Lectures on

Auslander-Reiten Theory

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Aim of these lectures is to provide a gentle introduction to the theory of almost split sequences and stable translation quivers. These lectures notes are mainly based [1], [2], [3], [5] and [8].

Comments are welcome. **No originality of proofs is claimed.**

1 Basic concepts

1.1 Preliminaries

We assume throughout that k is an algebraically closed field and (Λ, ι) denotes a finite-dimensional k-algebra. By defintion (Λ, ι) consists of a unitary ring Λ and a ring homomorphism $\iota\colon k \to \Lambda$ such that

(*a*) $\operatorname{im}\iota \subseteq \{a \in \Lambda \mid \forall b \in \Lambda\colon ab = ba\} = C(\Lambda)$, and

(*b*) Λ obtains the structure of a finite-dimensional $\iota(k) \cong k$-module.

From now on we suppress ι and do not distinguish between k and its image under ι. We denote with $\operatorname{Mod}\Lambda$ the category of all left Λ-modules and with $\operatorname{mod}\Lambda$ the full subcategory of all finitely generated Λ-modules. Note that $M \in \operatorname{Mod}\Lambda$ is finitely generated if and only if M is a finitely generated k-module, i.e. finite-dimensional over k. We denote with Λ^{op} the opposite algebra of Λ with multiplication $a * b = ba$ for all $a, b \in \Lambda^{\mathrm{op}}$.

Definition. A Λ-module M is called

(*a*) **indecomposable**, provided $M \neq 0$ and $M = U \oplus V$ implies $U = 0$ or $V = 0$.

(*b*) **simple**, provided $M \neq 0$ and for every submodule $U \subseteq M$ we have $U \in \{0, M\}$.

Definition. A Λ-linear map $f\colon A \to B$ is called

(*a*) **split monomorphism** (or **section**) if there exists $g \in \operatorname{Hom}_\Lambda(B, A)$ such that $g \circ f = \mathrm{id}_A$.

(*b*) **split epimorphism** (or **retraction**) if there exists $g \in \operatorname{Hom}_\Lambda(B, A)$ such that $f \circ g = \mathrm{id}_B$.

Lemma 1.1. *Let* $\delta\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be a short exact sequence in* $\operatorname{mod}\Lambda$. *The following statements are equivalent:*

(*a*) *f is a split monomorphism.*

(*b*) *g is a split epimorphism.*

(*c*) *There exists a submodule $D \subseteq B$ such that $B = \ker g \oplus D$.*

(*d*) *$B \cong A \oplus C$.*

Proof. We only prove $(d) \implies (a)$. We apply the left-exact functor $\operatorname{Hom}_\Lambda(C, -)$ to δ and get the exact sequence $0 \to \operatorname{Hom}_\Lambda(C, A) \xrightarrow{f_*} \operatorname{Hom}_\Lambda(C, B) \xrightarrow{g_*} \operatorname{Hom}_\Lambda(C, C)$. As $\dim_k \operatorname{Hom}_\Lambda(C, B) = \dim_k \operatorname{Hom}_\Lambda(C, A \oplus C) = \dim_k \operatorname{Hom}_\Lambda(C, A) + \dim_k \operatorname{Hom}_\Lambda(C, C)$ we conclude with exactness that $\dim_k \operatorname{im}\operatorname{Hom}_\Lambda(C, g) = \dim_k \operatorname{Hom}_\Lambda(C, B)$. Hence $g_*\colon \operatorname{Hom}_\Lambda(C, B) \to \operatorname{Hom}_\Lambda(C, C)$ is surjective and there exists $h\colon C \to B$ such that $g \circ h = g_*(h) = \mathrm{id}_C$. □

Lemma 1.2 (Schur's Lemma)**.** *Let S be a simple Λ-module, then* $\operatorname{End}_\Lambda(S) = \{\lambda \,\mathrm{id}_S \mid \lambda \in k\}$.

Proof. For $f \in \operatorname{End}_\Lambda(S)$ we find an eigenvalue, say λ. Then $\ker(f - \lambda \,\mathrm{id}_S) \neq 0$ and therefore $\ker(f - \lambda \,\mathrm{id}_S) = S$, i.e. $f = \lambda \,\mathrm{id}_S$. □

Lemma 1.3 (Fitting's Lemma)**.** *Let $M \in \operatorname{mod}\Lambda$ and $f \in \operatorname{End}_\Lambda(M)$. There is a unique decomposition $M = M_0(f) \oplus M_1(f)$ into f-invariant submodules such that f restricted to $M_0(f)$ is nilpotent and f restricted to $M_1(f)$ is invertible. Moreover, $M_0(f) = \bigcup_{n\in\mathbb{N}} \ker f^n$ and $M_1(f) = \bigcap_{n\in\mathbb{N}} \operatorname{im} f^n$.*

Lemma 1.4. *A finite-dimensional Λ-module M is indecomposable if and only if $\operatorname{End}_\Lambda(M)$ is local, i.e. $\operatorname{End}_\Lambda(M)$ has a unique maximal left ideal, which we denote with $\mathfrak{J}(\operatorname{End}_\Lambda(M))$. In this case $\mathfrak{J}(\operatorname{End}_\Lambda(M))$ is a maximal right ideal. The ideal $\mathfrak{J}(\operatorname{End}_\Lambda(M))$ consists of all non-invertible elements.*

Corollary 1.5. *Let M be an indecomposable Λ-module, then*

$$\mathfrak{J}(\operatorname{End}_\Lambda(M)) = \{f \in \operatorname{End}_\Lambda \mid \mathrm{id}_M - g \circ f \text{ is invertible for all } g \in \operatorname{End}_\Lambda(M)\}.$$

Proof. Let $I := \mathfrak{J}(\operatorname{End}_\Lambda(M))$ then

$$\begin{aligned} f \in I \Leftrightarrow f \text{ not invertible} &\overset{I \text{ ideal}, \mathrm{id}_M \in \operatorname{End}_\Lambda(M)}{\Leftrightarrow} \forall g \in \operatorname{End}_\Lambda(M)) : g \circ f \text{ not invertible} \\ &\Leftrightarrow \forall g \in \operatorname{End}_\Lambda(M)) : \mathrm{id}_M - g \circ f \text{ invertible} \end{aligned}$$

□

1.2 The radical of the module category

Motivated by Corollary 1.5 we define:

Definition. Let $A, B \in \operatorname{mod}\Lambda$, then

$$\operatorname{Rad}_\Lambda(A, B) := \{f \in \operatorname{Hom}_\Lambda(A, B) \mid \mathrm{id}_A - g \circ f \text{ is invertible for all } g \in \operatorname{Hom}_\Lambda(B, A)\}.$$

Lemma 1.6.

$$\operatorname{Rad}_\Lambda(A, B) = \{f \in \operatorname{Hom}_\Lambda(A, B) \mid \mathrm{id}_B - f \circ g \text{ is invertible for all } g \in \operatorname{Hom}_\Lambda(B, A)\}.$$

Proof. Let $f \in \operatorname{Rad}_\Lambda(A, B)$ and $g \in \operatorname{Hom}_\Lambda(B, A)$. Take $\varphi \in \operatorname{End}_\Lambda(A)$ such that

$$(*) \qquad \varphi \circ (\mathrm{id}_A - g \circ f) = \mathrm{id}_A = (\mathrm{id}_A - g \circ f) \circ \varphi.$$

We set $\psi := \mathrm{id}_B + f \circ \varphi \circ g$, then

$$\begin{aligned} \psi \circ (\mathrm{id}_B - f \circ g) &= (\mathrm{id}_B + f \circ \varphi \circ g) \circ (\mathrm{id}_B - f \circ g) = \mathrm{id}_B - f \circ g + f \circ \varphi \circ g - f \circ \varphi \circ g \circ f \circ g \\ &= \mathrm{id}_B - f \circ (\mathrm{id}_A - \varphi + \varphi \circ g \circ f) \circ g = \mathrm{id}_Y - f \circ 0_A \circ g = \mathrm{id}_B \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id}_B - f \circ g) \circ \psi &= (\mathrm{id}_B - f \circ g) \circ (\mathrm{id}_B + f \circ \varphi \circ g) = \mathrm{id}_B - f \circ g + f \circ \varphi \circ g - f \circ g \circ f \circ \varphi \circ g \\ &= \mathrm{id}_B - f \circ (\mathrm{id}_A - \varphi + g \circ f \circ \varphi) \circ g = \mathrm{id}_B - f \circ 0_A \circ g = \mathrm{id}_B . \end{aligned}$$

Now assume there is $f' \in \operatorname{Hom}_\Lambda(B, A)$ such that for all $g' \in \operatorname{Hom}_\Lambda(B, A)$ we find $\varphi' \in \operatorname{End}_\Lambda(B)$ such that

$$\varphi' \circ (\mathrm{id}_B - f' \circ g') = \mathrm{id}_B = (\mathrm{id}_B - f' \circ g') \circ \varphi'.$$

We set $\psi' := \mathrm{id}_A + g' \circ \varphi' \circ f'$ and compute in the same fashion that $f' \in \operatorname{Rad}_\Lambda(A, B)$. □

Lemma 1.7. *Let $A, B, C \in \operatorname{mod}\Lambda$ and $f \in \operatorname{Rad}_\Lambda(A,B)$.*
(a) *For $h \in \operatorname{Hom}_\Lambda(B,C)$ we have $h \circ f \in \operatorname{Rad}_\Lambda(A,C)$.*
(b) *For $h \in \operatorname{Hom}_\Lambda(C,A)$ we have $f \circ h \in \operatorname{Rad}_\Lambda(C,B)$.*
(c) *$\operatorname{Rad}_\Lambda(A,B)$ is a k-subspace of $\operatorname{Hom}_\Lambda(A,B)$.*

Proof. (a) Let $g \in \operatorname{Hom}_\Lambda(C,A)$ then

$$\operatorname{id}_A - g \circ h \circ f = \operatorname{id}_A - (g \circ h) \circ f$$

is invertible and $h \circ f \in \operatorname{Rad}_\Lambda(A,C)$
(b) This follows as in (a) using Lemma 1.6.
(c) Cleary $\operatorname{Rad}_\Lambda(A,B)$ is closed under scalar multiplication. Let $f_1, f_2 \in \operatorname{Rad}_\Lambda(A,B)$ and $g \in \operatorname{Hom}_\Lambda(B,A)$. Let $\varphi_1 \in \operatorname{End}_\Lambda(A)$ such that

$$\varphi_1 \circ (\operatorname{id}_A - g \circ f_1) = \operatorname{id}_A = (\operatorname{id}_A - g \circ f_1) \circ \varphi_1.$$

Moreover, let $\varphi_2 \in \operatorname{End}_\Lambda(A)$ such that

$$\varphi_2 \circ (\operatorname{id}_A - \varphi_1 \circ g \circ f_2) = \operatorname{id}_A = (\operatorname{id}_A - \varphi_1 \circ g \circ f_2) \circ \varphi_2.$$

We conclude

$$\varphi_2 \circ \varphi_1 \circ (\operatorname{id}_A - g \circ (f_1 + f_2)) = \varphi_2 \circ (\operatorname{id}_A - \varphi_1 \circ g \circ f_2) = \operatorname{id}_A .$$

Hence $\operatorname{id}_A - g \circ (f_1 + f_2)$ has the left inverse $\varphi_2 \circ \varphi_1$. By (b) $f_2 \circ \varphi_1$ is an element in $\operatorname{Rad}_\Lambda(A,B)$, so we find $\varphi_3 \in \operatorname{End}_\Lambda(A)$ such that $(\operatorname{id}_A - g \circ (f_2 \circ \varphi_1)) \circ \varphi_3 = \operatorname{id}_A$. We conclude

$$(\operatorname{id}_A - g \circ (f_1 + f_2)) \circ (\varphi_1 \circ \varphi_3) = (\operatorname{id}_A - g \circ f_2 \circ \varphi_1) \circ \varphi_3 = \operatorname{id}_A .$$

Moreover,

$$\varphi_2 \circ \varphi_1 = \varphi_2 \circ \varphi_1 \circ (\operatorname{id}_A - g \circ (f_1 + f_2)) \circ \varphi_1 \circ \varphi_3 = \varphi_1 \circ \varphi_3.$$

□

Lemma 1.8. *Let $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ be in $\operatorname{mod}\Lambda$. The following statements hold for $f \in \operatorname{Hom}_\Lambda(A,B)$.*
(a) *$f \in \operatorname{Rad}_\Lambda(A,B)$ if and only if $f \circ \iota_i \in \operatorname{Rad}_\Lambda(A_i,B)$ for all $i \in \{1,2\}$, where $\iota_i \colon A_i \to A$ denotes the canonical injection. In other words, $\operatorname{Rad}_\Lambda(A_1 \oplus A_2, B) \cong \operatorname{Rad}_\Lambda(A_1,B) \oplus \operatorname{Rad}_\Lambda(A_2,B)$.*
(b) *$f \in \operatorname{Rad}_\Lambda(A,B)$ if and only if $\pi_i \circ f \in \operatorname{Rad}_\Lambda(A,B_i)$ for all $i \in \{1,2\}$, where $\pi_i \colon A \to B_i$ denotes the canonical projection. In other words, $\operatorname{Rad}_\Lambda(A, B_1 \oplus B_2) = \operatorname{Rad}_\Lambda(A,B_1) \cong \operatorname{Rad}_\Lambda(A,B_2)$.*

Proof. We only prove (a). Let f be in $\operatorname{Rad}_\Lambda(A,B)$, then $f \circ \iota_i \in \operatorname{Rad}_\Lambda(A_i,B)$ by 1.7. Now assume that $f \circ \iota_i \in \operatorname{Rad}_\Lambda(A_i,B)$ for $i \in \{1,2\}$, then $f \circ \iota_i \circ \pi_i \in \operatorname{Rad}_\Lambda(A,B)$ and therefore

$$f = f \circ \operatorname{id}_A = f \circ (\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \in \operatorname{Rad}_\Lambda(A,B).$$

□

Proposition 1.9. *Let A, B be indecomposable Λ-modules. The following statments hold.*
(a) *$f \in \operatorname{Rad}_\Lambda(A,B)$ if and only if f is not an isomorphism.*

(b) *For* $A \ncong B$ *we have* $\mathrm{Rad}_\Lambda(X,Y) = \mathrm{Hom}_\Lambda(A,B)$.

Proof. If f is an isomorphism we know that $h \circ f = \mathrm{id}_A$ for some $h \in \mathrm{Hom}_\Lambda(B,A)$ and conclude that $\mathrm{id}_A - h \circ f = 0$ is not invertible. Let f be not an isomorphism and $g \in \mathrm{Rad}_\Lambda(B,A)$. We have to show that $h := \mathrm{id}_A - g \circ f$ is invertible. At first show that $g \circ f$ is not invertible. Assume to the contrary that there exists $l \in \mathrm{End}_\Lambda(A)$ such that $l \circ (g \circ f) = \mathrm{id}_A$, then f is a split monomorphism. Hence $\ker(l \circ g) \oplus \mathrm{im}(f) = B$. As B is indecomposable and $\mathrm{im}\, f \neq 0$, we conclude that f is surjective and therefore bijective, a contradiction.
Hence $g \circ f \in \mathfrak{J}(\mathrm{End}_\Lambda(A))$ and since $\mathrm{id}_A \notin \mathfrak{J}(\mathrm{End}_\Lambda(A))$, we conclude $\mathrm{id}_A - g \circ f \in \mathfrak{J}(\mathrm{End}_\Lambda(A)) = \mathrm{Rad}_\Lambda(A,A)$.

□

Lemma 1.10. *Let* $f\colon A \to B$ *be a non-zero homomorphism in* $\mathrm{mod}\,\Lambda$ *with* B *being indecomposable. The following statements are equivalent:*
(a) f *is not a split epimorphism.*
(b) $f \in \mathrm{Rad}_\Lambda(A,B)$.
(c) $\mathrm{im}\,\mathrm{Hom}_\Lambda(B,f) \subseteq \mathfrak{J}(\mathrm{End}_\Lambda(B))$.

Proof. (a) $\implies$ (b). We write $A = \bigoplus_i^n A_i$ with $A_1, \ldots, A_n$ indecomposable and assume that $f_i := f \circ \iota_i\colon A_i \to B$ is an isomorphism. Then $(f \circ \iota_i) \circ (f_i)^{-1} = \mathrm{id}_B$ and f a split epimorphism, a contradiction. Hence $f_i \in \mathrm{Rad}_\Lambda(A_i,B)$ for all $i \in \{1,\ldots,n\}$ and therefore $f = \sum_i^n f_i \circ \pi_i \in \mathrm{Rad}_\Lambda(A,B)$.
(b) $\implies$ (c) Note as before that $f \in \mathrm{Rad}_\Lambda(A,B)$ implies $g \circ f \in \mathrm{Rad}_\Lambda(B,B)$ for each $g \in \mathrm{Hom}_\Lambda(B,A)$.
(c) $\implies$ (a) If f is a split epimorphism, we have $\mathrm{id}_C \in \mathrm{im}\,\mathrm{Hom}_\Lambda(B,f) \setminus \mathfrak{J}(\mathrm{End}(B))$.

□

1.3 The Theorem of Krull-Remak-Schmidt

Definition. For $X \in \mathrm{mod}\,\Lambda$ indecomposable we denote with $[X]$ the isomorphism class of X. Moreover, $\mathrm{ind}\,\Lambda$ denotes the collection of all isomorphism classes.

Theorem 1.11 (Krull-Remak-Schmidt, [6])**.** *Let* $M \in \mathrm{mod}\,\Lambda$ *be a finite-dimensional* Λ*-module. There are* $n \in \mathbb{N}$*, pairwise non-isomorphic indecomposable* Λ*-submodules* $M_1, \ldots, M_n$ *of* M *and* $a_1, \ldots, a_n \in \mathbb{N}$ *such that* $M = \bigoplus_{i=1}^n M_i^{a_i}$.
If $M = \bigoplus_{i=1}^m N_i^{b_i}$ *is another such a decomposition, then* $n = m$ *and there exists a permutation* $\sigma \in \mathcal{S}_n$ *such that* $M_i \cong N_{\sigma(i)}$ *and* $a_i = b_{\sigma(i)}$ *for all* $i \in \{1,\ldots,n\}$.

Proof. The existence of such a decomposition is obvious. We fix two decompositions $\bigoplus_{i=1}^n M_i^{a_i} = M = \bigoplus_{i=1}^m N_i^{b_i}$ into indecomposable submodules with $M_i \ncong M_j$ and $N_i \ncong N_j$ for $i \neq j$ and $a_i, b_i > 0$ with $m \leq n$. Let N be an indecomposable Λ-module. Then $\mathrm{Hom}_\Lambda(N,N) \neq \mathrm{Rad}_\Lambda(N,N)$ by 1.9 and therefore the map

$$\varphi\colon \mathrm{ind}\,\Lambda \to \mathbb{R}, [N] \mapsto \frac{\dim_k \mathrm{Hom}_\Lambda(M,N) - \dim_k \mathrm{Rad}_\Lambda(M,N)}{\dim_k \mathrm{Hom}_\Lambda(N,N) - \dim_k \mathrm{Rad}_\Lambda(N,N)}$$

is well-defined. Since $M = \bigoplus_{i=1}^n M_i^{a_i}$, 1.8 yields

$$\varphi([N]) = \frac{\sum_{i=1}^n a_i(\dim_k \mathrm{Hom}_\Lambda(M_i,N) - \dim_k \mathrm{Rad}_\Lambda(M_i,N))}{\dim_k \mathrm{Hom}_\Lambda(N,N) - \dim_k \mathrm{Rad}_\Lambda(N,N)}.$$

We conclude for $j \in \{1, \ldots, m\}$ that $\varphi([M_j]) = a_j$. By the same token, we have

$$0 < a_j = \varphi([M_j]) = \frac{\sum_{i=1}^{m} b_i(\dim_k \operatorname{Hom}_\Lambda(N_i, M_j) - \dim_k \operatorname{Rad}_\Lambda(N_i, M_j))}{\dim_k \operatorname{Hom}_\Lambda(M_j, M_j) - \dim_k \operatorname{Rad}_\Lambda(M_j, M_j)}.$$

Proposition 1.9 yields the existence of a unique $\sigma(j) \in \{1, \ldots, m\}$ such that $N_{\sigma(j)} \cong M_j$ and therefore $a_j = \varphi([M_j]) = b_{\sigma(j)}$. As $\sigma(j) \neq \sigma(i)$ for $i \neq j$, the map $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ is injective and therefore bijective since $m \leq n$. In particular, $m = n$. $\square$

2 Auslander-Reiten sequences

From now on all considered modules are of finite dimension.

2.1 Definition and first properties

Definition. A short exact sequence $0 \to A \to B \xrightarrow{g} C \to 0$ in $\operatorname{mod}\Lambda$ is called **Auslander-Reiten sequence** or **almost split sequence**, provided:

(a) The sequence does not split.

(b) A is indecomposable.

(c) Every Λ-linear map $h\colon D \to C$, which is not a split epimorphism, factors through g, i.e. there exists $h'\colon D \to B$ such that $g \circ h' = h$.

$$\begin{array}{ccccccccc} & & & & & & D & & \\ & & & & & \swarrow^{\exists h'} & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

Remark. If the sequence is almost split, then C is indecomposable: Assume that $C = C_1 \oplus C_2$ with $C_1, C_2 \neq 0$ and consider the natural embeddings $\iota_1\colon C_1 \to C$ and $\iota_2\colon C_2 \to C$. Since ι_1, ι_2 are not split epimorphisms, we find $h_1\colon C_1 \to B$ and $g_2\colon C_2 \to B$ such that $\iota_1 = g \circ h_1$ and $\iota_2 = g \circ h_2$. We conclude

$$\mathrm{id}_C = \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = g \circ (h_1 \circ \iota_1 + h_2 \circ \iota_2).$$

This is a contradiction since δ does not split.

Lemma 2.1 (Uniqueness of Auslander-Reiten sequences)**.** *Assume that $\delta = 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $\delta' = 0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C \to 0$ are Auslander-Reiten sequences. Tthere is a commutative diagram with isomorphisms φ, ψ.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & \longrightarrow & 0 \end{array}$$

In other words, δ and δ' are isomorphic short exact sequences.

