Steiner bundles on Grassmannians and cones of relative projective Kronecker representations

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Zusammenfassung

Es sei $\Bbbk$ ein algebraisch abgeschlossener Körper und $r \in \mathbb{N}_{\geq 2}$. Verbindungen zwischen Darstellungen des Kronecker-Köchers

$$K_r = \quad 1 \overset{\gamma_1}{\underset{\gamma_r}{\vdots}} 2$$

mit r Pfeilen $\gamma_i \colon 1 \to 2$ und Vektorbündeln über $\mathbb{P}^{r-1} = \mathbb{P}^{r-1}(\Bbbk)$ für Körper der Charakteristik 0 sind wohlbekannt und kamen vermutlich erstmals systematisch in Huleks Artikel [Hul80] zur Anwendung.

In dieser Arbeit vertiefen wir die Untersuchung dieses Zusammenhangs für *Steiner-Bündel* über der Graßmann-Varietät $\mathrm{Gr}_d(A_r)$ von d-dimensionalen Unterräumen des *Pfeilraums* $A_r \coloneqq \bigoplus_{i=1}^{r} \Bbbk\gamma_i$ über Körpern beliebiger Charakteristik.

Fußend auf dem kategoriellen Ansatz von Jardim und Prata [JP15] zeigen wir, dass ein rechtsexakter Funktor $\widetilde{\Theta}_d \colon \mathrm{rep}(K_r) \longrightarrow \mathrm{Coh}(\mathrm{Gr}_d(A_r))$ existiert, der jeder Kronecker-Darstellung

$$M = (M_1, M_2, \psi_M \colon A_r \otimes_\Bbbk M_1 \longrightarrow M_2)$$

eine kohärente Garbe $\widetilde{\Theta}_d(M)$ auf $\mathrm{Gr}_d(A_r)$ derart zuordnet, sodass die Restriktion auf die volle Unterkategorie $\mathcal{X} \coloneqq \{M \in \mathrm{rep}(K_r) \mid \forall\, \mathfrak{v} \in \mathrm{Gr}_d(A_r) \colon \mathrm{rk}(\psi_M|_{\mathfrak{v} \otimes_\Bbbk M_1}) = d \dim_\Bbbk M_1\}$ eine exakte Äquivalenz $\widetilde{\Theta}_d \colon \mathcal{X} \longrightarrow \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ zwischen $\mathcal{X}$ und der Kategorie der Steiner-Bündel über $\mathrm{Gr}_d(A_r)$ induziert.

Wir beweisen, dass $\mathcal{X}$ mit der vollen Unterkategorie $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ *relativ d-projektiver* Darstellungen übereinstimmt, das heißt für jedes $M \in \mathcal{X}$ und jede $\Bbbk$-lineare injektive Abbildung $\alpha \colon A_d \longrightarrow A_r$ ist die Darstellung $(M_1, M_2, \psi_M \circ (\alpha \otimes \mathrm{id}_{M_1})) \in \mathrm{rep}(K_d)$ projektiv.

Weiter charakterisieren wir $\mathcal{X} = \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ als die Klasse von Darstellungen, die rechts Hom-orthogonal zu einer gewissen algebraischen Familie von "Testdarstellungen" in $\mathrm{rep}(K_r)$ sind. Diese Testdarstellungen gewinnen wir mithilfe einer neuen Konstruktionsmethode, indem wir adjungierte Paare von Funktoren zwischen $\mathrm{rep}(K_d)$ und $\mathrm{rep}(K_r)$ für $1 \leq d < r$ nutzen, die sich aus Restriktions-, Inflations- und Spiegelungsfunktoren zusammensetzen. Unser Ansatz vereint frühere Ad-hoc-Konstruktionen von Testdarstellungen für Kronecker-Köcher (vgl. [HU91; Wor13a; Bis20]) und liefert uns eine Familie von Testdarstellungen für jede unzerlegbare präprojektive Darstellung des Köchers K_d.

