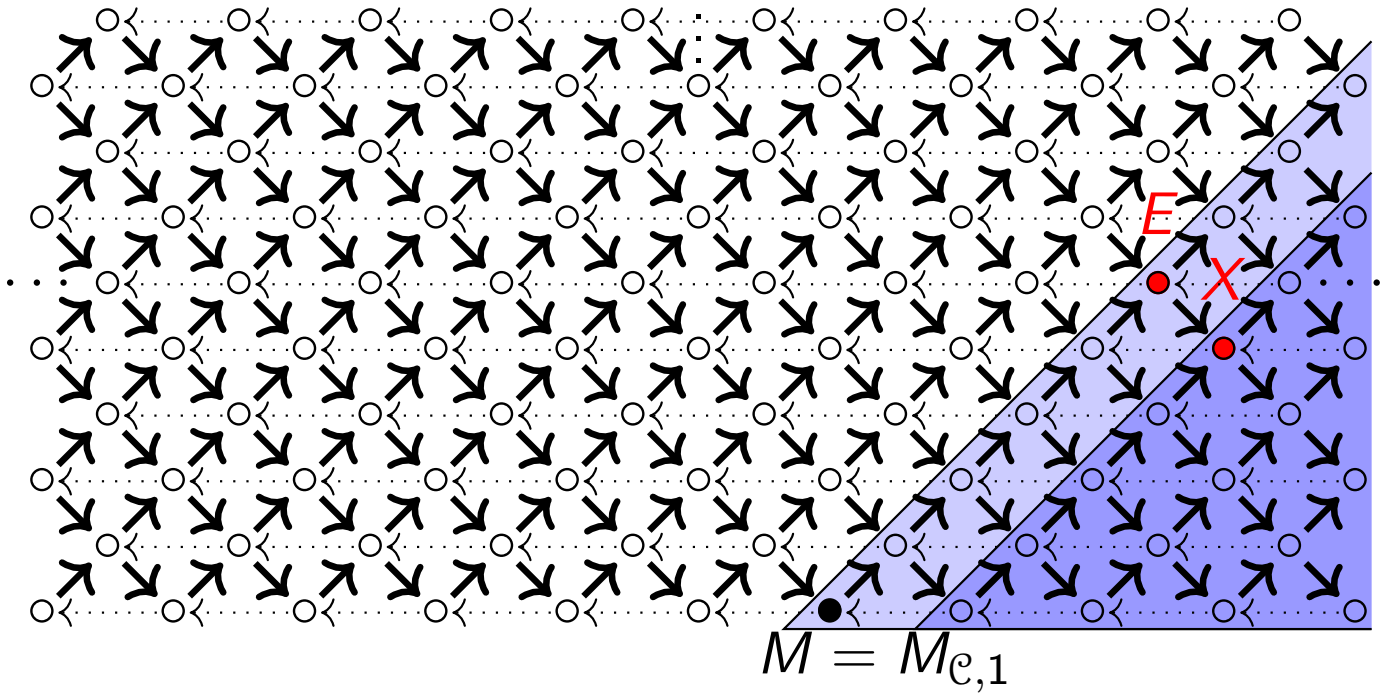
Application, I

Let $n \in \mathbb{N}$ and $M = (M_1, M_2, \psi_M)$ with the mentioned properties. Then $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1) \setminus \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$ is regular and quasi-simple.



Application, I

Let $n \in \mathbb{N}$ and $M = (M_1, M_2, \psi_M)$ with the mentioned properties. Then $M \in \mathrm{rep}_{\mathrm{inj}}(K_r, 1) \setminus \mathrm{rep}_{\mathrm{inj}}(K_r, 2)$ is regular and quasi-simple.



$$\Longrightarrow \quad 0 \to M \to E \to X \to 0.$$

Application, II

Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$, then

$$\alpha^*(E) \cong a_n P_n \oplus b_n P_{n+1} \oplus \Delta_X(2) P_0 \oplus \dim_{\Bbbk} X_1 P_1.$$

Corollary

Let $n \in \mathbb{N}$. There exist $c_0, c_1, c_n, c_{n+1} \in \mathbb{N}$ and an indecomposable, uniform but non-homogeneous $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ s.t.

$$\mathcal{F}|_{\mathbb{P}(A_2)} = c_0 \mathcal{O}_{\mathbb{P}(A_2)} \oplus c_1 \mathcal{O}_{\mathbb{P}(A_2)}(1) \oplus c_n \mathcal{O}_{\mathbb{P}(A_2)}(n) \oplus c_{n+1} \mathcal{O}_{\mathbb{P}(A_2)}(n+1).$$

$\rightsquigarrow$ New indecomposable uniform but non-homogeneous Steiner bundles with of disconnected type (having arbitrarily wide gaps).