Proof. As $g\colon B \to C$ is not a split epimorphism, there exists a Λ-linear map $\psi\colon B \to B'$ such that $g = g' \circ \psi$. By exactness of δ there results a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & \longrightarrow & 0 \end{array}$$

By the same token, we find $\psi' \colon B' \to B$ such that $g' = g \circ \psi'$ and get the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \psi'} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

We assume that there exists $n \in \mathbb{N}$ such that $(\varphi' \circ \varphi)^n = 0$. We set $h := (\psi' \circ \psi)^n$, then $h \circ f = (\psi \circ \psi')^n \circ f = f \circ (\varphi' \circ \varphi)^n = 0$. Hence $\ker g = \operatorname{im} f \subseteq \ker h$ and therefore the map $\overline{h} \colon C \to B, c \mapsto h(x_c)$, where $x_c \in g^{-1}(\{c\})$, is well-defined and Λ-linear. By definition of $\overline{h}$ we have $\overline{h} \circ g = h$ and conclude in conjunction with $g = h \circ g$ that $(g \circ \overline{h}) \circ g = g \circ h = g$. Now the surjectivity of g yields $g \circ \overline{h} = \mathrm{id}_C$, which is a contradiction since δ does not split.
By Fitting's Lemma $\varphi' \circ \varphi$ is an isomorphism. By the same token, it follows that $\varphi' \circ \varphi$ is an isomorphism. Hence φ is an isomorphism and the five lemma yields that ψ is an isomorphism. □

Lemma 2.2. *Let* $0 \to A \to B \xrightarrow{g} C \to 0$ *be a short exact sequence in* $\operatorname{mod}\Lambda$ *with* A *indecomposable. The following statements are equivalent:*

(*a*) The sequence is an Auslander-Reiten sequence.
(*b*) For every indecomposable module D we have

$$\operatorname{im} g_* = \operatorname{im}\operatorname{Hom}_\Lambda(D, g) = \operatorname{Rad}_\Lambda(D, C) \begin{cases} = \operatorname{Hom}_\Lambda(D, C), D \ncong C \\ \cong \mathfrak{J}(\operatorname{End}(C)), D \cong C. \end{cases}$$

Proof. (*a*) $\implies$ (*b*) Let D be an indecomposable module. Clearly $\operatorname{Rad}_\Lambda(D, C) \subseteq \operatorname{im}\operatorname{Hom}_\Lambda(D, g)$, which is just $\operatorname{Hom}_\Lambda(D, C)$ for $D \ncong C$. If $D \cong C$ and $f \notin \operatorname{Rad}_\Lambda(D, C)$, then f is an isomorphism by Fitting's Lemma, hence $f \notin \operatorname{im}\operatorname{Hom}_\Lambda(D, g)$ since the sequence does not split.
(*b*) $\implies$ (*a*) Now let D be in $\operatorname{mod}\Lambda$ and $\pi \colon D \to C$ be not a split epimorphism. We decompose $D = \bigoplus_i^n D_i$ into indecomposable direct summands. By 1.10 and 1.7 we know $h \in \operatorname{Rad}_\Lambda(D, C)$ and $h \circ \iota_i \in \operatorname{Rad}_\Lambda(D_i, C)$ for all $i \in \{1, \ldots, n\}$. Hence we find $\eta_i \colon D_i \to B$ such that $g \circ \eta_i = h \circ \iota_i$ for all i. We conclude

$$h = h \circ \mathrm{id}_D = \sum_{i=1}^n h \circ \iota_i \circ \pi_i = \sum_{i=1}^n g \circ \eta_i \circ \pi_i = g \circ (\sum_{i=1}^n \eta_i \circ \pi_i).$$

It remains to show that the sequence does not split. Assume that $\mathrm{id}_C \in \operatorname{im}\operatorname{Hom}_\Lambda(C, g)$. Hence we find $h \colon C \to B$ such that $\mathrm{id}_C = g \circ h$. By Lemma 1.8 some $\mathrm{id}_C \circ \iota_i \colon C_i \to C$ is not in $\operatorname{Rad}_\Lambda(C_i, C)$, but $\mathrm{id}_C \circ \iota_i = g \circ (h \circ \iota_i) \in \operatorname{Hom}_\Lambda(C_i, g)$, a contradiction. □

Corollary 2.3. *Let $0 \to A \to B \to C \to 0$ be an Auslander-Reiten sequence and $D \in \operatorname{mod}\Lambda$ indecomposable then the induced sequence*

$$0 \to \operatorname{Hom}_\Lambda(D, A) \to \operatorname{Hom}_\Lambda(D, B) \to \operatorname{Rad}_\Lambda(D, C) \to 0$$

is exact.

Examples. Let $\delta\colon 0 \to A \to B \to C \to 0$ be a short exact sequence.

(a) The sequence is not almost split if A is injective or C is projective.

(b) Let $\Lambda := k[X]/(X^2)$, $x := X + (X^2)$ and consider the short exact sequence

$$\delta : 0 \to (x) \xrightarrow{\iota} \Lambda \xrightarrow{g} (x) \to 0,$$

where $g\colon \Lambda \to (x)$ is the left multiplication with x. The sequence does not split since Λ is local with maximal ideal (x). As indecomposable Λ-modules are given by the number of Jordan blocks J with the property $J^2 = 0$, i.e. (0) and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we know that Λ and (x) are the only indecomposable Λ-modules (up to isomorphism). As Λ is projective, we have $\operatorname{im}\operatorname{Hom}_\Lambda(\Lambda, g) = \operatorname{Hom}_\Lambda(\Lambda, (x)) = \operatorname{Rad}_\Lambda(\Lambda, (x))$. As δ does not split, we know that $\operatorname{im}\operatorname{Hom}_\Lambda((x), g) \subseteq \operatorname{Rad}_\Lambda((x), (x))$. By Schur's Lemma we have $\operatorname{Rad}_\Lambda((x), (x)) = 0$. Hence $\operatorname{im}\operatorname{Hom}_\Lambda((x), g) = \operatorname{Rad}_\Lambda((x), (x))$, so δ is almost split.

2.2 The Auslander-Reiten translation

Definition. Let M be in $\operatorname{mod}\Lambda$. An exact sequence $P_1 \xrightarrow{\varepsilon_1} P_0 \xrightarrow{\varepsilon_0} M \to 0$ is called **minimal projective presentation** of M if (P_0, ε_0) is a projective cover of M and (P_1, ε_1) is a projective cover of $\ker\varepsilon_0$.

We consider the contravariant functor $(-)^t\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$, $M \mapsto \operatorname{Hom}_\Lambda(M, \Lambda)$ and also denote the contravariant functor $\operatorname{mod}\Lambda^{\mathrm{op}} \to \operatorname{mod}\Lambda$, $M \mapsto \operatorname{Hom}_\Lambda(M, \Lambda^{\mathrm{op}})$ with $(-)^t$.
Let P be an indecomposable projective Λ-module, then we find an idempotent $e \in \Lambda$ such that $P \cong \Lambda e$ and we have natural isomorphisms

$$(*) \qquad P^t \cong \operatorname{Hom}_\Lambda(\Lambda e, \Lambda) \cong e\Lambda = \Lambda^{\mathrm{op}} * e$$

of Λ^{op}-modules. In particular, P^t is a projective Λ^{op}-module. Since $(-)^t$ interchanges with direct sums, we conclude that $(-)^t$ sends projective Λ-module to projective Λ^{op}-modules and vice versa. Moreover the functor $(-)^t \circ (-)^t\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}} \to \operatorname{mod}\Lambda$ induces a functor $\operatorname{proj}\Lambda \to \operatorname{proj}\Lambda$ natural equivalent to the identity functor. The corresponding natural equivalence is given by the evalution morphisms $\tau_P\colon P \to (P^t)^t$, $\tau_P(p)(f) := f(p)$ for all $f \in P^t$ and $p \in P$. Given $M \in \operatorname{mod}\Lambda$ we fix a minimal projective presentation $P_1 \xrightarrow{\varepsilon_1} P_0 \xrightarrow{\varepsilon_0} M \to 0$ and get the exact sequence in $\operatorname{mod}\Lambda^{\mathrm{op}}$

$$0 \to M^t \xrightarrow{\varepsilon_0^t} P_0^t \xrightarrow{\varepsilon_1^t} P_1^t \xrightarrow{c} \operatorname{coker}\varepsilon_1^t \to 0$$

and define $\operatorname{Tr}(M) := \operatorname{coker}\varepsilon_1^t$. Note that $\operatorname{Tr}(M)$ does not depend on the minimal projective resolution.

Definition. Let M in $\operatorname{mod}\Lambda$ be a Λ-module. By Krull-Remak-Schmidt the module M has a unique (up to isomorphism) decomposition $M = M_{\mathrm{pf}} \oplus P_M$ into a projective module P_M and a module M_{pf} without projective direct summands. The module M_{pf} is called the **projective free part** of M. We write $M \in \operatorname{mod}_{\mathrm{pf}}\Lambda$ if $M = M_{\mathrm{pf}}$ or $M = 0$.

Lemma 2.4. *Let $M \in \operatorname{mod}\Lambda$.*
(*a*) $\operatorname{Tr}(M) = 0$ *if and only if M is projective.*
(*b*) $\operatorname{Tr}(M) \in \operatorname{mod}_{\mathrm{pf}}\Lambda^{\mathrm{op}}$.

Proof. (*a*) Assume that $\operatorname{Tr}(M) = 0$. By definition $\varepsilon_1^t\colon P_0^t \to P_1^t$ is surjective and therefore a split epimorphism as $(-)^t$ takes projective Λ-modules to projective Λ^{op}-modules. Hence we find $g\colon P_1^t \to P_0^t$ such that $\varepsilon_1^t \circ g = \mathrm{id}_{P_1^t}$ and $g^t \circ (\varepsilon_1^t)^t = \mathrm{id}_{(P_1^t)^t}$. The natural transformation $\tau\colon \mathrm{id} \to (-)^t \circ (-)^t$ yields the commutative diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{\varepsilon_1} & P_0 \\ \downarrow{\scriptstyle \tau_{P_1}} & & \downarrow{\scriptstyle \tau_{P_0}} \\ (P_1^t)^t & \xrightarrow{(\varepsilon_1^t)^t} & (P_0^t)^t \end{array}$$

and we conclude $(\tau_{P_1}^{-1} \circ g^t \circ \tau_{P_0}) \circ \varepsilon_1 = \tau_{P_1}^{-1} \circ (g^t \circ (\varepsilon_1^t)^t) \circ \tau_{P_1} = \mathrm{id}_{P_1}$. Hence $\varepsilon_1\colon P_1 \to P_0$ is a split monomorphism and $P_0 = \operatorname{im}(\varepsilon_1) \oplus U = \ker(\varepsilon_0) \oplus U$ for some submodule U. As $\varepsilon_0\colon P_0 \to M$ is a projective cover, the module $\ker(\varepsilon_0)$ is superfluous, hence $\ker(\varepsilon_0) = 0$ and therefore $M \cong P_0$.
(*b*) Let P be a projective direct summand of $\operatorname{Tr}(M)$ and let $\pi\colon \operatorname{Tr}(M) \to P$ be the canonical projection. Then $\pi \circ c\colon P_1^t \to \operatorname{Tr}(M) \to P$ is a split epimomorphism and we find $\eta\colon P \to P_1^t$ such that $\pi \circ c \circ \eta = \mathrm{id}_{P^t}$. We apply $(-)^t$ and conclude with

$$\mathrm{id}_{P^t} = (\pi \circ c \circ \eta)^t = \eta^t \circ (\pi \circ c)^t$$

that $(\pi \circ c)^t\colon P^t \to (P_1^t)^t$ is a split monomorphism. We consider the commutative diagram

$$\begin{array}{ccccc} & & P_1 & \xrightarrow{\varepsilon_1} & P_1 \\ & & \downarrow{\scriptstyle \tau_{P_1}} & & \downarrow{\scriptstyle \tau_{P_0}} \\ P^t & \xrightarrow{(\pi\circ c)^t} & (P_1^t)^t & \xrightarrow{(\varepsilon_1^t)^t} & (P_0^t)^t, \end{array}$$

and obtain on the one hand that $\tau_{P_1}^{-1} \circ (\pi \circ c)^t$ is a split monomorphism. On the other hand,

$$\varepsilon_1 \circ \tau_{P_1}^{-1} \circ (\pi \circ c)^t = \tau_{P_0}^{-1} \circ (\varepsilon_1^t)^t \circ (\pi \circ c)^t = \tau_{P_0}^{-1} \circ (\pi \circ c \circ \varepsilon_1^t)^t = 0.$$

Hence $\operatorname{im}(\tau_{P_1}^{-1} \circ (\pi \circ c)^t)$ is a direct summand of P_1 in the kernel of ε_1. Since $\ker \varepsilon_1$ is superfluous, we conclude $\operatorname{im}(\tau_{P_1}^{-1} \circ (\pi \circ c)^t) = 0$, which implies that $(\pi \circ c)^t = 0$. We consider the exact sequence

$$0 \to \ker(\pi \circ c) \to P_1^t \xrightarrow{\pi \circ c} P \to 0$$

and conclude with left-exactness of $(-)^t$ that $P^t = 0$, which implies $0 = (P^t)^t \cong P$. □

Recall that for $M \in \operatorname{mod}\Lambda$, the **radical** $\operatorname{Rad}_\Lambda(M)$ is defined as the intersection of all maximal submodules of M and $\mathfrak{J}(\Lambda) := \operatorname{Rad}_\Lambda(\Lambda)$ is called the **Jacobson radical** of Λ. By general theory (see [4, 2.6,6.1], [1, 1.3]) we know that $\mathfrak{J}(\Lambda)M = \operatorname{Rad}_\Lambda(M)$, $N \subseteq M$ is superfluous if and only if $N \subseteq \operatorname{Rad}_\Lambda(M)$ and $\mathfrak{J}(\Lambda) = \operatorname{Rad}_{\Lambda^{\mathrm{op}}}(\Lambda^{\mathrm{op}}) = \mathfrak{J}(\Lambda^{\mathrm{op}})$ is a two-sided ideal.

Lemma 2.5. (*a*) *For $M \in \operatorname{mod}_{\mathrm{pf}}\Lambda$ we have $M^t = \operatorname{Hom}_\Lambda(M, \mathfrak{J}(\Lambda))$.*
(*b*) *For $P \in \operatorname{mod}\Lambda$ we have $\operatorname{Rad}_{\Lambda^{\mathrm{op}}}(P^t) = \operatorname{Hom}_\Lambda(P, \mathfrak{J}(\Lambda))$.*

Proof. (a) We consider the canonical sequence $0 \to \mathfrak{J}(\Lambda) \xrightarrow{\iota} \Lambda \to \Lambda/\mathfrak{J}(\Lambda) \to 0$. We apply $\operatorname{Hom}_\Lambda(M,-)$ and conclude that $\iota_*\colon \operatorname{Hom}_\Lambda(M,\mathfrak{J}(\Lambda)) \to M^t$ is injective. Let $f\colon M \to \Lambda$ be Λ-linear. We write $\Lambda = \bigoplus_{i=1}^n P_i$ as sum of indecomposable direct summands. For each $i \in \{1,\dots,n\}$ we let $\pi_i\colon \Lambda \to P_i$ be the natural projection. Since P_i is projective and $M \in \operatorname{mod}_{\mathrm{pf}}\Lambda$, we conclude that $\pi_i \circ f$ is not surjective for each $i \in \{1,\dots,n\}$. Since each P_i has the unique maximal submodule $\mathfrak{J}(\Lambda)P_i$, we conclude $\operatorname{im} f \subseteq \bigoplus_{i=1}^n \mathfrak{J}(\Lambda)P_i = \mathfrak{J}(\Lambda)\Lambda = \mathfrak{J}(\Lambda)$.
(b) At first we consider the case that P is indecomposable. Then $P = \Lambda e$ for a primitive idempotent e. We have already seen that $P^t \cong \Lambda^{\mathrm{op}} * e$. Since $\mathfrak{J}(\Lambda) = \mathfrak{J}(\Lambda^{\mathrm{op}})$, we conclude that

$$\begin{aligned}\dim_k \operatorname{Hom}_\Lambda(P,\mathfrak{J}(\Lambda)) &= \dim_k \operatorname{Hom}_\Lambda(\Lambda e,\mathfrak{J}(\Lambda)) = \dim_k e\,\mathfrak{J}(\Lambda)\\ &= \dim_k \mathfrak{J}(\Lambda^{\mathrm{op}}) * e = \dim_k \operatorname{Rad}_{\Lambda^{\mathrm{op}}}(\Lambda^{\mathrm{op}} * e) = \dim_k \operatorname{Rad}_{\Lambda^{\mathrm{op}}}(P^t).\end{aligned}$$

Since $\operatorname{Rad}_{\Lambda^{\mathrm{op}}}(P^t)$ is the unique maximal submodule of P^t and $\operatorname{Hom}_\Lambda(P,\mathfrak{J}(\Lambda) \subseteq P^t)$, we conclude $\operatorname{Rad}_{\Lambda^{\mathrm{op}}}(P^t) = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P,\mathfrak{J}(\Lambda))$. Now the general case follows since $\operatorname{Rad}_{\Lambda^{\mathrm{op}}}(-)$ and $\operatorname{Hom}_\Lambda(-,\mathfrak{J}(\Lambda))$ interchange with direct sums. □

Proposition 2.6. *If M is in $\operatorname{mod}_{\mathrm{pf}}\Lambda$ and $P_1 \xrightarrow{\varepsilon_1} P_0 \xrightarrow{\varepsilon_0} M \to 0$ a minimal projective presentation, then $P_0^t \xrightarrow{\varepsilon_1^t} P_1^t \xrightarrow{c} \operatorname{Tr}(M) \to 0$ is a minimal projective resolution of $\operatorname{Tr}(M)$ in $\operatorname{mod}\Lambda^{\mathrm{op}}$.*

Proof. Let $\lambda \in \ker c = \operatorname{im}\varepsilon_1^t$, then there exists $\gamma\colon P_1 \to \Lambda$ such that $\lambda = \gamma \circ \varepsilon_1$. Since $\ker\varepsilon_1 \subseteq \operatorname{Rad}_\Lambda(P_0)$, we conclude

$$\operatorname{im}\lambda = (\gamma\circ\varepsilon_1)(P_0) \subseteq \gamma(\operatorname{Rad}_\Lambda(P_0)) = \gamma(\mathfrak{J}(\Lambda)P_0) \subseteq \mathfrak{J}(\Lambda)\Lambda = \mathfrak{J}(\Lambda).$$

Now Lemma 2.5(b) implies $\lambda \in \operatorname{Rad}_{\Lambda^{\mathrm{op}}}(P_1)$. Hence $\ker c$ is superfluous and $c\colon P_1^t \to \operatorname{Tr}(M)$ a projective cover.
Let $\mu \in \ker\varepsilon_1^t = \operatorname{im}\varepsilon_0^t$. Hence there is $\varphi\colon M \to \Lambda$ such that $\mu = \varphi\circ\varepsilon_0$. Since $M \in \operatorname{mod}_{\mathrm{pf}}\Lambda$, we conclude with 2.5$(a)$ that $\operatorname{im}\varphi \subseteq \mathfrak{J}(\Lambda)$, hence $\operatorname{im}\mu \subseteq \mathfrak{J}(\Lambda)$ and $\mu \in \operatorname{Hom}_\Lambda(P_1,\mathfrak{J}(\Lambda)) = \operatorname{Rad}_{\Lambda^{\mathrm{op}}}(P_1^t)$. Hence $\ker\varepsilon_1^t$ is superfluous and (P_1^t,ε_1^t) a projective cover of $\ker c$. □

Lemma 2.7. (a) *$\operatorname{Tr}(\operatorname{Tr}(M)) \cong M$ if and only if $M \in \operatorname{mod}_{\mathrm{pf}}\Lambda$.*
(b) *For $M,N \in \operatorname{mod}_{\mathrm{pf}}\Lambda$ we have $\operatorname{Tr}(M) \cong \operatorname{Tr}(N)$ if and only if $M \cong N$.*

Proof. (a) The implication $\operatorname{Tr}(\operatorname{Tr}(M)) \cong M \implies M \in \operatorname{mod}_{\mathrm{pf}}\Lambda$ follows from Lemma 2.4. Now let M be in $\operatorname{mod}_{\mathrm{pf}}\Lambda$. Then Proposition 2.6 implies that $P_0^t \xrightarrow{\varepsilon_1^t} P_1^t \xrightarrow{c} \operatorname{Tr}(M) \to 0$ is a minimal projective presentation of $\operatorname{Tr}(M)$. Application of $(-)^t$ yields the exact sequence

$$0 \to \operatorname{Tr}(M)^t \xrightarrow{c^t} (P_1^t)^t \xrightarrow{(\varepsilon_1^t)^t} (P_0^t)^t \xrightarrow{c'} \operatorname{Tr}(\operatorname{Tr}(M)) \to 0,$$

and the commuting diagram

$$\begin{array}{ccccccc}
(P_1^t)^t & \xrightarrow{\eta_1} & (P_0^t)^t & \xrightarrow{c'} & \operatorname{Tr}(\operatorname{Tr}(M)) & \longrightarrow & 0\\
\uparrow{\scriptstyle \tau_{P_1}} & & \uparrow{\scriptstyle \tau_{P_0}} & & & & \\
P_1 & \xrightarrow{\varepsilon_1} & P_0 & \xrightarrow{c} & M & \longrightarrow & 0.
\end{array}$$

As $(c'\circ\tau_{P_0})\circ\varepsilon_1 = c'\circ\eta_1\circ\tau_{P_1} = 0$, the universal property of the cokernel yields a Λ-linear map $d\colon M \to \operatorname{Tr}(\operatorname{Tr}(M))$ such that $c'\circ\tau_{P_0} = d\circ c$. In particular, d is surjective. By the same token, we get an epimorphism $d'\colon \operatorname{Tr}(\operatorname{Tr}(M)) \to M$ such that $d'\circ c' = c\circ\tau_{P_0}^{-1}$. We conclude that $\operatorname{Tr}(\operatorname{Tr}(M)) \cong M$.
(b) This follows immediately from (a). □

Let $M, N \in \operatorname{mod}\Lambda$. We define

$$\mathcal{P}(M,N) := \{f \in \operatorname{Hom}_\Lambda(M,N) \mid \exists P \text{ projective}, \ h\colon M \to P, g\colon P \to N\colon g \circ h = f\}$$

Direct computation shows that $\mathcal{P}(-,-)$ is an ideal of $\operatorname{mod}\Lambda$:

Lemma 2.8. *Let $M, M', N, N' \in \operatorname{mod}\Lambda$. The following statements hold:*
(a) *Let $f_1, f_2 \in \mathcal{P}(M,N)$, then $f_1 + f_2 \in \mathcal{P}(M,N)$.*
(b) *Let $f \in \mathcal{P}(M,N)$, $g \in \operatorname{Hom}_\Lambda(N,N')$ and $h \in \operatorname{Hom}_\Lambda(M',M)$, then $f \circ h \in \mathcal{P}(M',N)$ and $g \circ f \in \mathcal{P}(M,N')$.*
(c) *The quotient $\underline{\operatorname{Hom}}_\Lambda(M,N) := \operatorname{Hom}_\Lambda(M,N)/\mathcal{P}(M,N)$ obtains the structure of an abelian group and for $[f] \in \underline{\operatorname{Hom}}_\Lambda(M,N), [g] \in \underline{\operatorname{Hom}}_\Lambda(N,N')$ the multiplication $[g] \circ [f] := [g \circ f]$ is well-defined.*

Definition. We denote with $\underline{\operatorname{mod}}\Lambda$ the category which has the same objects as $\operatorname{mod}\Lambda$ and for M, N the morphism space $\operatorname{Hom}_{\underline{\operatorname{mod}}\Lambda}(M,N)$ is given by $\underline{\operatorname{Hom}}_\Lambda(M,N)$.
For M, N we also define

$$\mathcal{I}(M,N) := \{f \in \operatorname{Hom}_\Lambda(M,N) \mid \exists I \text{ injective}, \ h\colon M \to I, g\colon I \to N\colon g \circ h = f\}$$

and get in the same fashion the category $\overline{\operatorname{mod}}\Lambda$.