Die neuen Beschreibungen von $\mathcal{X}$ ermöglichen die Anwendung von modernen Methoden der Darstellungstheorie auf die Kategorie der Steiner-Bündel. Eine besondere Rolle spielt dabei die Auslander-Reiten-Theorie, die uns erlaubt, unzerlegbare Steiner-Bündel in einem Translationsköcher in unendlichen Kegeln so anzuordnen, dass die Untersuchung von Steiner-Bündeln mit spezifischen Eigenschaften wie Uniformität, Homogenität und Stabilität vereinfacht wird.

Umgekehrt nutzen wir Erkenntnisse und algo-geometrische Methoden aus der Theorie der Vektorbündel, um die Kategorie der Kronecker-Darstellungen zu untersuchen. Insbesondere untersuchen wir die kleinste Klasse von Kronecker-Darstellungen, die unter Erweiterungen und Isomorphismen abgeschlossen und für das wilde Verhalten der Kategorie $\mathrm{rep}(K_r)$ für $r \geq 3$ verantwortlich ist: die sogenannten *elementaren* Darstellungen.

Abstract

Let $\Bbbk$ be an algebraically closed field, and let $r \in \mathbb{N}_{\geq 2}$ be a natural number. Connections between representations of the generalized Kronecker quiver

$$K_r = \qquad 1 \overset{\gamma_1}{\underset{\gamma_r}{\rightrightarrows}} 2$$

with r arrows $\gamma_i \colon 1 \to 2$ and vector bundles on $\mathbb{P}^{r-1} = \mathbb{P}^{r-1}(\Bbbk)$ for fields of characteristic 0 have been known for some time and were apparently first systematically exploited in Hulek's article [Hul80].

This thesis addresses a specific aspect of this relationship, focusing on *Steiner bundles* on the Grassmann variety $\mathrm{Gr}_d(A_r)$, which consists of d-dimensional subspaces of the *arrow space* $A_r := \bigoplus_{i=1}^{r} \Bbbk\gamma_i$ over fields of arbitrary characteristic.

Building on the categorical approach of Jardim and Prata [JP15] we show that there exists a right exact functor $\widetilde{\Theta}_d \colon \mathrm{rep}(K_r) \longrightarrow \mathrm{Coh}(\mathrm{Gr}_d(A_r))$ that assigns to each Kronecker representation

$$M = (M_1, M_2, \psi_M \colon A_r \otimes_{\Bbbk} M_1 \longrightarrow M_2)$$

a coherent sheaf $\widetilde{\Theta}_d(M)$ on $\mathrm{Gr}_d(A_r)$ in such a way that the restriction to the full subcategory

$$\mathcal{X} := \{M \in \mathrm{rep}(K_r) \mid \forall\, \mathfrak{v} \in \mathrm{Gr}_d(A_r) \colon \mathrm{rk}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = d \dim_{\Bbbk} M_1\} \subseteq \mathrm{rep}(K_r)$$

induces an exact equivalence $\widetilde{\Theta}_d \colon \mathcal{X} \longrightarrow \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ between $\mathcal{X}$ and the category of Steiner bundles on $\mathrm{Gr}_d(A_r)$. By establishing several equivalent characterizations of $\mathcal{X}$, we gain insights into the category $\mathcal{X}$ and transport our findings, using $\widetilde{\Theta}_d$, into the category of Steiner bundles.

We show that $\mathcal{X}$ coincides with the full subcategory $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ of *relative d-projective* representations, i.e. for $M \in \mathcal{X}$ and every $\Bbbk$-linear injective map $\alpha \colon A_d \longrightarrow A_r$, the representation $(M_1, M_2, \psi_M \circ (\alpha \otimes \mathrm{id}_{M_1})) \in \mathrm{rep}(K_d)$ is projective. Moreover, we give a description of $\mathcal{X} = \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ as being the collection of representations that are right Hom-orthogonal to a certain algebraic family of "test representations" in $\mathrm{rep}(K_r)$. This family of test representations is obtained using a new construction method based on adjoint pairs of functors between $\mathrm{rep}(K_d)$ and $\mathrm{rep}(K_r)$ for $1 \leq d < r$ that are composites of restriction, inflation, and reflection functors. Our approach unifies former ad hoc constructions of test representations for Kronecker quivers (cf. [HU91; Wor13a; Bis20]) and produces a family of test representations for every indecomposable preprojective K_d-representation.