Lemma 2.9. *Let $M, N \in \operatorname{mod}\Lambda$ and $f \in \operatorname{Hom}_\Lambda(M,N)$. The following statements hold.*
(a) *$f \in \mathcal{P}(M,N)$ if and only if f factors through a projective cover (P,ε) of N.*
(b) *$f \in \mathcal{I}(M,N)$ if and only if f factors through an injective hull (I,ι) of M.*

Proof. We only prove (a). Assume that $f \in \mathcal{P}(M,N)$ and let $f = M \xrightarrow{g} Q \xrightarrow{h} N$ be a factorization with Q projective. Let (P,ε) be a projective cover of N. Since $\operatorname{Hom}_\Lambda(Q,-)$ is exact, the k-linear map

$$\varepsilon_*\colon \operatorname{Hom}_\Lambda(Q,P) \to \operatorname{Hom}_\Lambda(Q,N)$$

is surjective. Hence h factors through ε.

□

Lemma 2.10. *Let $M, N \in \operatorname{mod}\Lambda$*
(a) *We have $M \cong M_{\mathrm{pf}}$ in $\underline{\operatorname{mod}}\Lambda$.*
(b) *$M \cong N$ in $\underline{\operatorname{mod}}\Lambda$ if and only if $M_{\mathrm{pf}} \cong N_{\mathrm{pf}}$ in $\operatorname{mod}\Lambda$.*
(c) *$M \cong 0$ in $\underline{\operatorname{mod}}\Lambda$ if and only if M is projective.*

Proof. It sufficies to prove (a). We write $M = M_{\mathrm{pf}} \oplus P_M$ and let $i\colon M_{\mathrm{pf}} \to M, j\colon M_P \to M$ be the natural embeddings and $p\colon M \to M_{\mathrm{pf}}, q\colon M \to M_P$ the canonical projections. Then $i \circ p + \mathcal{P}(M,M) = \mathrm{id}_M - j \circ q + \mathcal{P}(M,M) = \mathrm{id}_M + \mathcal{P}(M,M)$ and $p \circ i = \mathrm{id}_{M_{\mathrm{pf}}} - q \circ j + \mathcal{P}(M,M) = \mathrm{id}_{M_{\mathrm{pf}}}$. Hence $M \cong M_{\mathrm{pf}}$ in $\underline{\operatorname{mod}}\Lambda$.

□

Proposition 2.11. (a) *Let $M, N \in \operatorname{mod}\Lambda$, then* Tr *induces a natural k-linear map*

$$\operatorname{Tr}_{M,N}\colon \operatorname{Hom}_\Lambda(M,N) \to \underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(N), \operatorname{Tr}(M)).$$

(b) *The transpose induces a k-linear map $\underline{\operatorname{Tr}}_{M,N}\colon \underline{\operatorname{Hom}}_\Lambda(M,N) \to \underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(N), \operatorname{Tr}(M))$.*

Proof. Let $M, N \in \operatorname{mod}\Lambda$ and $f\colon M \to N$ be Λ-linear. We take minimal projective presentations of M and N and get the following commutative diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{\varepsilon_1} & P_0 & \xrightarrow{\varepsilon_0} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ P_0' & \xrightarrow{\varepsilon_1'} & P_0' & \xrightarrow{\varepsilon_0'} & N & \longrightarrow & 0. \end{array}$$

After application of $(-)^t$ we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^t & \longrightarrow & P_0^t & \xrightarrow{\varepsilon_1^t} & P_1^t & \xrightarrow{c} & \operatorname{Tr}(M) \longrightarrow 0 \\ & & \uparrow{\scriptstyle f^t} & & \uparrow{\scriptstyle f_0^t} & & \uparrow{\scriptstyle f_1^t} & & \uparrow{\scriptstyle d} \\ 0 & \longrightarrow & N^t & \longrightarrow & (P_1')^t & \xrightarrow{(\varepsilon_1')^t} & (P_1')^t & \xrightarrow{c'} & \operatorname{Tr}(N) \longrightarrow 0. \end{array}$$

The universal property of the cokernel yields a unique map $d \in \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(N), \operatorname{Tr}(M))$ such that $d \circ c' = c \circ f_1^t$. We set $\operatorname{Tr}_{f_0,f_1}(f) := d$.
We now show that $[\operatorname{Tr}_{f_0,f_1}(f)]$ does not depend on the maps f_0 and f_1: Assume that g_1, g_0 are Λ-linear maps such that we have a commutative diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{\varepsilon_1} & P_0 & \xrightarrow{\varepsilon_0} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle f} & & \\ P_1' & \xrightarrow{\varepsilon_1'} & P_0' & \xrightarrow{\varepsilon_0'} & N & \longrightarrow & 0. \end{array}$$

We extend $P_1 \to P_0 \to M \to 0$ and $P_1' \to P_0' \to N \to 0$ to projective resolutions $(\mathbb{P}_M, \varepsilon)$ and $(\mathbb{P}_N, \varepsilon')$. By general theory we get two chain maps $\eta, \delta\colon (\mathbb{P}_M, \varepsilon) \to (\mathbb{P}_N, \varepsilon')$ with $\eta_{-1} = f = \delta_{-1}$, $\eta_0 = f_0$, $\eta_1 = f_1$, $\delta_0 = g_0$ and $\delta_1 = g_1$. Hence $\gamma := \eta - \delta$ is chain map with $\gamma_{-1} = 0$. By the Comparison theorem, γ is unique up to homotopy. As 0 is another such a chain map, we find $(s_n\colon P_n \to P_{n+1}')_{n\in\mathbb{Z}}$ such that $\delta_n - \eta_n = \varepsilon_{n+1}' \circ s_n + s_{n-1} \circ \varepsilon_n$ for all $n \in \mathbb{Z}$ and $s_l = 0$ for all $l < 0$. In particular, we have

$$f_1 - g_1 = \delta_1 - \eta_1 = \varepsilon_2' \circ s_1 + s_0 \circ \varepsilon_1 = (\varepsilon_2', s_0) \circ \begin{pmatrix} s_1 \\ \varepsilon_1 \end{pmatrix} : P_1 \xrightarrow{\lambda} P_2' \oplus P_0 \xrightarrow{\nu} P_1',$$

i.e. $f_1 - g_1$ factors through the projective module $P_2' \oplus P_0$. We let $\operatorname{Tr}_{g_0,g_1}(f) =: e \in \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(N), \operatorname{Tr}(M))$ be the unique morphism such that $e \circ c' = c \circ (g_1)^t$ and conclude

$$(d - e) \circ c' = c \circ (f_1^t - g_1^t) = c \circ (f_1 - g_1)^t = c \circ (\nu \circ \lambda)^t = c \circ \lambda^t \circ \nu^t.$$

Moreover, we have $\varepsilon_1' \circ \nu = \varepsilon_1' \circ \varepsilon_2' + \varepsilon_1' \circ s_0 = \varepsilon_1' \circ s_0 = s_{-1} \circ \varepsilon_0$ as $s_{-1} = 0$.
We conclude $0 = (\varepsilon_1' \circ \nu)^t = \nu^t \circ (\varepsilon_1')^t$, so the universal property of the cokernel implies the existence of $s \in \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(N), (P_2' \oplus P_0)^t)$ such that $\nu^t = s \circ c'$. We summarize our findings:

$$(d - e) \circ c' = c \circ \lambda^t \circ \nu^t = c \circ \lambda^t \circ s \circ c',$$

and conclude $\operatorname{Tr}_{f_0,f_1}(f) - \operatorname{Tr}_{g_0,g_1}(f) = d - e = c \circ \lambda^t \circ s\colon \operatorname{Tr}(N) \to (P_2' \oplus P_0)^t \to P_1^t \to \operatorname{Tr}(M)$, which shows that $\operatorname{Tr}_{f_0,f_1}(f) - \operatorname{Tr}_{g_0,g_1}(f) \in \mathcal{P}(\operatorname{Tr}(N), \operatorname{Tr}(M))$.
Hence we have a well-defined map

$$\operatorname{Tr}_{M,N}\colon \operatorname{Hom}_\Lambda(M, N) \to \underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(N), \operatorname{Tr}(M)), f \mapsto [\operatorname{Tr}_{f_0,f_1}(f)].$$

Now direct computation shows that $\mathrm{Tr}_{M,N}$ is k-linear.
(b) Let $f \in \mathcal{P}(N,M)$. By (a) we just have to show that $\mathrm{Tr}_{M,N}(f) = [0]$. As $f \in \mathcal{P}(M,N)$ we find $q \in \mathrm{Hom}_\Lambda(M, P_0')$ such that $f = q \circ \varepsilon_0$. Hence we obtain the commutative diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{\varepsilon_1} & P_0 & \xrightarrow{\varepsilon_0} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & \swarrow{\scriptstyle q} & \downarrow{\scriptstyle f} & & \\ P_0' & \xrightarrow{\varepsilon_1'} & P_0' & \xrightarrow{\varepsilon_0'} & N & \longrightarrow & 0 \end{array}$$

and can assume that $f_0 = 0$ since $\varepsilon_1' \circ f_0 = f_1 \circ \varepsilon_1 = q \circ \varepsilon_1 \circ \varepsilon_0 = 0$. By definition $\mathrm{Tr}_{f_0,f_1}(f)$ is the unique Λ^{op}-linear map such that $\mathrm{Tr}_{f_0,f_1}(f) \circ c' = c \circ (f_1)^t = 0$, i.e. $\mathrm{Tr}_{f_0,f_1}(f) = 0$ and $\mathrm{Tr}_{M,N}(f) = [\mathrm{Tr}_{f_0,f_1}(f)] = [0]$. We denote the corresponding k-linear map by

$$\underline{\mathrm{Tr}}_{M,N} \colon \underline{\mathrm{Hom}}_\Lambda(M,N) \to \underline{\mathrm{Hom}}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(N), \mathrm{Tr}(M)).$$

□

By construction we have for $f \colon M \to N$ and $g \colon N \to L$ that $\mathrm{Tr}_{M,N}(f) \circ \mathrm{Tr}_{N,L}(g) = \mathrm{Tr}_{M,L}(g \circ f)$ and $\mathrm{Tr}_{M,M}(\mathrm{id}_M) = [\mathrm{id}_{\mathrm{Tr}(M)}]$.

Definition. The k-linear functor $\mathrm{Tr} \colon \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda^{\mathrm{op}}$ given by

$$M \mapsto \mathrm{Tr}(M) \text{ and } [f] \in \underline{\mathrm{Hom}}_\Lambda(M,N) \mapsto \mathrm{Tr}([f]) := \underline{\mathrm{Tr}}_{M,N}([f])$$

is called the **transpose** of Λ. We denote the transpose of Λ^{op} also with Tr.

Lemma 2.12. *We have a duality*

$$\underline{\mathrm{mod}}\Lambda \underset{\mathrm{Tr}}{\overset{\mathrm{Tr}}{\rightleftarrows}} \underline{\mathrm{mod}}\Lambda^{\mathrm{op}}.$$

Proof. We show that $\mathrm{Tr} \circ \mathrm{Tr} \cong \mathrm{id}_{\underline{\mathrm{mod}}\Lambda}$. The proof for $\mathrm{Tr} \circ \mathrm{Tr} \cong \mathrm{id}_{\underline{\mathrm{mod}}\Lambda^{\mathrm{op}}}$ is similiar.

For $X \in \mathrm{mod}_{\mathrm{pf}}\Lambda$ with minimal projective presentation $Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} X \to 0$ we get a commutative diagram $(*)$:

$$\begin{array}{ccccccc} Q_1 & \xrightarrow{d_1} & Q_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tau_{Q_1}} & & \downarrow{\scriptstyle \tau_{Q_0}} & & \downarrow{\scriptstyle \nu_X} & & \\ (Q_1^t)^t & \xrightarrow{(d_1^t)^t} & (Q_0^t)^t & \xrightarrow{(d_0^t)^t} & \mathrm{Tr}(\mathrm{Tr}(X)). & \longrightarrow & 0 \end{array}$$

Moreover, $\mathrm{Tr}(X) \in \mathrm{mod}_{\mathrm{pf}}\Lambda^{\mathrm{op}}$ in conjunction with Lemma 2.7 imply that the bottom row is a minimal projective presentation of $\mathrm{Tr}(\mathrm{Tr}(X))$. We get an isomorphism $\nu_X \colon X \to \mathrm{Tr}(\mathrm{Tr}(X))$ such that the diagram commutes.
We now show that $\nu = ([\nu_X])_{X \in \underline{\mathrm{mod}}\Lambda}$ is a natural transformation. Let $M, N \in \underline{\mathrm{mod}}\Lambda$. As $X \cong X_{\mathrm{pf}}$ in $\underline{\mathrm{mod}}\Lambda$, we can assume $M, N \in \mathrm{mod}_{\mathrm{pf}}\Lambda$. Let $f \colon M \to N$ be Λ-linear. We consider a commutative diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{\varepsilon_1} & P_0 & \xrightarrow{\varepsilon_0} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ P_1' & \xrightarrow{\varepsilon_1'} & P_0' & \xrightarrow{\varepsilon_0'} & N & \longrightarrow & 0, \end{array}$$

such that the rows are minimal projective presentations of M and N respectively. Application of $(-)^t$ yields the commutative diagram

$$\begin{array}{ccccccc} P_0^t & \xrightarrow{\varepsilon_0^t} & P_1^t & \xrightarrow{\varepsilon_1^t} & \mathrm{Tr}(M) & \longrightarrow & 0 \\ \uparrow{\scriptstyle f_0^t} & & \uparrow{\scriptstyle f_1^t} & & \uparrow{\scriptstyle \mathrm{Tr}_{f_0,f_1}(f)} & & \\ (P_0')^t & \xrightarrow{(\varepsilon_0')^t} & (P_1')^t & \xrightarrow{(\varepsilon_1')^t} & \mathrm{Tr}(N) & \longrightarrow & 0 \end{array}$$

which rows are minimal projective presentations. We obtain

$$\begin{array}{ccccccc} (P_1^t)^t & \xrightarrow{(\varepsilon_1^t)^t} & (P_0^t)^t & \xrightarrow{(\varepsilon_0^t)^t} & \mathrm{Tr}(\mathrm{Tr}(M)) & \longrightarrow & 0 \\ \downarrow{\scriptstyle (f_1^t)^t} & & \downarrow{\scriptstyle (f_0^t)^t} & & \downarrow{\scriptstyle \mathrm{Tr}_{f_0^t,f_1^t}(\mathrm{Tr}_{f_0,f_1}(f))=:g} & & \\ (P_1')^{tt} & \xrightarrow{(\varepsilon_1')^{tt}} & (P_0')^{tt} & \xrightarrow{(\varepsilon_0')^{tt}} & \mathrm{Tr}(\mathrm{Tr}(N)) & \longrightarrow & 0 \end{array}$$

and conclude

$$\begin{aligned} f \circ \varepsilon_0 = \varepsilon_0' \circ f_0 = \varepsilon_0' \circ (\tau_{P_0'}^{-1} \circ (f_0^t)^t \circ \tau_{P_0}) &\overset{(*)}{=} \nu_N^{-1} \circ (\varepsilon_0')^{tt} \circ (f_0^t)^t \circ \tau_{P_0} \\ = \nu_N^{-1} \circ g \circ (\varepsilon_0^t)^t \circ \tau_{P_0} &\overset{(*)}{=} \nu_N^{-1} \circ g \circ \nu_M \circ \varepsilon_0. \end{aligned}$$

The surjectivity of ε_0 yields $f = \nu_N^{-1} \circ g \circ \nu_M$, i.e. $\nu_N \circ f = g \circ \nu_M$. Hence

$$[\nu_N] \circ [f] = [g] \circ [\nu_M] = (\mathrm{Tr} \circ \mathrm{Tr})([f]) \circ [\nu_M].$$

□

We denote by $D = \mathrm{Hom}_k(-, k)$ the standard duality between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$. As D takes projective modules to injective modules and vice versa, we conclude:

Theorem 2.13. *We have mutually inverse equivalences*

$$\underline{\operatorname{mod}}\Lambda \underset{\mathrm{Tr}\, D}{\overset{D\,\mathrm{Tr}}{\rightleftarrows}} \overline{\operatorname{mod}}\Lambda.$$

Corollary 2.14. *The Auslander-Reiten translation induces a bijection*

$$\{[M] \in \operatorname{ind}\Lambda \mid M \text{ not projective}\} \underset{\mathrm{Tr}\, D}{\overset{D\,\mathrm{Tr}}{\rightleftarrows}} \{[M] \in \operatorname{ind}\Lambda \mid M \text{ not injective}\}.$$

Proof. Let M be indecomposable and not projective, then $M \in \operatorname{mod}_{\mathrm{pf}}\Lambda$. Hence

$$((\mathrm{Tr} \circ D) \circ (D \circ \mathrm{Tr}))(M) \cong (\mathrm{Tr} \circ \mathrm{Tr})(M) \overset{2.7}{\cong} M.$$

If N is not injective and indecomposable, then $DN \in \operatorname{mod}_{\mathrm{pf}}\Lambda$ and

$$((D \circ \mathrm{Tr}) \circ (\mathrm{Tr} \circ D))(N) \cong D(DN) \cong N.$$

□

2.3 The defect of an exact sequence (a weak version)

Proposition 2.15. *Let $P_1 \xrightarrow{\varepsilon_1} P_0 \xrightarrow{\varepsilon_0} X \to 0$ be a minimal projective presentation. There exists an exact sequence*

$$0 \to \mathrm{Hom}_\Lambda(Y, X) \to \mathrm{Hom}_\Lambda(P_0, Y) \to \mathrm{Hom}_\Lambda(P_1, Y) \to \mathrm{Tr}(X) \otimes_\Lambda Y \to 0$$

for all $Y \in \mathrm{mod}\,\Lambda$.

Proof. For $i \in \{0, 1\}$ we consider the k-linear map

$$\alpha_i \colon \mathrm{Hom}_\Lambda(P_i, \Lambda) \otimes_\Lambda Y \to \mathrm{Hom}_\Lambda(P_i, Y)$$

given by $\alpha_i(f \otimes y)(p) = f(p)y$ for all $y \in Y, p \in P_i$. We apply $\mathrm{Hom}_\Lambda(-, Y)$ to the minimal projective presentation and observe on the one hand,

$$[(\varepsilon_1^* \circ \alpha_0)(f \otimes y)](p_1) = (\varepsilon_1^*(\alpha_0(f \otimes y)))(p_1) = \alpha_0(f \otimes y)(\varepsilon_1(p_1)) = f(\varepsilon_1(p_1))y$$

for all $f \in \mathrm{Hom}_\Lambda(P_0, Y)$, $y \in Y$ and $p_1 \in P_1$. On the other hand, we have

$$[(\alpha_1 \circ (\varepsilon_1^* \otimes \mathrm{id}_Y))(f \otimes y)](p_1) = (f \circ \varepsilon_1(p_1))(y) = f(\varepsilon_1(p_1))y.$$

We get the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_\Lambda(P_0, \Lambda) \otimes_\Lambda Y & \xrightarrow{\varepsilon_1^* \otimes \mathrm{id}_Y} & \mathrm{Hom}_\Lambda(P_1, \Lambda) \otimes_\Lambda Y \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} \\ \mathrm{Hom}_\Lambda(P_0, Y) & \xrightarrow{\varepsilon_1^*} & \mathrm{Hom}_\Lambda(P_1, Y). \end{array}$$

Observe that the upper row is obtained from applying $(- \otimes_\Lambda Y) \circ (-)^t$ to the minimal projective presentation. Since $(- \otimes_\Lambda Y)$ is right exact, we have the commutative diagram

$$\begin{array}{ccccccccc} & & & & P_0^t \otimes_\Lambda Y & \xrightarrow{\varepsilon_1^* \otimes \mathrm{id}_Y} & P_1^t \otimes_\Lambda Y & \xrightarrow{c} & \mathrm{Tr}(X) \otimes_\Lambda Y \longrightarrow 0 \\ & & & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \\ 0 \longrightarrow & \mathrm{Hom}_\Lambda(X, Y) & \xrightarrow{\varepsilon_0^*} & \mathrm{Hom}_\Lambda(P_0, Y) & \xrightarrow{\varepsilon_{1*}} & \mathrm{Hom}_\Lambda(P_1, Y). & & & \end{array}$$

For $Z \in \mathrm{mod}\,\Lambda$ we consider the k-linear map

$$\alpha_Z \colon \mathrm{Hom}_\Lambda(Z, \Lambda) \otimes_\Lambda Y \to \mathrm{Hom}_\Lambda(Z, Y), \alpha_Z(f \otimes y)(z) = f(z)y$$

and observe that α_Λ is an isomorphism. Hence α_{Λ^n} is an isomorphism for each $n \in \mathbb{N}$. As each projective module P is isomorphic to a direct summand of some Λ^n, we conclude that $\alpha_{P_i} = \alpha_{P_i}$, $i \in \{0, 1\}$ are isomorphisms.
Hence the sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(X, Y) \xrightarrow{\varepsilon_0^*} \mathrm{Hom}_\Lambda(P_0, Y) \xrightarrow{\varepsilon_1^*} \mathrm{Hom}_\Lambda(P_1, Y) \xrightarrow{c \circ \alpha_1^{-1}} \mathrm{Tr}(X) \otimes_\Lambda Y \longrightarrow 0$$

is exact. □

Corollary 2.16. *Let $P_1 \to P_1 \to X \to 0$ a minimal projective presentation. We have*

$$\dim_k \operatorname{Hom}_\Lambda(X, M) - \dim_k \operatorname{Hom}_\Lambda(M, D\operatorname{Tr}(X)) = \dim_k \operatorname{Hom}_\Lambda(P_0, M) - \dim_k \operatorname{Hom}_\Lambda(P_1, M)$$

for all $M \in \operatorname{mod}\Lambda$.

Proof. By the adjoint isomorphism theorem (see [7, 2.76]) we have

$$\operatorname{Hom}_k(\operatorname{Tr}(X) \otimes_\Lambda M, k) \cong \operatorname{Hom}_\Lambda(M, \operatorname{Hom}_k(\operatorname{Tr}(X), k)) = \operatorname{Hom}_\Lambda(M, D\operatorname{Tr}(X)).$$

In particular, $\dim_k \operatorname{Tr}(X) \otimes_k M = \dim_k \operatorname{Hom}_\Lambda(M, D\operatorname{Tr}(X))$. Now apply 2.15. □

Definition. Let $\delta\colon 0 \to A \to B \to C \to 0$ be an exact sequence and $X \in \operatorname{mod}\Lambda$. We define

$$\delta^*(X) := \operatorname{coker}(\operatorname{Hom}_\Lambda(B, X) \to \operatorname{Hom}_\Lambda(A, X))$$

and

$$\delta_*(X) := \operatorname{coker}(\operatorname{Hom}_\Lambda(X, B) \to \operatorname{Hom}_\Lambda(X, C)).$$

Theorem 2.17 (The defect formula (a weak version))**.** *Let $\delta\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact, then*

$$\dim_k(\delta^*(D\operatorname{Tr}(X)) = \dim_k(\delta_*(X))$$

for all $X \in \operatorname{mod}\Lambda$.

Proof. Let $X \in \operatorname{mod}\Lambda$, then by definition we have

$$\begin{aligned}\dim_k \delta_*(X) &= \dim_k \operatorname{Hom}_\Lambda(X, C) - \dim_k \operatorname{im} g_* \\ &= \dim_k \operatorname{Hom}_\Lambda(X, C) - (\dim_k \operatorname{Hom}_\Lambda(X, B) - \dim_k \operatorname{Hom}_\Lambda(X, A))\end{aligned}$$

and

$$\begin{aligned}\dim_k \delta^*(D\operatorname{Tr}(X)) &= \dim_k \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X)) - \dim_k \operatorname{im} f^* \\ &= \dim_k \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X)) - (\dim_k \operatorname{Hom}_\Lambda(B, D\operatorname{Tr}(X)) - \dim_k \operatorname{Hom}_\Lambda(C, D\operatorname{Tr}(X)))\end{aligned}$$

Let $P_1 \to P_0 \to X \to 0$ be a minimal projective presentation. For $Y \in \{A, B, C\}$ we define $z(Y) := \dim_k \operatorname{Hom}_\Lambda(X, Y) - \dim_k \operatorname{Hom}_\Lambda(Y, D\operatorname{Tr}(X))$. We conclude

$$\begin{aligned}\dim_k \delta_*(X) - \dim_k \delta^*(D\operatorname{Tr}(X)) &= z(C) - z(B) + z(A) \\ &\overset{2.16}{=} \dim_k \operatorname{Hom}_\Lambda(P_0, C) - \dim_k \operatorname{Hom}_\Lambda(P_0, B) + \dim_k \operatorname{Hom}_\Lambda(P_0, A) \\ &\quad + \dim_k \operatorname{Hom}_\Lambda(P_1, C) - \dim_k \operatorname{Hom}_\Lambda(P_1, B) + \dim_k \operatorname{Hom}_\Lambda(P_0, A) \\ &= \delta_*(P_0) + \delta_*(P_1)\end{aligned}$$

But since P_i is projective the functor $\operatorname{Hom}_\Lambda(P_i, -)$ is exact. Hence $\dim_k \delta^*(P_i) = 0$ for $i \in \{0, 1\}$. □

Corollary 2.18. *Let $\delta\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence and $X \in \operatorname{mod}\Lambda$. The following statements are equivalent.*

(a) *$g_*\colon \operatorname{Hom}_\Lambda(X, B) \to \operatorname{Hom}_\Lambda(X, C)$ is surjective.*

(b) *$f^*\colon \operatorname{Hom}_\Lambda(B, D\operatorname{Tr}(X)) \to \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X))$ is surjective.*

Proof. We have the following equivalences:

$$\operatorname{im} g_* = \operatorname{Hom}_\Lambda(X, C) \Leftrightarrow \delta_*(X) = 0 \overset{2.17}{\Leftrightarrow} \delta^*(D\operatorname{Tr}(X)) = 0 \Leftrightarrow \operatorname{im} f^* = \operatorname{Hom}_\Lambda(A, \operatorname{Tr}(X)).$$

□

Corollary 2.19 (The Auslander-Reiten formulas, a weak version)**.** *Let $A, B \in \operatorname{mod}\Lambda$, then*

$$\dim_k \underline{\operatorname{Hom}}_\Lambda(A, B) = \dim_k \operatorname{Ext}^1_\Lambda(B, D\operatorname{Tr} A)$$

and

$$\dim_k \overline{\operatorname{Hom}}_\Lambda(B, D\operatorname{Tr}(A)) = \dim_k \operatorname{Ext}^1_\Lambda(A, B).$$

Proof. Consider $\delta\colon 0 \to X \overset{\iota}{\to} P \overset{\varepsilon}{\to} B \to 0$ exact with a projective cover (P, ε) of B. Then $\operatorname{im}\varepsilon_*$ is clearly contained in $\mathcal{P}(A, B)$. As (P, ε) a projective cover, we conclude with Lemma 2.9 that $\operatorname{im}\varepsilon_* = \mathcal{P}(A, B)$. Hence

$$\delta_*(A) = \underline{\operatorname{Hom}}_\Lambda(A, B).$$

As P is projective, the long exact sequence yields the exact sequence

$$\operatorname{Hom}_\Lambda(P, D\operatorname{Tr}(A)) \overset{\iota^*}{\to} \operatorname{Hom}_\Lambda(X, D\operatorname{Tr}(A)) \to \operatorname{Ext}^1_\Lambda(B, D\operatorname{Tr}(A)) \to 0.$$

We conclude $\operatorname{Ext}^1_\Lambda(B, D\operatorname{Tr}(A)) \cong \operatorname{Hom}_\Lambda(X, D\operatorname{Tr}(A))/\operatorname{im}\iota^* = \delta^*(D\operatorname{Tr}(A))$. Now 2.17 yields

$$\dim_k \operatorname{Ext}^1_\Lambda(B, D\operatorname{Tr}(A)) = \dim_k \delta^*(D\operatorname{Tr}(A)) = \dim_k \delta_*(A) = \dim_k \underline{\operatorname{Hom}}_\Lambda(A, B).$$

Now consider $\delta\colon 0 \to B \overset{\iota}{\to} I \overset{\varepsilon}{\to} Y \to 0$ with an injective hull (I, ι) of B. Then $\delta_*(A) = \operatorname{Ext}^1_\Lambda(A, B)$ and

$$\delta^*(D\operatorname{Tr}(A)) = \overline{\operatorname{Hom}}_\Lambda(B, D\operatorname{Tr}(A)).$$

□

Corollary 2.20 (The Auslander-Reiten formulas, a second (weak) version)**.** *Let $A, B \in \operatorname{mod}\Lambda$, then*

$$\dim_k \underline{\operatorname{Hom}}(\operatorname{Tr} D(B), A) = \dim_k \operatorname{Ext}^1_\Lambda(A, B) = \dim_k \overline{\operatorname{Hom}}_\Lambda(B, D\operatorname{Tr}(A)).$$

Proof. We have

$$\dim_k \underline{\operatorname{Hom}}(\operatorname{Tr} D(B), A) = \dim_k \operatorname{Ext}^1_\Lambda(A, D\operatorname{Tr}(\operatorname{Tr} D(B))).$$

We write $B = B_{\text{if}} \oplus I_B$, with I_B injective and B_{if} injective free. As

$$\operatorname{Tr} D(B) \cong \operatorname{Tr} D(B_{\text{if}}) \oplus \operatorname{Tr} D(I_B) = \operatorname{Tr} D(B_{\text{if}}),$$

we conclude

$$\operatorname{Ext}^1_\Lambda(A, D\operatorname{Tr}(\operatorname{Tr} D(B))) \cong \operatorname{Ext}^1_\Lambda(A, D\operatorname{Tr}(\operatorname{Tr} D(B_{\text{if}}))) \cong \operatorname{Ext}^1_\Lambda(A, B_{\text{if}}) \cong \operatorname{Ext}^1_\Lambda(A, B)$$

since $\operatorname{Ext}^1_\Lambda(A, -)$ does not see injectives. This proves the first equality.
For the other equality write $B = B_{\text{pf}} \oplus P_B$ and consider

$$\dim_k \overline{\operatorname{Hom}}_\Lambda(B, D\operatorname{Tr}(A)) = \dim_k \operatorname{Ext}^1_\Lambda(\operatorname{Tr} D(D\operatorname{Tr}(A)), B).$$

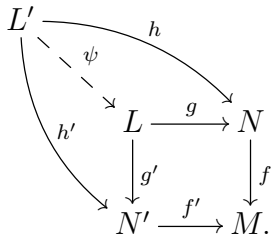
□

2.4 Pushouts and pullbacks

Definition. A commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{g} & N \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f} \\ N' & \xrightarrow{f'} & M \end{array}$$

in $\operatorname{mod}\Lambda$ is called **pullback** or **pullback diagram** if for every pair $h\colon L' \to N$, $h'\colon L' \to N'$ in $\operatorname{mod}\Lambda$ with the property $f \circ h = f' \circ h'$ there exists a unique Λ-linear map $\psi\colon L' \to L$ such that $h = g \circ \psi$ and $h' = h' \circ \psi$. In other words, we get the following diagram



Lemma 2.21. *The following statements are equivalent.*

(*a*) *The commutative diagram*

$$\begin{array}{ccc} L & \xrightarrow{g} & N \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f} \\ N' & \xrightarrow{f'} & M \end{array}$$

is a pullback diagram.

(*b*) *The sequence* $0 \longrightarrow L \xrightarrow{\begin{pmatrix} g \\ -g' \end{pmatrix}} N \oplus N' \xrightarrow{(f \;\; f')} M$ *is exact.*

Proof. (*a*) $\implies$ (*b*) Let $h := \begin{pmatrix} g \\ -g' \end{pmatrix}$ and denote with $\iota\colon \ker h \to L$ the natural embedding. Then

$$f \circ (g \circ \iota) = (f \circ g) \circ \iota = (f' \circ g') \circ \iota = f' \circ (g' \circ \iota).$$

The pullback property yields a unique Λ-linear map $\psi\colon \ker h \to L$ such that $g \circ \iota = g \circ \psi$ and $g' \circ \iota = g' \circ \psi$. Cleary ι also has this property. Since $\ker h \subseteq \ker g \cap \ker g'$, we know that $0 \in \operatorname{Hom}_\Lambda(\ker h, L)$ also has this property. By uniqueness we conclude $0 = \psi = \iota$ and therefore $0 = \ker h$. Hence h is injective.
Let $L' := \ker(f, f')$, $\pi_N\colon N \oplus N' \to N$ and $\pi_{N'}\colon N \oplus N' \to N'$ be the canonical projections. Then $f \circ \pi_{N|L'} = f' \circ (-\pi_{N'|L'})$. Hence we get a unique Λ-linear map $\psi\colon L' \to L$ such that $\pi_{N|L'} = g \circ \psi$ and $\pi_{N'|L'} = -g' \circ \psi$. Let $(x, y) \in L'$, then

$$(x, y) = (\pi_{N'|L'}, \pi_{N'|L'})(x, y) = \begin{pmatrix} g \\ -g' \end{pmatrix} (\psi(x, y)) \in \operatorname{im} \begin{pmatrix} g \\ -g' \end{pmatrix}.$$

$(b) \implies (a)$ Let $h\colon L' \to N$ and $h'\colon L' \to N'$ such that $f \circ h = f \circ h'$, then $\operatorname{im}\begin{pmatrix} h \\ -h' \end{pmatrix} \subseteq \ker(f, f') = \operatorname{im}\begin{pmatrix} g \\ -g' \end{pmatrix}$ and there exists a unique Λ-linear map $\psi\colon L' \to L$ such that $\psi \circ \begin{pmatrix} g \\ -g' \end{pmatrix} = \begin{pmatrix} h \\ -h' \end{pmatrix}$. Hence $\psi \circ g = h$ and $\psi \circ g' = h'$. □

Corollary 2.22. *Let $f\colon N \to M$ and $f'\colon N' \to M$ be Λ-linear. There exists a pullback diagram*

$$\begin{array}{ccc} L & \xrightarrow{g} & N \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f} \\ N' & \xrightarrow{f'} & M. \end{array}$$

Proof. We define $L := \{(n, n') \in N \oplus N' \mid f(n) = f'(n')\}$ and denote by $\pi_N\colon N \oplus N' \to N$ and $\pi_{N'}\colon N \oplus N' \to N'$ the canonical projections. We define $g := \pi_{N|L}$ and $g' := \pi_{N'|L}$. The sequence $0 \longrightarrow L \xrightarrow{\begin{pmatrix} g \\ -g' \end{pmatrix}} N \oplus N' \xrightarrow{(f \;\; f')} M$ is exact. □

Corollary 2.23. *Let*

$$0 \to A \xrightarrow{\iota} B \xrightarrow{f} C \to 0$$

be a short exact sequence and $f' \in \operatorname{End}_\Lambda(C)$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota'} & L & \xrightarrow{g'} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{f} & C & \longrightarrow & 0 \end{array}$$

with exact rows such that the right square is a pullback diagram.

Proof. Let

$$\begin{array}{ccc} L & \xrightarrow{g'} & C \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{f} & C \end{array}$$

be a pullback diagram. As $f \circ \iota = f' \circ 0_{\operatorname{Hom}_\Lambda(A,C)}$ there exists a unique map $\iota'\colon A \to L$ such that $g \circ \iota' = \iota$ and $0_{\operatorname{Hom}_\Lambda(A,C)} = g' \circ \iota'$. We have established the desired commutative diagram. It remains to prove that the upper row is exact. Cleary ι' is injective and $\operatorname{im}\iota' \subseteq \ker g'$. For $c \in C$ we find $b \in B$ such that $f'(c) = f(b)$, hence

$$(c, b) \in \ker\begin{pmatrix} f' \\ -f \end{pmatrix} = \operatorname{im}(g', g),$$

i.e. we find $l \in L$ such that $(c, b) = (g'(l), g(l))$, i.e. g' is surjective.

We consider $h := g_{|\ker g'}\colon \ker g' \to B$ and note that h is injective as $\begin{pmatrix} g' \\ -g \end{pmatrix} : L \to C \oplus B$ is injective. Now let $x \in \ker g'$, then

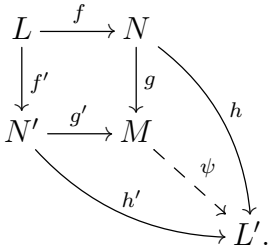
$$0 = (f' \circ g')(x) = (f \circ g)(x),$$

hence $g(x) \in \ker f = \operatorname{im}\iota = \operatorname{im}(g \circ \iota')$ and we find $a \in A$ such that $x - \iota'(a) \in \ker g$. As $x, \iota'(a) \in \ker g'$, we conclude that $x - \iota'(a) \in \ker g \cap \ker g' = \ker h = 0$, i.e. $x \in \operatorname{im}\iota'$. □

Definition. A commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} \\ N' & \xrightarrow{g'} & L \end{array}$$

in $\operatorname{mod}\Lambda$ is called **pushout** or **pushout diagram** if for every pair $h\colon N \to L'$, $h'\colon N' \to L'$ in $\operatorname{mod}\Lambda$ with the property $h \circ f = h' \circ f'$ there exists a unique Λ-linear map $\psi\colon L \to L'$ such that $h = \psi \circ g$ and $h' = \psi \circ g'$. In other words, we get the following diagram



Lemma 2.24. *The following statements are equivalent.*

(*a*) *The commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} \\ N' & \xrightarrow{g'} & L \end{array}$$

is a pushout diagram.

(*b*) *The sequence* $M \xrightarrow{\begin{pmatrix} f \\ -f' \end{pmatrix}} N \oplus N' \xrightarrow{\begin{pmatrix} g & g' \end{pmatrix}} L \longrightarrow 0$ *is exact.*

Proof. □

Corollary 2.25. *Let $f\colon M \to N$ and $f'\colon M \to N'$ be Λ-linear. There exists a pushout diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} \\ N' & \xrightarrow{g'} & L. \end{array}$$

2.5 The existence of Auslander-Reiten sequences

Theorem 2.26. (*a*) *Let C be an indecomposable non-projective Λ-module. There exists an Auslander-Reiten sequence $0 \to A \to B \to C \to 0$, with and $A \cong D\operatorname{Tr}(C)$.*

(*b*) *Let A be an indecomposable non-injective Λ-module. There exists an Auslander-Reiten sequence $0 \to A \to B \to C \to 0$, with $C \cong \operatorname{Tr} D(A)$.*

Proof. (*a*) As C is indecomposable and not projective, the module $D\operatorname{Tr}(C) \in \operatorname{mod}\Lambda$ is indecomposable and not injective. Hence we find a short exact sequence

$$0 \to D\operatorname{Tr}(C) \xrightarrow{i} X \xrightarrow{p} Y \to 0$$

that does not split. We apply $\mathrm{Hom}_\Lambda(C,-)$ and consider for $\Gamma := \mathrm{End}_\Lambda(C)^{\mathrm{op}}$ the exact sequence

$$\delta\colon \quad \mathrm{Hom}_\Lambda(C,X) \xrightarrow{p_*} \mathrm{Hom}_\Lambda(C,Y) \xrightarrow{c} \operatorname{coker} p_* \to 0$$

in $\operatorname{mod}\Gamma$.
The assumption $\operatorname{coker} p_* = 0$ yields that every morphism $C \to Y$ factors through p. It then follows from 2.18 that every morphism $D\operatorname{Tr}(C) \to D\operatorname{Tr}(C)$ factors through $i\colon D\operatorname{Tr}(C) \to X$. In particular, we find $g\colon X \to D\operatorname{Tr}(C)$ such that $g \circ i = \mathrm{id}_{\operatorname{Tr}(C)}$, a contradiction since the sequence does not split.
Hence we find a simple Γ-submodule $T \subseteq \operatorname{coker} p_*$. By the surjectivity of c we find $t \in \mathrm{Hom}_\Lambda(C,Y)$ such that $T = \Gamma.c(t) = \Gamma.\bar{t}$. Corollary 2.23 yields the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D\operatorname{Tr}(C) & \xrightarrow{\iota} & E & \xrightarrow{\pi} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & D\operatorname{Tr}(C) & \xrightarrow{i} & X & \xrightarrow{p} & C & \longrightarrow & 0.
\end{array}$$

We show now the upper row is an almost split sequence. The assumption that π is a split epimorphism yields the existence $\gamma\colon C \to E$ such that $\pi \circ \gamma = \mathrm{id}_C$. We conclude

$$p \circ (\varepsilon \circ \gamma) = (t \circ \pi) \circ \gamma = t.$$

Hence $t \in \operatorname{im} p_*$. The exactness of δ implies $c(t) = 0$, but $\Gamma.c(t) = T \neq 0$, a contradiction. This shows that π is not a split epimorphism.
Let $s\colon Z \to C$ be Λ-linear and not be a split epimorphism. We have to show that s factors through π. Corollary 2.23 yields the commutative diagram Δ:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D\operatorname{Tr}(C) & \xrightarrow{j} & L & \xrightarrow{v} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & D\operatorname{Tr}(C) & \xrightarrow{\iota} & E & \xrightarrow{\pi} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & D\operatorname{Tr}(C) & \xrightarrow{i} & X & \xrightarrow{p} & C & \longrightarrow & 0
\end{array}$$

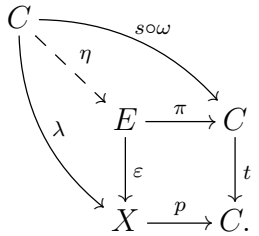
Let $\omega\colon C \to Z$ be a morphism. As s is not a split epimorphism we conclude with Lemma 1.10 that $s \in \mathrm{Rad}_\Lambda(Z,C)$ and $\omega * s = s \circ \omega \in \mathrm{Rad}_\Lambda(C,C) = \mathfrak{J}(\mathrm{End}_\Lambda(C))$. As Γ is local with unique maximal ideal $\mathfrak{J}(\Gamma) = \mathfrak{J}(\mathrm{End}_\Lambda(C))$ the Γ-linear map

$$\varphi\colon \Gamma \to T, x \mapsto x.\bar{t},$$

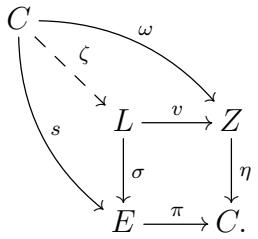
together with the simplicitiy of T implies $\omega * s \in \ker\varphi$. Hence $0 = (\omega * s).\bar{t}$ and we conclude with

$$0 = (\omega \circ s).\bar{t} = (\omega * s).c(t) \overset{c\ \Gamma-\text{linear}}{=} c((\omega * s).t) = c(t \circ s \circ \omega)$$

that $t \circ s \circ \omega \in \ker c = \operatorname{im} p_*$. Hence $t \circ s \circ \omega = p \circ \lambda$ for some $\lambda \in \mathrm{Hom}_\Lambda(C,X)$.
As the right bottom square in Δ is a pullback diagram, we obtain a commutative diagram



Now the pullback diagram in the top right of Δ yields $\zeta\colon C \to L$ and a commutative diagram



□

Hence $v \circ \zeta = \omega$, so every morphism in $\mathrm{Hom}_\Lambda(C, Z)$ factors through v. We now apply 2.18 to the exact sequence

$$0 \to D\,\mathrm{Tr}(C) \xrightarrow{j} L \xrightarrow{v} Z \to 0$$

and conclude that every morphism in $\mathrm{Hom}_\Lambda(D\,\mathrm{Tr}(C), D\,\mathrm{Tr}(C))$ factors through j. Hence the sequence splits, i.e. we find $v'\colon Z \to L$ such that $v \circ v' = \mathrm{id}_Z$. As an upshot we have

$$s = s \circ v \circ v' = \pi \circ (\sigma \circ v').$$

Hence s factors through π. Since C is indecomposable we conclude with 2.14 that $D\,\mathrm{Tr}(C)$ is indecomposable. Hence

$$0 \to D\,\mathrm{Tr}(C) \xrightarrow{\iota} E \xrightarrow{\pi} C \to 0$$

is in fact an almost split sequence endig in C.
(b) As A is indecomposable and non-injective, the module $C := \mathrm{Tr}\,D(A)$ is indecomposable and non-projective. Now (a) yields an almost split sequence

$$0 \to D\,\mathrm{Tr}(C) \to E \to C \to 0.$$

By 2.14 we know that $D\,\mathrm{Tr}(C) \cong \mathrm{Tr} \circ \mathrm{Tr}(A) \cong A$.

2.6 A symmetric definition

We have studied the following morphims:

Definition. Let $f\colon M \to N$ be Λ-linear.
- (a) f is called **right almost split**, if f is not a split epimorphism and any morphism $X \to N$ which is not a split epimorphism factors through f.
- (b) f is called **left almost split**, if f is not a split monomorphism and any morphism $M \to N$ which is not a split monomorphism factors through f.

In other words, a short exact sequence $\delta\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an almost split sequence, provided A is indecomposable and g is right almost split.

Remark. Clearly $f \in \mathrm{Hom}_\Lambda(M, N)$ is right almost split if and only if $D(f) \in \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(DN, DM)$ is left almost split.

Lemma 2.27. *Let $f\colon M \to N$ be Λ-linear.*
- (a) *If f is right almost split, N is indecomposable.*
- (b) *If f is left almost split, M is indecomposbable.*

Proof. See Remark 2.1. □

Theorem 2.28. *Let $\delta\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. The following statements are equivalent:*

(*a*) *δ is an almost split sequence.*

(*b*) *C is indecomposable and for every indecomposable module $X \in \operatorname{mod}\Lambda$ we have*

$$\operatorname{im}(f^*\colon \operatorname{Hom}_\Lambda(B, X) \to \operatorname{Hom}_\Lambda(A, X)) = \operatorname{Rad}_\Lambda(A, X)$$

(*c*) *C is indecomposable and f is left almost split.*

Proof. (*a*) $\implies$ (*b*) As δ is almost split we know by Lemma 2.1 and Theorem 2.26 that $A \cong D\operatorname{Tr}(C)$. We assmue w.l.o.g. $A = D\operatorname{Tr}(C)$. Let X be indecomposable. If X is injective, we know by definition that f^* is surjective, hence $\operatorname{im} f^* = \operatorname{Hom}_\Lambda(X, A)$. As A is not injective, we get $X \ncong A$ and conclude $\operatorname{im} f^* = \operatorname{Rad}_\Lambda(A, X)$.
Now assume that X is not injective and define $Y := \operatorname{Tr} D(X)$ which is indecomposable by 2.14. If $Y \ncong C$, then 2.2 implies $\operatorname{im} g_* = \operatorname{Rad}_\Lambda(Y, C) = \operatorname{Hom}_\Lambda(Y, C)$. Hence g_* is surjective which by Corollary 2.18 is equivalent to the surjectivity of f^* at $D\operatorname{Tr} Y$, i.e.

$$\operatorname{im} f^* = \operatorname{Hom}_\Lambda(A, D\operatorname{Tr} Y) = \operatorname{Hom}_\Lambda(A, X).$$

As $Y \ncong C$, we have $X = D\operatorname{Tr}(Y) \ncong D\operatorname{Tr}(C) = A$. It follows $\operatorname{im} f^* = \operatorname{Rad}_\Lambda(A, X)$.
If $Y = C$, we have $\operatorname{im} g_* = \operatorname{Rad}_\Lambda(C, C) = \mathfrak{J}(\operatorname{End}_\Lambda(C))$ and therefore $\delta_*(Y) = \operatorname{End}_\Lambda(C)/\mathfrak{J}(\operatorname{End}_\Lambda(C))$, which is a finite-dimensional division algebra over k. Since k is algebraically, we get $\delta_*(Y) \cong k$, hence

$$1 = \dim_k \delta_*(Y) \overset{2.17}{=} \dim_k \delta^*(D\operatorname{Tr}(Y)) = \dim_k \delta^*(X).$$

Since $X \cong D\operatorname{Tr}(Y) = D\operatorname{Tr}(C) \cong A$ and $\operatorname{End}_\Lambda(A)$ is local with unique maximal ideal

$$\mathfrak{J}(\operatorname{End}_\Lambda(A)) = \operatorname{Rad}_\Lambda(A, A) \cong \operatorname{Rad}_\Lambda(A, X),$$

we conclude from $\operatorname{im} f^* \subseteq \operatorname{Rad}_\Lambda(A, X)$ (δ does not split) and $\dim_k \delta_*(X) = 1$ that $\operatorname{im} f^* = \operatorname{Rad}_\Lambda(A, X)$.
(*b*) $\implies$ (*c*) Follows as in Lemma 2.2 using the dual arguments.
(*c*) $\implies$ (*a*) We apply the standard duality $D\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$. Now (*c*) implies that

$$0 \to D(C) \xrightarrow{D(g)} D(B) \xrightarrow{D(f)} D(A) \to 0$$

is almost split in $\operatorname{mod}\Lambda^{\mathrm{op}}$ and (*a*) $\implies$ (*b*) and (*b*) $\implies$ (*c*) show that $D(g)$ is left almost split and that $D(A)$ is indecomposable. Therefore $g = D(D(g))$ is right almost split and $C \cong D(D(C))$ indecomposable. □

Definition. Let $f\colon X \to Y$ be Λ-linear.

(*a*) f is called **right minimal** if for every endomorphism $g \in \operatorname{End}_\Lambda(X)$ the equality $f \circ g = f$ implies that g is an automorphism. If f is right minimal and right almost split, we call f **right minimal almost split**.

(*b*) f is called **left minimal** if for every endomorphism $g \in \operatorname{End}_\Lambda(Y)$ the equality $g \circ f = g$ implies that g is an automorphism. If f is left minimal and left almost split, we call f **left minimal almost split**.

Proposition 2.29. *Let $f\colon B \to C$ be an epimorphism that is not a split epimorphism such that* $\ker f$ *is indecomposable. Then f is right minimal.*

Proof. Let $\gamma\colon M \to M$ be Λ-linear such that $f \circ \gamma = f$. We consider the fitting decomposition $M = M_0(\gamma) \oplus M_1(\gamma)$. As $f \circ \gamma^n = f$ for all $n \in \mathbb{N}_0$ we have $M_0(\gamma) = \bigcup_{n\in\mathbb{N}} \ker \gamma^n \subseteq \ker f$. We consider $\lambda\colon \ker f \to \ker f, m \mapsto \gamma(m)$. The assumption that λ is nilpotent yields $M_0(\gamma) = \ker f$. Hence $f_{|M_1(\gamma)}\colon M_1(\gamma) \to M$ is injective and by assumption bijective. This is a contradiction, since f is not a split epimorphism.
As $\mathrm{End}_\Lambda(\ker f)$ is local, we conclude that λ is bijective. We conclude with $\ker \gamma \subseteq M_0(\gamma) \subseteq \ker f$ that $\ker \gamma \subseteq \ker \lambda = 0$, i.e. $\gamma \in \mathrm{End}_\Lambda(M)$ is bijective.

□

Proposition 2.30. *Let $g\colon B \to C$ be right minimal almost split (and therefore C is not projective and indecomposable). The following statements hold.*

(*a*) *g is an epimorphism.*

(*b*) *The short exact sequence $0 \to \ker g \xrightarrow{\iota} B \xrightarrow{g} C \to 0$ has the following properties:*

  (*a*) $\ker g \cong D\,\mathrm{Tr}(C)$

  (*b*) *ι is a left minimal almost split morphism.*

Proof. (*a*) Let (P, ε) be a projective cover of C. As C is not projective, $\varepsilon\colon P \to C$ is not a split epimorphism. Hence $\varepsilon = g \circ h$ for some $h \in \mathrm{Hom}_\Lambda(P, B)$. In particular, $\mathrm{im}\, g \supseteq \mathrm{im}\, \varepsilon = C$.
(*b*) We first show that $\ker g$ is indecomposable. We decompose $\ker g = \bigoplus_{i=1}^n D_i$ into indecomposable direct summands. As $\iota\colon \ker g \to B$ is not a split monomorphism, we conclude with $\mathrm{id}_{\ker g} = \sum_{i=1}^n \iota_i \circ \pi_i$ that some $\pi_i\colon \ker g \to D_i$ does not factor through ι. We set $D := D_i$ and $\pi_D := \pi_i$. The dual result of Corollary 2.23 yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker g & \xrightarrow{\iota} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_D} & & \downarrow{\scriptstyle v} & & \| & & \\ 0 & \longrightarrow & D & \xrightarrow{s} & E & \xrightarrow{t} & C & \longrightarrow & 0 \end{array}$$

such that the left square is a pushout diagram. As π_D does not factor through ι, s is not a section and therefore t not a retraction. As g is a right almost split, every morphism $X \to C$ that is not a split epimorphism factors through t. Hence t is right almost split as well. As D is indecomposable we conclude with 2.29 that t is right minimal almost split. In particular, t is not a retraction and therefore factors through g. Let $\eta\colon E \to B$ such that $t = g \circ \eta$. We conclude

$$g = t \circ v = g \circ \eta \circ v = g \circ (\eta \circ v).$$

As g is right minimal, we conclude that $\eta \circ v$ is an isomorphism. Hence v is injective and since $s \circ \pi_D = v \circ \iota$ is injective, we conclude that π_D is injective. It follows that $\ker g \cong D$ is indecomposable.
We have proven that $A := \ker g$ and C are indecomposable. Since the sequence does not split, we conclude that $0 \to A \to B \to C \to 0$ is an almost split sequence and therefore $A \cong D\,\mathrm{Tr}(C)$ by Lemma 2.1. Moreover, Theorem 2.28 implies that ι is left almost split. The dual result of 2.29 yields that ι is left minimal. □

Theorem 2.31. *The following statements are equivalent for a short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

(a) *The sequence is almost split.*
(b) *f is left almost split and g is right almost split.*
(c) *f is left minimal almost split.*
(d) *g is right minimal almost split.*

Proof. (a) $\Leftrightarrow$ (b) Follows from 2.27 and 2.28.
(b) $\Rightarrow$ (c) This follows from 2.30.
(c) $\Rightarrow$ (b) Follows from 2.27 and 2.28.
(d) $\Rightarrow$ (a) Follows from 2.27 and 2.28.
(a) $\Rightarrow$ (d) Follows using the dual result of 2.30.

□

2.7 Irreducible morphisms

In this section we study the building blocks of right and left minimal almost split morphisms.

Definition. A morphism $f\colon X \to Y$ in $\operatorname{mod}\Lambda$ is called **irreducible** provided:
(a) f is neither a split monomorphism nor a split epimorphim.
(b) If $f = g \circ h$, then h is a split monomorphism or g is a split epimorphism.

Lemma 2.32. *An irreducible morphism is injective or surjective* (*but not bijective*).

Proof. Let $f\colon X \to Y$ be irreducible and assume that $\ker f \neq 0$. We consider the natural decomposition

$$f\colon X \xrightarrow{\pi} X/\ker f \xrightarrow{\overline{f}} Y.$$

As π is not injective, $\overline{f}$ has to be a split epimorphism. Now $\operatorname{im}\overline{f} = \operatorname{im} f$ implies that f is surjective. □

Lemma 2.33. *Let $f\colon M \to N$ be Λ-linear. If f is right minimal almost split or left minimal almost split, then f is irreducible.*

Proof. We assume that f is right minimal almost split. Then f is not a split epimorphism and by 2.27 N is indecomposable. Hence f is not a split monomorphism since otherwise $M \mid N$ would imply $M \cong N$ and it would follow that f is an isomorphism. Now assume $f = h \circ g$ and that $h\colon X \to N$ is not a split epimorphism. We find $l\colon X \to M$ such that $h = f \circ l$ and conclude $f = h \circ g = f \circ l \circ g$. As f is right minimal, $l \circ g\colon M \to M$ is an automorphism. Therefore g is a split monomorphism. □

Corollary 2.34. *Let N be an indecomposable module. There exists a right minimal almost split morphism ending in N and a left minimal almost split morphism starting in N.*

Proof. We only show the existence of right minimal almost split morphisms that end in N. In view of Theorem 2.26 and Proposition 2.29 it remains to prove the existence for $N = P$ projective. We consider the embedding $\iota\colon \operatorname{Rad}_\Lambda(P) \to P$, then clearly f is not a split epimorphism. Let $g\colon X \to P$ be not a split epimorphism, then $\operatorname{im} g \neq P$ since P is projective. Hence $\operatorname{im} g \subseteq \operatorname{Rad}_\Lambda(P)$. Hence g factors through ι. Moreover, $\iota \circ l = \iota$ implies the injectivity of $l \in \operatorname{End}_\Lambda(P)$ which shows that l is an isomorphism.

□

Remark. Observe that if $f\colon N \to X$ and $g\colon M \to X$ are right minimal almost split, then we find and isomorphim $h\colon N \to M$ such that $g \circ h = f$.

Theorem 2.35. (*a*) *Let C be indecomposable. A morphism $g\colon B \to C$ is irreducible if and only if there exists a morphism $g'\colon B' \to C$ such that $(g, g')\colon B \oplus B' \to C$ is right minimal almost split.*

(*b*) *Let A be indecomposable. A morphism $f\colon A \to B$ is irreducible if and only if there exists a morphism $f'\colon C \to B'$ such that $\begin{pmatrix} f \\ f' \end{pmatrix} : C \to B \oplus B'$ is left minimal almost split.*

Proof. (*a*) First we assume that $g\colon B \to C$ is irreducible. By 2.34 there exists a right minimal almost split morphism $h\colon X \to C$ and by definition we find $l\colon B \to X$ such that $g = h \circ l$. Since g is irreducible and h not a split epimorphism, we conclude that l is a split monomorphism. Hence

$$0 \to B \xrightarrow{l} X \xrightarrow{c} \operatorname{coker} l \to 0$$

splits and we find $s\colon \operatorname{coker} l \to X$ such that $c \circ s = \mathrm{id}_{\operatorname{coker} l}$. Hence we have a commutative diagram

$$\begin{array}{ccc} B \oplus \operatorname{coker} l & \xrightarrow{(g, h\circ s)} & C \\ \downarrow{\scriptstyle (l,s)} & & \| \\ X & \xrightarrow{\quad h \quad} & C \end{array}$$

with the vertical morphism being isomorphisms. We conclude that $(g, h \circ s)$ is right minimal almost split.

Now we assume that $l\colon X \to C$ is right minimal almost split and consider a decomposition $X = X' \oplus X$ with $X' \neq 0 \neq X$ and let $g := l \circ \iota_{X'}\colon X' \to C$, where $\iota_{X'}$ denotes the natural embedding. By 1.7 we have $g \in \operatorname{Rad}_\Lambda(X', C)$ and 1.10 implies that g is not a retraction. As C is indecomposable it follows that g is not a split monomorphism.

We consider a decomposition $g = X' \xrightarrow{v} Y \xrightarrow{u} C$ and assume that u is not a split epimorphism. We find $w\colon Y \to X$ such that $u = l \circ w$. We write

$$\mathrm{id}_X = (\iota_{X'}, \iota_{X''}) \circ \begin{pmatrix} \pi_{X'} \\ \pi_{X''} \end{pmatrix}$$

and conclude

$$\begin{aligned} l = l \circ \mathrm{id}_X &= (l \circ \iota_{X'}, l \circ \iota_{X''}) \circ \begin{pmatrix} \pi_{X'} \\ \pi_{X''} \end{pmatrix} = (g, l \circ \iota_{X''}) \circ \begin{pmatrix} \pi_{X'} \\ \pi_{X''} \end{pmatrix} \\ &= (l \circ w \circ v, l \circ \iota_{X''}) \circ \begin{pmatrix} \pi_{X'} \\ \pi_{X''} \end{pmatrix} = l \circ (w \circ v, \iota_{X''}) \circ \begin{pmatrix} \pi_{X'} \\ \pi_{X''}. \end{pmatrix} \end{aligned}$$

As l is right minimal, we conclude that $(w \circ v, \iota_{X''}) \circ \begin{pmatrix} \pi_{X'} \\ \pi_{X''} \end{pmatrix} \in \operatorname{End}_\Lambda(X)$ is an automorphism. As $\begin{pmatrix} \pi_{X'} \\ \pi_{X''} \end{pmatrix}$ is an isomorphism, we conclude that $(w \circ v, \iota_{X''})\colon X' \oplus X'' \to X$ is an isomorphism and therefore also $w \circ v$. We conclude that v is a split monomorphism. □

Definition. Let $X, Y \in \operatorname{mod} \Lambda$ be indecomposable, then $\operatorname{Rad}^2_\Lambda(A, B)$ is the k-vector subspace of $\operatorname{Hom}_\Lambda(X, Y)$ generated by morphisms $g \circ f$ with $f \in \operatorname{Rad}_\Lambda(X, Z), g \in \operatorname{Rad}_\Lambda(Z, Y)$ and Z indecomposable.

Lemma 2.36. *Let X, Y be indecomposable. A morphism $f\colon X \to Y$ is irreducible if and only if $f \in \operatorname{Rad}_\Lambda(X,Y) \setminus \operatorname{Rad}^2_\Lambda(X,Y)$.*

Proof. If $f \in \operatorname{Rad}_\Lambda(X,Y) \setminus \operatorname{Rad}^2_\Lambda(X,Y)$, then f is neither a section nor a retraction. Assume that $f = g \circ h$ with $h\colon X \to Z$, $g\colon Z \to Y$. If g is not a retraction and h not a section, we conclude with 1.10 and its dual result that $g \in \operatorname{Rad}_\Lambda(Z,Y)$ and $h \in \operatorname{Rad}_\Lambda(X,Z)$. We decompose $Z = \bigoplus_{i=1}^n Z_i$ into indecomposable direct summands and conclude with 1.7

$$f = g \circ h = \sum_{i=1}^{n} (g \circ \iota_i) \circ (\pi_i \circ h) \in \operatorname{Rad}^2_\Lambda(X,Y),$$

a contradiction. Hence f is irreducible.
Now we assume that f is irreducible. Cleary this implies $f \in \operatorname{Rad}_\Lambda(X,Y)$. The assumption $f \in \operatorname{Rad}^2_\Lambda(X,Y)$ yields a decomposition $f = \sum_{i=1}^n g_i \circ h_i$ with $h_i \in \operatorname{Rad}_\Lambda(X,Z_i)$ and $g_i \in \operatorname{Rad}_\Lambda(Z_i,Y)$ for indecomposable modules Z_i. We write $Z := \bigoplus_{i=1}^n Z_i$ and consider $f = g \circ h$ with $g = (g_1, \dots, g_n)$ and $h = (h_1, \dots, h_n)$. Now 1.8 implies $g \in \operatorname{Rad}_\Lambda(Z,Y)$ and $h \in \operatorname{Rad}_\Lambda(X,Z)$. We conclude with 1.10 and its dual result that g is not a retraction and h is not a section, a contradiction. □

Definition. Let X, Y be indecomposable. The **space of irreducible morphisms** of X and Y is defined as

$$\operatorname{Irr}_\Lambda(X,Y) := \operatorname{Rad}_\Lambda(X,Y) / \operatorname{Rad}^2_\Lambda(X,Y).$$

Corollary 2.37. *If $g\colon Y \to Z$ is irreducible, Z indecomposable and $l\colon X \to Y$ a split monomorphism, then $g \circ l$ is irreducible.*

Proof. Since g is irreducible we find Y' and $g'\colon Y' \to Z$ such that $(g, g')\colon Y \oplus Y' \to Z$ is right minimal almost split. Since l is a split monomorphism we find $\pi\colon Y \to X$ such that $\pi \circ l = \mathrm{id}_X$. Hence $\varphi\colon X \oplus \ker \pi \oplus Y' \to Y \oplus Y'$, $(x,p,y) \mapsto (l(x)+p, y)$ is an isomorphism with $(g,g') \circ \varphi = (g \circ l, g_{|\ker \pi}, g')$. We conclude that $(g \circ l, g_{|\ker \pi}, g')$ is right minimal almost split. Now Theorem 2.38 implies that $g \circ l$ is irreducible. □

Theorem 2.38. *Let X, Y be Λ-modules and $X = \bigoplus_{i=1}^l X_i^{n_i}$ a decomposition into indecomposable modules $X_1, \dots, X_l$ such that $X_i \ncong X_j$ for $i \neq i$. The following statements hold.*

(a) *If $(g_1, \dots, g_l)^t\colon X \to Y$ is right minimal almost split, then*
  (i) *for all $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, n_i\}$ we have $g_{ij} \in \operatorname{Rad}_\Lambda(X_i, Y)$.*
  (ii) *for all $i \in \{1, \dots, l\}$ the set $\{\overline{g}_{i1}, \dots, \overline{g}_{in_i}\}$ is a k-basis of $\operatorname{Irr}_\Lambda(X_i, Y)$.*
  (iii) *if Z is indecomposable such that $\operatorname{Irr}_\Lambda(Z,Y) \neq 0$, then $Z \cong X_i$ for some $i \in \{1, \dots, l\}$.*

(b) *If morphism $(g_1, \dots, g_l)\colon Y \to X$ is left minimal almost split, then*
  (i) *for all $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, n_i\}$ we have $g_{ij} \in \operatorname{Rad}_\Lambda(Y, X_i)$.*
  (ii) *for all $i \in \{1, \dots, l\}$ the set $\{\overline{g}_{i1}, \dots, \overline{g}_{in_i}\}$ is a k-basis of $\operatorname{Irr}_\Lambda(Y, X_i)$.*
  (iii) *if Z is indecomposable such that $\operatorname{Irr}_\Lambda(Y,Z) \neq 0$, then $Z \cong X_i$ for some $i \in \{1, \dots, l\}$.*

Proof. (a) Assume first that g is right minimal almost split. By 2.35 each g_{ij} is irreducible and hence in $\operatorname{Rad}_\Lambda(X_i, Y)$. If Z is indecomposable and $\operatorname{Irr}_\Lambda(Z,Y) \neq 0$, then we find an irreducible homomorphism $h\colon Z \to Y$ that we can extend by 2.35 to a right minimal almost split morphism $(h, h')\colon Z \oplus Z' \to Y$. As neither (h, h') nor g a split epimorphisms, we get a commutative diagram

$$\begin{array}{ccc} Z \oplus Z' & \xrightarrow{(h,h')} & Y \\ {\scriptstyle u}\uparrow \quad \downarrow{\scriptstyle v} & & \| \\ X & \xrightarrow{g} & Y \end{array}$$

and conclude $g \circ v \circ u = (h, h') \circ u = g$. Since g is right minimal we conclude that $v \circ u$ is an isomorphism. By the same token, we conclude that $u \circ v$ is an isomorphism, hence $X \cong Z \oplus Z'$ and Krull-Remak-Schmidt yields $Z \cong X_i$ for some $i \in \{1, \ldots, l\}$.
Let $\psi\colon X_m \to Y$ be irreducible, then ψ factors through g, i.e. we find $u\colon X_m \to X$ and a commutative diagram

$$\begin{array}{ccc} X_m & \xrightarrow{\psi} & Y \\ & {\scriptstyle u}\searrow \quad \nearrow{\scriptstyle g} & \\ & X. & \end{array}$$

We write $u = [u_1, \ldots, u_l]$ and $u_i = [u_{i1}, \ldots, u_{in_i}]\colon X_m \to X_i$ for each $i \in \{1, \ldots, n_i\}$ with $u_{ij}\colon X_m \to X_i$.
We can write ψ as

$$\psi = g \circ u = \sum_{i=1}^{l} \sum_{j=1}^{n_i} g_{ij} u_{ij}.$$

Now let $i \neq m$, then $u_{ij} \in \operatorname{Hom}_\Lambda(X_m, X_i) = \operatorname{Rad}_\Lambda(X_m, X_i)$ and $g_{ij} \in \operatorname{Rad}_\Lambda(X_i, Y)$ implies that $g_{ij} \circ u_{ij} \in \operatorname{Rad}^2_\Lambda(X_m, Y)$. It follows

$$\overline{\psi} = \sum_{j=1}^{n_m} \overline{g}_{mj} \circ \overline{u}_{mj}.$$

Since k is algebraically closed and $\operatorname{End}_\Lambda(X_m)$ is local with unique maximal ideal $\mathfrak{J}(\operatorname{End}_\Lambda(X_m))$, we have $\operatorname{End}_\Lambda(X_m)/\mathfrak{J}(\operatorname{End}_\Lambda(X_m)) \cong k$. Therefore we find for each $j \in \{1, \ldots, n_m\}$ an element $w_{mj} \in \mathfrak{J}(\operatorname{End}_\Lambda(X_m)) = \operatorname{Rad}_\Lambda(X_m, X_m)$ and $\lambda_j \in k$ such that $u_{mj} = w_{mj} + \lambda_j \circ \mathrm{id}_{X_m}$. It follows

$$\overline{\psi} = \sum_{j=1}^{n_m} \lambda_j \overline{g}_{mj} + \overline{g}_{mj} \circ \overline{w}_{mj} = \sum_{j=1}^{n_m} \lambda_j \overline{g}_{mj}.$$

Hence $\operatorname{Irr}_\Lambda(X_m, Y) = \langle \overline{g}_{m1}, \ldots, \overline{g}_{mn_m} \rangle_k$.
Now assume that $\sum_{j=1}^{n_m} \beta_j \overline{g}_{mj} = 0$ with $\beta_1, \ldots, \beta_{n_m} \in k$, then $f := \sum_{j=1}^{n_m} \beta_j g_{mj} \in \operatorname{Rad}^2_\Lambda(X_m, Y)$. The assumption $\beta_j \neq 0$ yields that $l := [\beta_1, \ldots, \beta_{n_m}]\colon X_m \to X_m^{n_m}, x \mapsto (\beta_1 x, \ldots, \beta_{n_m} x)$ is a split monomorphism. By Corollary 2.37 $f = g_m \circ l$ is irreducible, this is a contradiction since $f \in \operatorname{Rad}^2_\Lambda(X_m, Y)$.

□

Corollary 2.39. *Let X, Y be indecomposable Λ-modules.*

(a) *If X is not injective and $0 \to X \to E \to \operatorname{Tr} D(X) \to 0$ an almost split sequence starting in X, then $\dim_k \operatorname{Irr}_\Lambda(X, Y) = \dim_k \operatorname{Irr}_\Lambda(Y, \operatorname{Tr} D(X))$ and $\dim_k \operatorname{Irr}_\Lambda(X, Y) \neq 0$ if and only if Y is isomorphic to a direct summand of E.*

(b) *If X is not projective and $0 \to D \operatorname{Tr}(X) \to E \to X \to 0$ an almost split sequence starting in X, then $\dim_k \operatorname{Irr}_\Lambda(X, Y) = \dim_k \operatorname{Irr}_\Lambda(D \operatorname{Tr}(X), Y)$ and $\dim_k \operatorname{Irr}_\Lambda(Y, X) \neq 0$ if and only if Y is isomorphic to a direct summand of E.*

Proof. We only prove (b). Let $0 \to D\operatorname{Tr}(X) \xrightarrow{f} E \xrightarrow{g} X \to 0$ be the considered almost split sequence. By Theorem 2.31 f is left minimal almost split and g right minimal almost split. Let $n \in \mathbb{N}_0$ be the maximal element with Y^n is isomorphic to a direct summand of E. In view of Theorem 2.38 we have

$$\dim_k \operatorname{Irr}_\Lambda(Y, X) = n = \dim_k \operatorname{Irr}_\Lambda(D\operatorname{Tr}(X), Y).$$

□

3 The Auslander-Reiten quiver

In this chapter we define the Auslander-Reiten quiver for a finite-dimensional k-algebra and discuss an approach used by Riedtmann to determine the possible Auslander-Reiten quivers for algebras of finite representation type.

3.1 Translation quivers

Definition. A **quiver** Γ is a tuple $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ consisting of sets Γ_0, Γ_1 and maps $s, t\colon \Gamma_1 \to \Gamma_0$ such that for all x in Γ_0 the fibres $s^{-1}(x)$ und $t^{-1}(x)$ are finite.
The elements in Γ_0 are called **vertices** and the elements in Γ_1 are called **arrows**. For $\alpha \in \Gamma_1$ we call $s(\alpha)$ the **start** of α and $t(\alpha)$ the **target** of α. By $\beta\colon x \to y$ we mean the element $\beta \in \Gamma_1$ with $s(\alpha) = x$ and $t(\alpha) = y$.

Definition. For a vertex $x \in \Gamma_0$ we denote with $x^+ := \{t(\alpha) \mid \alpha \in \Gamma_1, s(\alpha) = x\}$ and $x^- := \{s(\alpha) \mid \alpha \in \Gamma_1, t(\alpha) = x\}$ its **direct successors** and **direct predecessors**.

Definition. A **valued translation quiver** is a tuple (Γ, τ, ν) consisting of a quiver Γ and a bijection $\tau\colon \mathcal{P}^c \to \mathcal{I}^c$ of subsets $\mathcal{P}^c, \mathcal{I}^c \subseteq \Gamma_0$ such that the following properties are satisfied.

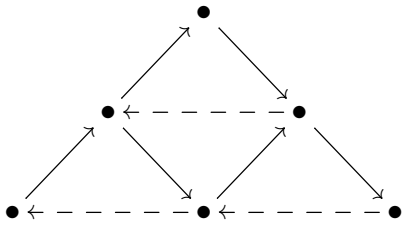
(*a*) $x \notin x^+$ for all $x \in \Gamma_0$ (no loops).
(*b*) For $\alpha \neq \beta \in \Gamma_1$ we have $s(\alpha) \neq s(\beta)$ or $t(\alpha) \neq t(\beta)$ (no multiple arrows).
(*c*) For all $x \in \mathcal{P}^c$ we have $(\tau(x))^+ = x^-$.
(*d*) For each arrow $\alpha\colon x \to y$ such that $y \in \mathcal{P}^c$ with $\nu(\alpha) = (a, b)$ we have $\nu(\tau(y) \to x) = (b, a)$.

A **translation quiver** is a given by the tuple (Γ, τ) satisfying $(a), (b)$ and (c). We call the vertices in $\Gamma_0 \setminus \mathcal{P}^c$ **projective** and the vertices in $\Gamma_0 \setminus \mathcal{I}^c$ **injective**. The translation quiver (Γ, τ) is called **stable** provided $\mathcal{I}^c = \mathcal{P}^c = \Gamma_0$.

Examples. The following illustrations show translation quivers. The translation τ is indicated by broken arrows.

(*a*)

$\Gamma_1 :=$



(*b*)

$\Gamma_2 :=$ 

Definition. Let Λ be a finite-dimensional k-algebra over an algebraically closed field. We define a quiver Γ_Λ by the following data.

- The vertices are given by the isomorphism classes of indecomposable Λ-modules, i.e. $(\Gamma_\Lambda)_0 = \operatorname{ind}\Lambda$.
- There exists an arrow $[M] \to [N]$ if and only if $\operatorname{Irr}_\Lambda(M,N) \neq 0$.

Furthermore, we set

$$\mathcal{P}^c_\Lambda := \{[M] \in \operatorname{ind}\Lambda \mid M \text{ not projective}\} \subseteq (\Gamma_\Lambda)_0,$$

$$\mathcal{I}^c_\Lambda := \{[M] \in \operatorname{ind}\Lambda \mid M \text{ not injective}\} \subseteq (\Gamma_\Lambda)_0$$

and define a map $\tau_\Lambda \colon \mathcal{P}^c \to \mathcal{I}^c$ via $\tau_\Lambda([M]) := [D\operatorname{Tr}(M)]$. Note that this is well-defined by 2.14. Moreover, we define $\nu_\Lambda \colon (\Gamma_\Lambda)_1 \to \mathbb{N}\times\mathbb{N}, (\alpha \colon [M] \to [N]) \mapsto (\dim_k \operatorname{Irr}_\Lambda(M,N), \dim_k \operatorname{Irr}_\Lambda(M,N))$.

Theorem 3.1. *The tuple $(\Gamma_\Lambda, \tau_\Lambda, \nu_\Lambda)$ is a valued translation quiver with $\nu_\Lambda(\tau_\Lambda(x)) = \nu_\Lambda(x)$ for all $x \in \mathcal{P}^c$. Under abuse of notation we write Γ_Λ for $(\Gamma_\Lambda, \tau_\Lambda, \nu_\Lambda)$.*

Proof. This follows immediately from 2.14, 2.39. □

Remark. The definition of ν_Λ seems strange. In the general case of artinian algebras the definition of ν_Λ is way more subtle since $\dim_k$ does not make any sense. Also ν is not of the form $\nu_\Lambda(x) = (a_x, a_x)$ for all $x \in (\Gamma_\Lambda)_1$.

3.2 Finite representation type

Recall that for a finite-dimensional k-algebra there exists a unique decomposition

$$\Lambda = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$$

into indecomposable two-sided ideals and for each $i \in \{1,\dots,n\}$ there exists a central idempotent e_i such that $\mathcal{B}_i = \Lambda e_i$. These idempotents satisfy $e_i e_j = 0$ for all $i \neq j$ and $1_\Lambda = e_1 + \cdots + e_n$. Let $M \in \operatorname{mod}\Lambda$ be indecomposable. Then $M = 1_\Lambda.M = e_1.M \oplus \cdots \oplus e_n.M$ as Λ-modules. Hence there exists a unique $i_0 \in \{1,\dots,n\}$ such that $e_{i_0}.M = M$. We say that M **belongs to the block** $\mathcal{B}_{i_0}$. Moreover, we have $\operatorname{Hom}_\Lambda(M,N) = 0$ if M and N belong to different blocks $\mathcal{B}_i$ and $\mathcal{B}_j$: For $f \colon e_i.M \to e_j.N$ and $m \in M$ we find m' such that $e_i.m' = m$ and $n \in N$ such that $f(e_i.m') = e_j.n$. We conclude

$$f(m) = f(e_i.m') = e_i f(m') = e_i(e_j.n) = 0.$$

In particular, the indecomposable modules in a "component" of the Auslander-Reiten quiver belong to the same block.
Aim of this section is to prove that if Λ is of finite representation type each Auslander-Reiten component contains all the modules belonging to the same block. Clearly it sufficies to show that if Λ is of finite representation type and **connected** (i.e. n = 1) the Auslander-Reiten quiver is connected as a graph.

Definition. Let S and T be simple Λ-modules. We say that S and T are **linked** if there exists a finite sequence $S = S_1, \ldots, S_n = T$ of simple modules such that for each $1 \leq i < n$ we find a simple module U such that

$$\mathrm{Hom}_\Lambda(P(U), P(S_i)) \neq 0 \neq \mathrm{Hom}_\Lambda(P(U), P(S_{i+1})),$$

for its projective cover $\varepsilon_U \colon P(U) \to U$. In other words, $P(S_i)$ and $P(S_{i+1})$ have the common composition factor U.

Theorem 3.2. (*see* [4, II.3.4]) *The following statements are equivalent.*
(*a*) Λ *is connected.*
(*b*) *Any two simple modules are linked.*

Corollary 3.3. *Let* Λ *be a finite-dimensional and connected and* $\operatorname{ind}\Lambda = I \dot\cup J$ *be a disjoint union into non-empty sets* I *and* J*. There exist* $[X] \in I$ *and* $[Y] \in J$ *such that* $\mathrm{Hom}_\Lambda(X, Y) \neq 0$ *or* $\mathrm{Hom}_\Lambda(Y, X) = 0$*, i.e.* X *and* Y *are not* Hom-***orthogonal***.

Proof. Assume to the contrary that I and J are Hom-orthogonal. Fix $[A] \in I$ and $[B] \in J$. Take simple submodules $S_A \subseteq A$ and $S_B \subseteq B$. Since S_A and S_B are linked, we find a sequence $S_A = S_1, \ldots, S_n = S_B$ of simple modules and simple modules $U_1, \ldots, U_{n-1}$ such that in the sequence

$$S_A = S_1, P(S_1), P(U_1), P(S_2), P(U_2), \ldots, P(U_{n-2}), P(S_{n-1}), P(U_{n-1}), P(S_n), S_n = S_B$$

each consecutive elements are not Hom-orthogonal. But this is a contradiction since projective covers of simple modules are indecomposable and therefore the isomorphism class of each element in the sequence belongs to I or J. □

Proposition 3.4. *Let* Λ *be of finite representation type. Let* X, Y *be indecomposable* Λ*-modules, then* $\dim_k \mathrm{Irr}_\Lambda(X, Y) \leq 1$.

Proof. Assume that $l := \dim_k \mathrm{Irr}_\Lambda(X, Y) \geq 2$ and let $f \colon X \to Y$ be irreducible. Since f is injective or surjective but not bijective we conclude $\dim_k X \neq \dim_k Y$. We assume that $\dim_k X > \dim_k Y$, the other case follows by dual arguments. Now f is an epimorphism and Y not projective since otherwise f would be a split epimorphism. Hence there exists an almost split sequence

$$0 \to D\,\mathrm{Tr}(Y) \to X^l \oplus E \to Y \to 0.$$

We conclude

$$\dim_k D\,\mathrm{Tr}(Y) = l \dim_k X + \dim_k E - \dim_k Y > (l-1)\dim_k X \geq \dim_k X.$$

In view of 2.39 we also have $\dim_k \mathrm{Irr}_\Lambda(D\,\mathrm{Tr}(Y), X) = l > 1$. We conclude that X is not projective and repeat the argument to get $\dim_k D\,\mathrm{Tr}(X) > \dim_k D\,\mathrm{Tr}(Y)$, $\dim_k \mathrm{Irr}_\Lambda(D\,\mathrm{Tr}(X), D\,\mathrm{Tr}(Y)) =$

l and the non-projectivity of $D\operatorname{Tr}(Y)$. Repeating this argument yields an infinite ascending sequence

$$\dim_k Y < \dim_k X < \dim_k D\operatorname{Tr}(Y) < \dim_k D\operatorname{Tr}(X) < \dim_k (D\operatorname{Tr})^2(Y) < \dim_k (D\operatorname{Tr})^2(X) < \cdots .$$

But this is a contradiction since $(D\operatorname{Tr})^i(Y)$ and $(D\operatorname{Tr})^i(X)$ are indecomposable and Λ of finite representation type. □

Remark. This result shows us that the translation quiver Γ_Λ encodes already all information about an algebra Λ of finite representation type, since $\operatorname{im}\nu_\Lambda = \{(1,1)\}$.

Theorem 3.5 (Auslander's Theorem, see [3])**.** *If Λ is connected and of finite representation type, then the underlying graph $\overline{\Gamma}_\Lambda$ of Γ_Λ is connected.*

Proof. Let $Q \subseteq \Gamma_\Lambda$ be such that $\overline{Q}$ is a connected component of $\overline{\Gamma}_\Lambda$. We assume that $Q_0 \neq (\Gamma_\Lambda)_0$, otherwise there is nothing to show. In view of 3.3 we find $[A] \in \operatorname{ind}\Lambda \setminus Q_0$ such that $\operatorname{Hom}_\Lambda(A,B) \neq 0$ or $\operatorname{Hom}_\Lambda(B,A) \neq 0$ for some $[B] \in Q_0$.
Hence we find a component $\Gamma \subseteq \overline{\Gamma}_\Lambda$ and a vertex $[M] \in \operatorname{ind}\Lambda \setminus \Gamma_0$ such that $\operatorname{Hom}_\Lambda(M,G) \neq 0$ for some $[G] \in \Gamma_0$.

We consider the $\mathbb{C}$-vector space V with basis Γ_0 and define a $\mathbb{C}$-bilinear form on V by setting

$$\langle [X],[Y]\rangle := \dim_k \operatorname{Hom}_\Lambda(X,Y)$$

for all $[X],[Y] \in \Gamma_0$. Moreover we set

$$S_Y := \begin{cases} [Y] + [D\operatorname{Tr}(Y)] - \sum_{[N]\in[Y]^-}[N], & Y \text{ not projective} \\ [Y] - \sum_{[N]\in[Y]^-}[N], & Y \text{ projective.} \end{cases}$$

If Y is not projective, then 2.26, 2.38 and 3.4 yield an Auslander-Reiten sequence

$$0 \to D\operatorname{Tr}(Y) \to E_Y \to Y \to 0,$$

where $E := \bigoplus_{[N]\in[Y]^-}[N]$. We conclude in conjunction with 2.3 for $[X] \in \Gamma_0$ that

$$0 \to \operatorname{Hom}_\Lambda(X, D\operatorname{Tr}(Y)) \to \operatorname{Hom}_\Lambda(X,E_Y) \to \operatorname{Rad}_\Lambda(X,Y) \to 0$$

is exact. We get

$$\begin{aligned}\langle [X],S_Y\rangle &= \dim_k \operatorname{Hom}_\Lambda(X,Y) + \dim_k \operatorname{Hom}_\Lambda(X,D\operatorname{Tr}(Y)) - \dim_k \operatorname{Hom}_\Lambda(X,E_Y) \\ &= \dim_k \operatorname{Hom}_\Lambda(X,Y) - \dim_k \operatorname{Rad}_\Lambda(X,Y).\end{aligned}$$

Hence $\langle [X],S_Y\rangle = 0$ if and only if $X \ncong Y$.
Now assume that Y is projective. Then 2.34 implies that $\operatorname{Rad}_\Lambda(Y) = \bigoplus_{[N]\in[Y]^-} N$. Let $[X] \in \Gamma_0$ and $f\colon X \to Y$ be Λ-linear. As $Y/\operatorname{Rad}_\Lambda(Y)$ is simple, we conclude that $\operatorname{im} f \not\subseteq \operatorname{Rad}_\Lambda(Y)$ implies $\operatorname{im} f = Y$ and therefore $X \cong Y$, as X is indecomposable. Hence

$$\langle [X],S_Y\rangle = \dim_k \operatorname{Hom}_\Lambda(X,Y) - \dim_k \operatorname{Hom}_\Lambda(X,\operatorname{Rad}_\Lambda(Y))$$

implies $\langle [X],S_Y\rangle \neq 0$ if and only if $X \cong Y$. We summarize our finding so far:

Let X, Y be indecomposable Λ-modules such that $[X], [Y] \in \Gamma_0$, then

$$(*) \qquad \langle [X], S_Y \rangle \neq 0 \Leftrightarrow X \cong Y.$$

Now let $v \in V \setminus \{0\}$, we write $v = \sum_{[Z]\in\Gamma_0} \lambda_{[Z]}[Z]$ for uniquely determined $\lambda_{[Z]} \in \mathbb{C}$. Since $v \neq 0$ we find $[Z_0] \in \Gamma_0$ such that $\lambda_{[Z_0]} \neq 0$. It follows

$$\langle v, S_{[Z_0]} \rangle = \sum_{[Z]\in\Gamma_0} \lambda_{[Z]} \langle [Z], S_{[Z_0]} \rangle \overset{(*)}{=} \lambda_{[Z_0]} \langle [Z_0], S_{[Z_0]} \rangle \overset{(*)}{\neq} 0.$$

Hence $\langle\ ,\ \rangle \colon V \times V \to \mathbb{C}$ defines a non-degenerate $\mathbb{C}$-bilinear form.

Now fix $[M] \in \operatorname{ind} \Lambda \setminus \Gamma_0$ such that $\operatorname{Hom}_\Lambda(M, G) \neq 0$. We consider the k-linear map

$$\Phi \colon \mathbb{C}^{\Gamma_0} \to \mathbb{C}^{\Gamma_0}, \lambda \mapsto (\langle \sum_{[Z]\in\Gamma_0} \lambda_{[Z]}[Z], [N] \rangle)_{[N]\in\Gamma_0}.$$

Since $\langle\ ,\ \rangle$ is non-degenerate, we conclude that Φ is injective and therefore surjective. Hence we find λ such that

$$(\langle \sum_{[Z]\in\Gamma_0} \lambda_{[Z]}[Z], [N] \rangle)_{[N]\in\Gamma_0} = \Phi(\lambda) = (\dim_k \operatorname{Hom}_\Lambda(M, N))_{[N]\in\Gamma_0}.$$

Since $\operatorname{Hom}_\Lambda(M, G) \neq 0$, we conclude that $\lambda \neq 0$. Hence we find $[Y] \in \Gamma_0$ such that $\lambda_{[Y]} \neq 0$. In view of $(*)$ we conclude

$$\langle \sum_{[Z]\in\Gamma_0} \lambda_{[Z]}[Z], S_Y \rangle = \lambda_{[Y]} \langle [Y], S_Y \rangle \neq 0.$$

If Y is not projective, the almost split sequence

$$0 \to D \operatorname{Tr}(Y) \to E_Y \to Y \to 0$$

implies for $L := \sum_{[Z]\in\Gamma_0} \lambda_{[Z]}[Z]$ that

$$\begin{aligned} 0 \neq \langle L, S_Y \rangle &= \langle L, [D \operatorname{Tr}(Y)] \rangle + \langle L, [Y] \rangle - \sum_{N\in[Y]^-} \langle L, [N] \rangle \\ &= \dim_k \operatorname{Hom}_\Lambda(M, D \operatorname{Tr}(Y)) + \dim_k \operatorname{Hom}_\Lambda(M, Y) - \sum_{N\in[Y]^-} \dim_k \operatorname{Hom}_\Lambda(M, N) \\ &= \langle [M], S_Y \rangle. \end{aligned}$$

Now $(*)$ yields $M \cong Y$, a contradiction. Hence Y is projective. We get

$$\begin{aligned} 0 \neq \langle L, S_Y \rangle &= \langle L, [Y] \rangle - \sum_{[N]\in[Y]^-} \langle L, [N] \rangle \\ &= \dim_k \operatorname{Hom}_\Lambda(M, Y) - \sum_{[N]\in[Y]^-} \dim_k \operatorname{Hom}_\Lambda(M, N) \\ &= \dim_k \operatorname{Hom}_\Lambda(M, Y) - \dim_k \operatorname{Hom}_\Lambda(M, \operatorname{Rad}_\Lambda(Y)) = \langle [M], S_Y \rangle, \end{aligned}$$

and conclude $M \cong Y$, a contradiction. $\square$

3.3 Stable translation quiver

Definition. Let Γ be a translation quiver. A vertex is called **stable** if $\tau^n(x)$ is defined for all $n \in \mathbb{Z}$. The **stable translation quiver** Γ^s is obtained from Γ by deleting all non-stable vertices and all arrows ending or starting in these vertices.

Definition. The **stable Auslander-Reiten quiver** Γ_Λ^s is obtained from Γ_Λ by deleting all non-stable vertices and all arrows ending or starting in these vertices.

Examples. We continue with the examples Γ_1, Γ_2. Then $(\Gamma_1)^s$ is the empty quiver whereas $(\Gamma_2)^s$ is given by



The stable Auslander-Reiten quiver is of special interest since the translation is defined on every vertex. We have seen that the structure of Γ_Λ^s might be completely different from the structure of Γ_Λ (in fact Γ_1 is the Auslander-Reiten quiver of the upper triangular matrices of dimension 3). However, for selfinjective algebras the quivers Γ_Λ^s and Γ^s look "similiar":

Definition. A k-algebra Λ is called **self-injective** if Λ is an injective Λ-module.

Recall that there are only finitely many (up to isomorphism) simple Λ-modules $S_1, \dots, S_n$ and a projective/injective module is indecomposable if and only if it is the projective cover/injective hull of some S_i. In particular, the number of indecomposable projective/injective modules (up to isomorphism) is also n. Hence we have the equivalence

$$\Lambda \text{ is self-injective } \Leftrightarrow (\forall X \in \operatorname{mod}\Lambda\colon\ X \text{ is projective} \Leftrightarrow\ X \text{ is injective}).$$

Lemma 3.6. *Let Λ be a self-injective algebra. A vertex $[M]$ in Γ_Λ is stable if and only if M is not projective.*

Proof. Cleary every projective module is not stable. Now let $[M]$ be a vertex such that M is not projective and assume that $\tau_\Lambda^n([M])$ is not defined for some $n \in \mathbb{Z}$. Then $n \neq 0$. If $n > 0$, we find $l \in \mathbb{N}_0$ maximal such that $\tau_\Lambda^l([M]) =: [N]$ is still defined. Then N is not projective and we have an almost split sequence $0 \to D\operatorname{Tr}(N) \to E \to N \to 0$. In particular, $D\operatorname{Tr}(N)$ is not injective. Since Λ is self-injective, the module $D\operatorname{Tr}(N)$ is also not projective, a contradiction. If $n < 0$, consider $l \in \mathbb{N}_0$ maximal such that $\tau_\Lambda^{-l}([M]) =: [N]$ is still defined and conclude in the same fashion that $\operatorname{Tr} D(N)$ is not injective. □

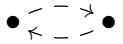
Definition. Let (Q, τ_Q) and (P, τ_P) be stable translation quivers.

(*a*) A **morphism** $\sigma\colon Q \to P$ **of stable translation quivers** is a quiver morphism such that $\sigma \circ \tau_Q = \tau_P \circ \sigma$.

(*b*) (Q, τ_Q) is a **connected translation quiver** if (Q, τ_Q) can not be written as a union of two stable subquivers.

(*c*) A **directed tree** T is a quiver such that $\overline{T}$ is a tree.

Remark. Clearly $\overline{Q}$ being connected implies that (Q, τ_Q) is connected but the stable translation quiver

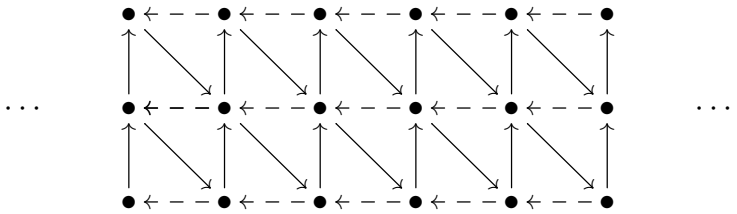


is connected and the quiver • • is not connected.

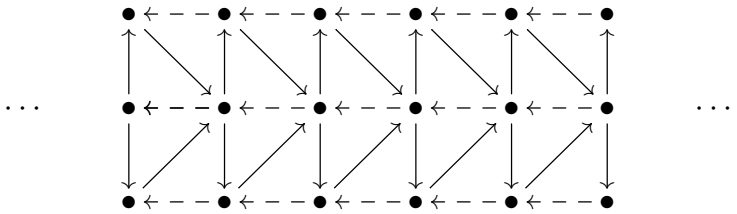
Definition. Let T be a directed tree. We define a stable translation quiver $\mathbb{Z}T$ in the following way:

- $(\mathbb{Z}T)_0 := \mathbb{Z} \times T_0$
- $(\mathbb{Z}T)_1 := \{(n, v) \to (n, w) \mid v \to w \in T_1, n \in \mathbb{Z}\} \cup \{(n, w) \to (n + 1, v) \mid v \to w \in T_1, n \in \mathbb{Z}\}$
- $\tau_T \colon \mathbb{Z}T \to \mathbb{Z}T, \tau_T(n, v) := (n - 1, v)$.

Examples. (*a*) Let $T_1 = 1 \to 2 \to 3$, then $\mathbb{Z}T_1$ looks as follows:



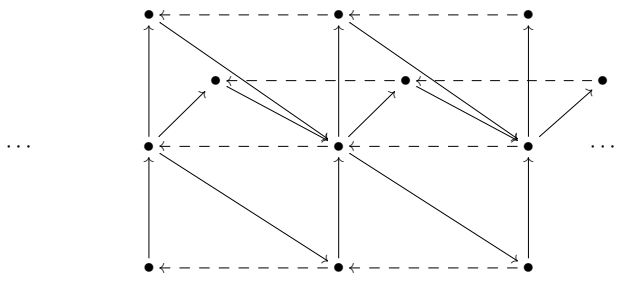
(*b*) Let $T_2 = 1 \leftarrow 2 \to 3$, then $\mathbb{Z}T_2$ looks as follows:



(*c*) Let

$$T_3 = \quad \begin{array}{ccc} & & 3 \\ & & \uparrow \\ 2 & \longrightarrow & 4, \\ \uparrow & & \\ 1 & & \end{array}$$

then $\mathbb{Z}T_3$ looks as follows:



(*d*) The quiver morphism $\sigma\colon \mathbb{Z}T_1 \to \mathbb{Z}T_2$ given by

$$\sigma((n,x)) = \begin{cases}(n,x), x \neq 1\\(n-1,x), x = 1\end{cases}$$

is an isomorphism of translation quivers. We will prove that this is not a coincidence.

Definition. Let (Q,τ_Q) be a stable translation quiver. The translation τ_Q acts in a natural way on the vertices and arrows of Q. The orbit graph $Q/\langle\tau_Q\rangle$ is the graph given by the following data:

- $(Q/\langle\tau_Q\rangle)_0 := \Gamma_0/\langle\tau_Q\rangle$.
- $[x]-[y] \Leftrightarrow \exists x_0 \in [x]\colon x_0 \in y^- \cup y^+ \Leftrightarrow \exists y_0 \in [y]\colon y_0 \in x^- \cup x^+$.

Lemma 3.7. *Let T be a directed tree and (Q,τ_Q) a stable translation quiver. For every quiver morphism $\Phi\colon T \to Q$ and every $n \in \mathbb{Z}$ there exists a unique morphism $\Phi_n\colon \mathbb{Z}T \to Q$ such that $\Phi_n(n,x) = \Phi(x)$ for all $x \in T_0$.*

Proof. Assume that Ψ is such a quiver morphism and let (m,x) be a vertex. Since $(m,x) = \tau_T^{n-m}(n,x)$, we conclude

$$\Psi(m,x) = \Psi(\tau_T^{n-m}(n,x)) = \tau_Q^{n-m}\Psi(n,x) = \tau_Q^{n-m}\Phi(x),$$

for all x and all $m \in \mathbb{Z}$. Hence we have to define $\Phi_n(m,x) := \tau_Q^{n-m}\Phi(x)$. By definition $\tau_Q \circ \Phi_n = \Phi_n \circ \tau_T$. It remains to show that for each arrow $(a,x) \to (b,y) \in (\mathbb{Z}T)_1$ we have $\Phi_n(a,x) \in \Phi_n(b,y)^-$. We have to consider two cases. If $x \to y \in T_1$, then $a = b$. Since $\Phi(x) \in \Phi(y)^-$ and $\tau_Q^{n-a} = \tau_Q^{n-b}$ is a quiver morphism, we conclude

$$\Phi_n(a,x) = \tau_Q^{n-a}\Phi(x) \in (\tau_Q^{n-b}\Phi(y))^- = (\Phi_n(b,y))^-.$$

If $y \to x \in T_1$, then $b = a+1$. We conclude

$$\Phi_n(b,y) = \tau_Q^{n-a-1}\Phi(y) \in (\tau_Q^{n-a-1}\Phi(x))^- = (\tau_Q^{n-a}\Phi(x))^+ = (\Phi_n(a,x))^+.$$

□

Proposition 3.8. *Let S,T be directed trees. The following statements are equivalent.*
(*a*) *The stable translation quivers $(\mathbb{Z}S,\tau_S)$, $(\mathbb{Z}T,\tau_T)$ are isomorphic.*
(*b*) *$\overline{S}$ and $\overline{T}$ are isomorphic graphs.*

Proof. Let $\Phi\colon \mathbb{Z}S \to \mathbb{Z}T$ be an isomorphism of stable translation quivers. As $\tau_T \circ \Phi = \Phi \circ \tau_S$, the map $\varphi\colon (\mathbb{Z}S)/\langle\tau_S\rangle \to (\mathbb{Z}T)/\langle\tau_T\rangle$, $[x] \mapsto [\Phi(x)]$, $[x]-[y] \mapsto [\Phi(x)]-[\Phi(y)]$ is a well-defined isomorphism of graphs. Since $\mathbb{Z}L/\langle\tau_L\rangle \cong L$ as graphs for each directed tree L, we conclude $\overline{S} \cong \overline{T}$.
Now assume that $\varphi\colon \overline{S} \to \overline{T}$ is an isomorphism of graphs. We consider the set

$$\mathcal{A} := \{(\psi,U) \mid U \subseteq S \text{ directed tree}, \psi\colon U \to \mathbb{Z}T \text{ morphism s.t. } \pi_T \circ \psi = \varphi_{|U}\},$$

where $\pi_T\colon \mathbb{Z}T \to T$ is the canonical quiver morphism. We define a partial order $\leq$ on $\mathcal{A}$ by defining

$$(\psi_1,U_1) \leq (\psi_2,U_2) :\Leftrightarrow U_1 \subseteq U_2 \text{ and } \Psi_{2|U_1} = \Psi_1.$$

Let $s \in S_0$ and define $\psi_s \colon \{s\} \to \mathbb{Z}T, s \mapsto (0, \varphi(s))$. Hence $(\psi_s, \{s\}) \in \mathcal{A}$ and standard arguments in conjunction with Zorn's Lemma yield a maximal element $(\psi, U) \in \mathcal{A}$.
We assume that $U \neq S$. Since S is a directed tree, there exists $s_0 \in S_0 \setminus U_0$ and $u_0 \in U_0$ such that $s_0 \to u_0 \in S_1$ or $u_0 \to s_0 \in S_1$. We assume that $u_0 \to s_0$, the case $s_0 \to u_0$ follows in the same fashion.
We consider the subquiver $V \subseteq S$ with vertex set $V_0 = U_0 \cup \{s_0\}$. Since $\psi(u_0) \in \pi_T^{-1}(\varphi(u_0))$, we find $n \in \mathbb{Z}$ such that $\psi(u_0) = (n, \varphi(u_0))$. Since φ is a morphism of graphs, we have $\varphi(s_0) \to \varphi(u_0) \in T_1$ or $\varphi(u_0) \to \varphi(s_0) \in T_1$.
We define a morphism $\eta \colon V \to \mathbb{Z}T$ via $\eta_{|U} = \psi$ and

$$\eta(s_0) := \begin{cases} (n, \varphi(s_0)), \varphi(u_0) \to \varphi(s_0) \in T_1, \\ (n+1, \varphi(s_0)), \varphi(s_0) \to \varphi(u_0) \in T_1. \end{cases}$$

This definition is well-defined: On the one hand, $\varphi(u_0) \to \varphi(s_0) \in T_1$ implies $(n, \varphi(u_0)) \to (n, \varphi(s_0)) \in (\mathbb{Z}T)_1$. On the other hand, $\varphi(s_0) \to \varphi(u_0) \in T_1$ implies $(n, \varphi(u_0)) \to (n+1, \varphi(s_0)) \in (\mathbb{Z}T)_1$.
By construction we have $(V, \eta) \in \mathcal{A}$ and $(U, \psi) < (V, \eta)$, a contradiction. Hence $U = S$ and 3.7 yields a uniquely determined morphism of translation quivers $\Phi \colon \mathbb{Z}S \to \mathbb{Z}T$ such that $\Phi(0, x) = \psi(x)$ for all $x \in S_0$. We conclude $\Phi(m, x) = \tau_T^{-m}\Phi(0, x) = \tau_T^{-m}\psi(x)$ for all $m \in \mathbb{Z}, x \in S_0$. Since $\pi_T \circ \psi = \varphi$, we conclude that for each $x \in S_0$ we find $n_x \in \mathbb{Z}$ such that $\psi(x) = (n_x, \varphi(x))$. It follows

$$\Phi(m, x) = \tau_T^{-m}\Phi(0, x) = \tau_T^{-m}\psi(x) = \tau_T^{-m}(n_x, \varphi(x)) = (m + n_x, \varphi(x))$$

for all $m \in \mathbb{Z}, x \in S_0$. Hence Φ is an isomorphism with invserve $(m, \varphi(x)) \mapsto (m - n_x, x)$ for all $m \in \mathbb{Z}, x \in S_0$. □

3.4 The Riedtmann Structure Theorem

Definition. Let (Q, τ_Q) be a stable translation quiver.
(*a*) A subgroup $\Pi \subseteq \mathrm{Aut}(Q, \tau_Q)$ is called **admissable** if

$$|\Pi y \cap (\{x\} \cup x^+)| \leq 1 \text{ and } |\Pi y \cap (\{x\} \cup x^-)| \leq 1$$

for all $x, y \in Q_0$.
(*b*) If $\Pi \subseteq \mathrm{Aut}(Q, \tau_Q)$ is an admissable subgroup, we define Q/Π to be the translation quiver given by the following data:

- $(Q/\Pi)_0 := Q_0/\Pi$.
- $(Q/\Pi)_1 := Q_1/\Pi$.
- $\tau_{Q/\Pi}([x]) := [\tau_Q(x)]$

Definition. A morphism of translation quivers $\Phi \colon (Q, \tau_Q) \to (P, \tau_P)$ is called **covering** if
(*a*) $\Phi \colon Q_0 \to \Phi_0$ is surjective, and
(*b*) for all $x \in Q_0$ the maps $\Phi_{|x^+} \colon x^+ \to \Phi(x)^+$ and $\Phi_{|x^-} \colon x^- \to \Phi(x)^-$ are bijective.

Example. Let $\Pi \subseteq \mathrm{Aut}(Q, \tau_Q)$ be an admissable subgroup, then the canoncial projection $\pi \colon Q \to Q/\Pi$ is a covering.

Definition. Let T be a directed tree, $\Phi\colon \mathbb{Z}T \to Q$ a covering (of stable translation quivers). The group $\Pi_\Phi := \{\varphi \in \mathrm{Aut}(\mathbb{Z}T, \tau_T) \mid \Phi\circ\varphi = \Phi\}$ is called the **group of deck transformations**.

Proposition 3.9. *Π_Φ is an admissable group.*

Proof. We have to show that

$$|\Pi_\Phi u \cap (\{v\} \cup v^+)| \leq 1 \text{ and } |\Pi_\Phi u \cap (\{v\} \cup v^-)| \leq 1 \text{ for all } u, v \in (\mathbb{Z}T)_0.$$

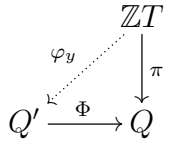
Let $u, v \in (\mathbb{Z}T)_0$ and assume there are $a, b \in \Pi_\Phi u \cap v^+$. Let $\psi, \psi' \in \Pi_\Phi$ such that $a = \psi(u)$, $b = \psi'(u)$. Then

$$\Phi(a) = \Phi(\psi(u)) = \Phi(u) = \Phi(\psi'(u)) = \Phi(b).$$

As $a, b \in v^+$ and Φ is locally injective we conclude $a = b$. Hence $|\Pi_\Phi u \cap v^+| \leq 1$.
Now assume that $|\Pi_\Phi u \cap v^+| = 1$ and $\Pi_\Phi u \cap \{v\} \neq \varnothing$. Hence we find $\psi, \psi' \in \Pi_\Phi$ such that $\psi(u) \in v^+$ and $\psi'(u) = v$. But this implies $\Phi(v) = \Phi(\psi'(u)) = \Phi(u) = \Phi(\psi(u))$ and therefore the arrow $v \to \psi(u)$ is mapped to a loop, a contradiction. By same token, one can show that $|\Pi_\Phi(u) \cap (\{v\} \cup v^-)| \leq 1$.

□

Proposition 3.10. *Let T be a directed tree, $\pi\colon \mathbb{Z}T \to Q$ a morphism of stable translation quivers, $\Phi\colon Q' \to Q$ a covering and $t = (n, \bar{t}) \in (\mathbb{Z}T)_0$. For every vertex $y \in Q'_0$ such that $\Phi(y) = \pi(t)$ there exists a unique morphism $\varphi_y\colon \mathbb{Z}T \to Q'$ such that $\Phi \circ \varphi_y = \pi$ and $\varphi_y(t) = y$.*



Proof. We denote with $L \subseteq \mathbb{Z}T$ the directed tree with vertex set $\{n\} \times T_0$. Now we define a (uniquely determined) morphism $\varphi\colon L \to Q'$ of quivers s.t. $\Phi \circ \varphi = \pi_{|L}$ and $\varphi(t) = y$ by induction on the distance $d(v, t) \in \mathbb{N}_0$ of $v \in L_0$ to t. At first we set $\varphi(t) := y$. Now let $v \in L_0$ with $d(v, t) > 0$, $w \in L_0$ be the unique vertex in L_0 with $d(w, t) = d(v, t) - 1$ and $d(v, w) = 1$. By induction $\varphi(w)$ is already defined and $\Phi(\varphi(w)) = \pi(w)$. We assume without loss of generality that $w \to v \in L_1$. As π is a quiver morphism we have $\pi(w) \to \pi(v) \in Q_1$. Since Φ is a covering, there exists a unique arrow $\alpha\colon \varphi(w) \to z \in Q'_1$ s.t. $\Phi(\alpha) = \pi(w) \to \pi(v)$. We set $\varphi(v) := z$.
Now 3.7 yields a unique morphism $\overline{\varphi}\colon \mathbb{Z}L \to Q'$ of stable translation quivers such that $\overline{\varphi}(n, x) = \varphi(x)$ for all $x \in L_0$. Let $\psi\colon \mathbb{Z}T \to \mathbb{Z}L$ be the natural isomorphism of translation quivers with $\psi(m, a) = \psi(m, (n, a))$ for all $m \in \mathbb{Z}$ and $a \in T_0$. We define $\varphi_y := \overline{\varphi} \circ \psi$. Then

$$\varphi_y(t) = \overline{\varphi}(n, t) = \varphi(t) = y.$$

By construction we have for all $x \in T_0$

$$(\Phi \circ \varphi_y)(n, x) = (\Phi \circ \overline{\varphi})(n, (n, x)) = \Phi(\varphi(n, x)) = \pi(n, x).$$

We conclude for $m \in \mathbb{Z}$ and $x \in T_0$

$$\begin{aligned}(\Phi \circ \varphi_y)(m, x) &= (\Phi \circ \varphi_y)(\tau_T^{m-n}(n, x)) = \tau_Q^{m-n}((\Phi \circ \varphi_y)(n, x)) \\ &= \tau_Q^{m-n}\pi(n, x) = \pi(\tau_T^{m-n}(n, x)) = \pi(m, x),\end{aligned}$$

hence $\Phi \circ \varphi_y = \pi$. If $\psi_y \colon \mathbb{Z}T \to Q'$ is another such morphism, we conclude that $\nu := \psi_{y|L} \colon L \to Q'$ is a quiver morphism with $\Phi \circ \nu = \pi_{|L} = \Phi \circ \psi$. Since φ is the uniquely determined with this property we conclude $\nu = \varphi$. Hence φ_y and ψ_y coincide on $L = \{n\} \times T_0$. Since φ_y and ψ_y are morphism of stable translation quivers, we conclude $\varphi_y = \psi_y$. □

Corollary 3.11. *Let $\Phi \colon \mathbb{Z}T \to Q$, $\Phi' \colon \mathbb{Z}S \to Q$ be coverings and $\varphi \colon \mathbb{Z}T \to \mathbb{Z}S$ a morphism of stable translation quivers such that $\Phi' \circ \varphi = \Phi$, then φ is an isomorphism. In particular, $\mathbb{Z}T \cong \mathbb{Z}S$ as stable translation quivers and $\overline{T} \cong \overline{S}$.*

Proof. Fix $x \in (\mathbb{Z}T)_0$ and let $\varphi(x) =: y$. Since $\Phi'(y) = \Phi(x)$ and $\Phi \colon \mathbb{Z}T \to Q$ is a covering we can apply 3.10 and get a morphism $\psi \colon \mathbb{Z}S \to \mathbb{Z}T$ such that $\Phi \circ \psi = \Phi'$ and $\psi(y) = x$. Hence $\psi \circ \varphi \colon \mathbb{Z}T \to \mathbb{Z}T$ fixes x and $\Phi \circ (\psi \circ \varphi) = \Phi$. Since $\mathrm{id}_{\mathbb{Z}T}$ also has these properties we conclude with 3.10 that $\psi \circ \varphi = \mathrm{id}_{\mathbb{Z}T}$. By the same token, we conclude $\varphi \circ \psi = \mathrm{id}_{\mathbb{Z}S}$. □

Corollary 3.12. *Let $\Phi \colon \mathbb{Z}T \to Q$ be a covering, then $\Pi_\Phi x = \Pi_\Phi y$ if and only if $\Phi(x) = \Phi(y)$. Moreover $\mathbb{Z}T/\Pi_\Phi \cong Q$ as stable translation quivers.*

Proof. Let $x, y \in (\mathbb{Z}T)_0$. If $\Pi_\Phi x = \Pi_\Phi y$, we find $\pi \in \Pi_\Phi$ such that $\pi(x) = y$ and conclude $\Phi(y) = \Phi(\pi(x)) = \Phi(x)$. Now assume that $\Phi(x) = \Phi(y)$, then 3.10 yields a morphism $\varphi \colon \mathbb{Z}T \to \mathbb{Z}T$ such that $\Phi \circ \varphi = \Phi$ and $\varphi(x) = y$. Now 3.11 yields $\varphi \in \Pi_\Phi$ and therefore $\Pi_\Phi x = \Pi_\Phi y$.
It remains to show that $\mathbb{Z}T/\Pi_\Phi \cong Q$. Observe that $\mathbb{Z}T/\Pi_\Phi$ is well-defined since Π_Φ is admissable. By the previous discussion we have well-defined morphisms $\varphi \colon \mathbb{Z}T/\Pi_\Phi \to Q; [x] \mapsto \Phi(x)$ and $\psi \colon Q \to \mathbb{Z}T/\Pi_\Phi$, $\Phi(x) \to [x]$ that are mutally inverse to each other. □

Corollary 3.13. *Let Q be a stable translation quiver and Π an admissable subgroup of $\mathrm{Aut}(\mathbb{Z}T, \tau_T)$ such that $\mathbb{Z}T/\Pi \cong Q$. We denote by $\pi \colon \mathbb{Z}T \to \mathbb{Z}T/\Pi$ the natural covering, $\varphi \colon \mathbb{Z}T/\Pi \to Q$ the given isomorphism and $\Phi := \varphi \circ \pi$. Then $\Pi = \Pi_\Phi$.*

Proof. Let $g \in \Pi$. Since $g(x)$ and x are in the same Π-orbit we conclude

$$(\Phi \circ g)(x) = \varphi(\pi(g(x)) = \varphi(\pi(x)) = \Phi(x),$$

for all $x \in (\mathbb{Z}T)_0$. Hence $\Pi \subseteq \Pi_\Phi$.
Now let $f \in \Pi_\Phi$. We fix $x \in (\mathbb{Z}T)_0$ and set $y := f(x)$. We get

$$(\varphi \circ \pi)(x) = \Phi(x) = \Phi(f(x)) = (\varphi \circ \pi \circ f)(x)$$

and conclude $\pi(x) = \pi(f(x))$, hence $[x] = [f(x)]$. By defintion there exists $h \in \Pi$ such that $h(f(x)) = x$. Since $h \in \Pi \subseteq \Pi_\Phi$, we conclude $h \circ f \in \Pi_\Phi$ with $(h \circ f)(x) = x$. Now 3.10 yields $h \circ f = \mathrm{id}_{\mathbb{Z}T}$, i.e. $f = h^{-1} \in \Pi$. □

Proposition 3.14. *Let $\Phi \colon \mathbb{Z}T_1 \to Q, \Phi' \colon \mathbb{Z}T_2 \to Q$ be coverings and Π_1, Π_2 admissable subgroups of $\mathbb{Z}T_1$ and $\mathbb{Z}T_2$ such that $\mathbb{Z}T_1/\Pi_1 \cong Q \cong \mathbb{Z}T_2/\Pi_2$ as stable translation quivers. There exists an isomorphism $\lambda \colon \mathbb{Z}T_1 \to \mathbb{Z}T_2$ such that $\Pi_1 = \lambda^{-1} \circ \Pi_2 \circ \lambda$.*

Proof. For $i \in \{1, 2\}$ let $\varphi_i \colon \mathbb{Z}T_i/\Pi_i \to Q$ isomorphisms and $\pi_i \colon \mathbb{Z}T_i \to \mathbb{Z}T_i/\Pi_i$ the natural coverings. The morphisms $\Phi_i := \varphi_i \circ \pi_i \colon \mathbb{Z}T_i \to Q$ are coverings and 3.13 yields $\Pi_i = \Pi_{\Phi_i}$.
We fix $x_i \in (\mathbb{Z}T_i)_0$ such that $\Phi_1(x_1) = \Phi_2(x_2)$. Now 3.10 and 3.11 yield an isomorphism $\lambda \colon \mathbb{Z}T_1 \to \mathbb{Z}T_2$ such that $\lambda(x_1) = x_2$ and $\Phi_2 \circ \lambda = \Phi_1$. For $g \in \Pi_{\Phi_1}$ we have $\Phi_2 \circ (\lambda \circ g \circ \lambda^{-1}) = \Phi_1 \circ g \circ \lambda^{-1} = \Phi_1 \circ \lambda^{-1} = \Phi_2$, hence

$$\Pi_{\Phi_1} \to \Pi_{\Phi_2}; g \mapsto \lambda \circ g \circ \lambda^{-1},$$

is a well-defined group homomorphism with inverse $f \mapsto \lambda^{-1} \circ f \circ \lambda$. We conclude

$$\Pi_1 = \Pi_{\Phi_1} = \lambda \circ \Pi_{\Phi_2} \circ \lambda^{-1} = \lambda \circ \Pi_2 \circ \lambda^{-1}.$$

□

We have done all the preparations to prove Riedtmanns's Structure Theorem.

Definition. Let (Q, τ_Q) be a stable translation quiver.

(a) An **oriented path of length** $n \geq 1$ in Q is a sequence of arrows $p = \alpha_1 \cdots \alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i < n$. The vertex $s(\alpha_1)$ is called the **start** of p.

(b) An oriented path $\alpha_1 \cdots \alpha_n$ is called **sectional** provided $\tau_Q(t(\alpha_{i+2})) \neq s(\alpha_i)$ for all $1 \leq i \leq n-2$.

(c) For each vertex $x \in Q_0$ we denote with ϵ_x the **trivial path** starting in x. By definition ϵ_x has length 0 and is sectional. If p is an oriented path of length $n \geq 1$ starting in x, we define $\varepsilon_x p := p$.

Theorem 3.15 (Riedtmann's Structure Theorem)**.** *Let (Q, τ_Q) be a stable and connected translation quiver. The following statements hold.*

(a) *There exists a directed tree T_Q and an admissable subgroup $\Pi \subseteq \mathrm{Aut}(\mathbb{Z}T_Q, \tau_{T_Q})$ such that $Q \cong \mathbb{Z}T_Q/\Pi$ as stable translation quivers.*

(b) *If S_Q is another oriented tree as in (a), then $\overline{S}_Q \cong \overline{T}_Q$.*

(c) *The group Π is uniquely determined up to conjugation.*

We call $\overline{T}_Q$ the tree class of Q.

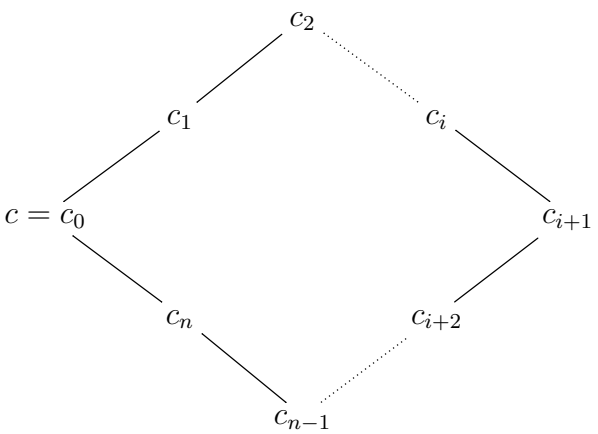
Proof. In view of 3.14 and 3.8 it remains to prove (a). We fix $x \in Q_0$ and consider the $\mathcal{P}_Q$ set of all sectional paths in Q of length ≥ 0 starting in x. We define a quiver T_Q given by the following data:

- $(T_Q)_0 := P_Q$
- $p \to q \in (T_Q)_1 \Leftrightarrow q = p\alpha$ for some $\alpha \in Q_1$.

We claim that T_Q is an oriented tree. Let $p = \alpha_1 \cdots \alpha_n \in P_Q$ be an oriented path of length $n \geq 1$. We set $p_i := \alpha_1 \cdots \alpha_i \in (T_Q)_0$ for $1 \leq i \leq n$ and have an oriented path in T_Q

$$\varepsilon_x \to p_1 \to p_2 \to \cdots \to p_n.$$

Hence every vertex p in $\overline{T}_Q$ is connected to ε_x and therefore $\overline{T}_Q$ is connected as a graph. Now assume that $\overline{T}_Q$ contains a cycle. By definition T_Q does not contain loops and cycles of length 2. We conclude that we have a cycle



with $n \geq 3$ such that all c_j are pairwise distinct. In the following we denote the length of a path p with $l(p) \in \mathbb{N}_0$. Since $l(p) = l(q) - 1$ for each arrow $p \to q$ in T_Q there exists $i \in \{0, \ldots, n-1\}$ such that we have arrows $c_i \to c_{i+1}$ and $c_{i+2} \to c_{i+1}$. But this implies $c_i = c_{i+2}$, a contradiction. This shows that T_Q is in fact a directed tree.
Now let $\varphi\colon T_Q \to Q$ be the quiver morphism with $\varphi(\alpha_1 \cdots \alpha_n) := t(\alpha_n)$ and $\varphi(\varepsilon_x) = x$. Lemma 3.7 implies the existence of a unique morphism of stable translation quivers $\Phi\colon \mathbb{Z}T_Q \to Q$ with $\Phi(0, v) = \varphi(v)$ for all $v \in (T_Q)_0$.
We claim that Φ is a covering. Note that is sufficies to show that

- $\Phi_{|v^+}\colon v^+ \to \Phi(v)^+$ is a bijection for each $v \in (\mathbb{Z}T_Q)_0$ and
- Φ is surjective on vertices.

Let $v \in (\mathbb{Z}T_Q)_0$, then we find $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $p = \alpha_1 \cdots \alpha_n \in (T_Q)_0$ with $v = (m, p)$, where $n = 0$ denotes $p = \varepsilon_x$. Let $(a, q), (b, q') \in v^+$ such that $\Phi(a, q) = \Phi(b, q')$. Without loss of generality we assume $b \geq a$. We conclude

$$\tau_Q^{-a}(\varphi(q)) = \tau_Q^{-a}\Phi(0, q) = \Phi(a, q) = \Phi(b, q') = \tau_Q^{-b}(\varphi(q')),$$

hence $\tau_Q^{b-a}(\varphi(q)) = \varphi(q')$. By definition of $\mathbb{Z}T_Q$ we have

$$v^+ = \{(m, r) \mid r \in p^+\} \cup \{(m+1, r) \mid r \in p^-\}.$$

We assume that $a \neq b$ and get $b = a + 1 = m + 1$, $q' \in p^-$ and $q \in p^+$. As

$$q' \in p^- = \begin{cases} \varnothing, n = 0 \\ \{\alpha_1 \cdots \alpha_{n-1}\}, n > 0 \end{cases}$$

we conclude $n > 0$ and $q' = \alpha_1 \cdots \alpha_{n-1}$. Moreover, $q \in p^+ \subseteq (T_Q)_0$ implies that there exists $\alpha \in Q_1$ such that $q = p\alpha = \alpha_1 \cdots \alpha_n \alpha$. It follows

$$s(\alpha_n) = t(\alpha_{n-1}) = \varphi(q') = \tau_Q(\varphi(q)) = \tau_Q(t(\alpha)),$$

but this is a contradiction since $q = p\alpha$ is a sectional path. Hence $a = b$ and $\varphi(q) = \varphi(q')$. Now either $a = b = m + 1$ and $q = q' = \alpha_1 \cdots \alpha_{n-1}$ or $a = b = m$ and $q, q' \in p^+$. In the first case there is nothing to show. In the second case we find $\beta, \beta' \in Q_1$ such that $q = p\beta, q' = p\beta'$. It follows $t(\beta) = \varphi(q) = \varphi(q') = t(\beta')$, hence $\beta = \beta'$. This shows that the map Φ is locally injective.
We now show that Φ is locally surjective. We first consider the special case that $m = 0$, i.e. $v = (0, p)$.
Let $y \in \Phi(v)^+ = \varphi(p)^+ = t(\alpha_n)^+$ and $\alpha\colon t(\alpha_n) \to y \in Q_1$. We distinguish two cases.

- $\tau_Q(y) \neq t(\alpha_{n-1})$, then $p\alpha = \alpha_1 \cdots \alpha_n \alpha \in (T_Q)_0$ and we have an arrow $p \to p\alpha \in (T_Q)_1$. Hence $(0, p\alpha) \in (0, p)^+$ with $\Phi(0, p\alpha) = \varphi(p\alpha) = t(\alpha) = y$.
- $\tau_Q(y) = t(\alpha_{n-1})$, then $n \geq 1$. We consider $\alpha_1 \cdots \alpha_{n-1} \in (T_Q)_0$. As $\alpha_1 \cdots \alpha_{n-1} \to p \in (T_Q)_1$ we get $(0, p) \to (1, \alpha_1 \cdots \alpha_{n-1}) \in (\mathbb{Z}T_Q)_1$ and

$$\Phi(1, \alpha_1 \cdots \alpha_{n-1}) = \Phi(\tau_{T_Q}^{-1}(0, \alpha_1 \cdots \alpha_{n-1})) = \tau_Q^{-1}(t(\alpha_{n-1})) = y.$$

Now we consider the general case $v = (m, p)$ and $y \in \Phi(v)^+$. Then $\tau_Q^m y \in \Phi(0, p)^+$ and we find $w \in (0, p)^+$ with $\Phi(w) = \tau_Q^m y$. We conclude $\Phi(\tau_{T_Q}^{-m} w) = \tau_Q^{-m} \tau_Q^m y$ and $\tau_{T_Q}^{-m} w \in (m, p)^+$. Hence Φ is locally surjective.
It remains to show that Φ is surjective on vertices. Let $q \in Q_0$. Since Φ interchanges with the translations, we conclude $q \notin \operatorname{im} \Phi$ if and only if $\tau_Q^l q \notin \operatorname{im} \Phi$ for all $l \in \mathbb{Z}$. Since Φ is locally bijective we also have $q \in \operatorname{im} \Phi$ if and only if $q \cup q^+ \cup q^- \in \operatorname{im} \Phi$. Hence the sets $\operatorname{im} \Phi$ and $Q_0 \setminus \operatorname{im} \Phi$ induce a decomposition of Q into stable subquivers. Since Q is connected, we conclude $\operatorname{im} \Phi = Q_0$.

□

3.5 Outlook

Definition. Let Q be a finite, connected and acyclic quiver. Then Q is called (see Figure 3.1)

(a) **directed Dynkin diagram**, if $\overline{Q}$ is a simply-laced Dynkin diagram, i.e. $Q \in \{A_n, D_n, E_6, E_7, E_8\}$.

(b) **directed Euclidean diagram**, if $\overline{Q}$ is a simply-laced Euclidean diagram, i.e. $Q \in \{\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$.

(c) **wild** in all other cases.

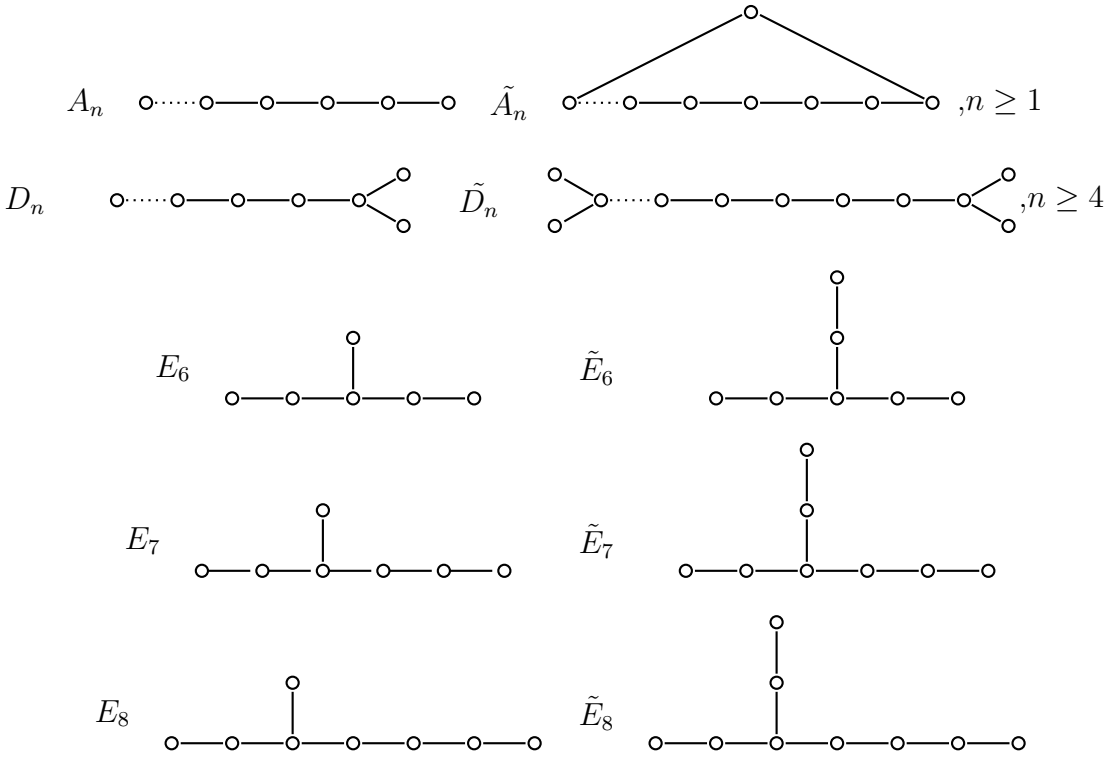


Figure 3.1: List of simply-laced Dynkin and Euclidean diagrams. By definition a graph X_n has n vertices and a graph $\tilde{X}_n$ has $n+1$ vertices.

Theorem 3.16 (Riedtmann)**.** *Let Λ be a finite-dimensional k-algebra of finite representation type. Let Γ be a connected component of the stable Auslander-Reiten quiver Γ_Λ^s of Λ. The tree class of Γ is a simply-laced Dynkin diagram.*

Definition. A component Γ of the Auslander-Reiten quiver Γ_Λ is called regular, provided that every vertex in Γ is stable.

Let Λ be a hereditary algebra. Recall that by definition submodules of projective modules are projective. If Λ is connected, the Gabriel quiver Q_Λ is connected and $\operatorname{mod}\Lambda \cong \operatorname{mod} kQ_\Lambda$, where kQ_Λ is path algebra of Q_Λ.

Theorem 3.17 (Auslander-Bautista-Platzeck-Reiten-Smalø,Ringel 1979)**.** *If Q_Λ is not a directed Dynkin diagram, then every stable component Γ is of type A_∞. If in addition Q_Λ is not a directed Euclidean diagram, then the admissable group Π_Γ is trivial, i.e. $\Gamma \cong \mathbb{Z}A_\infty$.*

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