With these new descriptions and tools at hand, we are able to employ representation-theoretic techniques provided by the adjoint pairs and Auslander-Reiten theory to organize the category of indecomposable Steiner bundles in infinite cones which facilitates the study of bundles that exhibit certain properties such as uniformity, homogeneity, and stability. Conversely, we use insights and algo-geometric methods from the theory of vector bundles to study the category of Kronecker representations. In particular, we investigate the smallest class of Kronecker representations that is closed under extensions and isomorphisms and is responsible for the wild behavior of the category $\mathrm{rep}(K_r)$ for $r \geq 3$: the so-called *elementary* representations.

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0. Introduction

Let $\Bbbk$ be an algebraically closed field. Algebraic vector bundles on projective varieties over $\Bbbk$ are fundamental objects that are studied to get a better understanding of the geometry and topology of the underlying variety. Vector bundles on $\mathbb{P}^1$ are completely classified: Grothendieck (see e.g. [OSS80, p. 2.1.1]) has shown that every vector bundle $\mathcal{F}$ on $\mathbb{P}^1$ is a direct sum of Serre twists of the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$, i.e. we find a (up to isomorphism) unique decomposition

$$\mathcal{F} = \bigoplus_{i\in\mathbb{Z}} a_i \mathcal{O}_{\mathbb{P}^1}(i).$$

For $n \geq 2$ the situation is quite different. Although numerous results for vector bundles on $\mathbb{P}^n$ with $n \in \mathbb{N}_{\geq 2}$ have been obtained and a variety of methods and tools have been developed for their investigation, a comprehensive description in the above sense is not in sight (see e.g. [Har79]). For example, it is still unknown whether there exists an indecomposable vector bundle of rank 2 on $\mathbb{P}^n$ for $n \geq 6$ in any characteristic (see [Ben16, (6.3.3)]). This motivates the study of algebraic families of vector bundles on $\mathbb{P}^n$ and other varieties with additional specific characteristics and properties.

These include the extensively studied class of *cokernel bundles* (cf. [Bra08; CHS22; JP15; Mar12]) on a smooth projective variety X. The definition of a cokernel bundle on X depends on a fixed so-called *strongly exceptional pair* $(\mathcal{E}_1, \mathcal{E}_2)$ of vector bundles on X (see [Bra08] for the definition) and the cokernel bundles $\mathcal{F} \in \operatorname{Vect}(X)$ of type $(\mathcal{E}_1, \mathcal{E}_2)$ are characterized by the property that they fit into a short exact sequence

$$0 \longrightarrow V_1 \otimes_{\Bbbk} \mathcal{E}_1 \longrightarrow V_2 \otimes_{\Bbbk} \mathcal{E}_2 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $V_1, V_2 \in \operatorname{mod}\Bbbk$ are finite dimensional $\Bbbk$-vector spaces. For fields of characteristic 0, Jardim and Prata [JP15] have shown that the category of cokernel bundles on X of type $(\mathcal{E}, \mathcal{F})$ with $r := \dim_{\Bbbk} \operatorname{Hom}_X(\mathcal{E}, \mathcal{F})$ is equivalent to an abstractly described full subcategory of the category of representations $\operatorname{rep}(K_r)$ for the generalized Kronecker quiver

$$K_r = \qquad 1 \overset{\gamma_1}{\underset{\gamma_r}{\rightrightarrows}} 2.$$

The most prominent representatives of cokernel bundles are *Steiner bundles* on $\mathbb{P}^n$, i.e. cokernel bundles of type $(\mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n})$, where $\mathcal{O}_{\mathbb{P}^n}(-1)$ denotes the Serre twisting sheaf of the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$.

The definition of Steiner bundles goes back to Dolgachev & Kapranov [DK93] and was generalized by Miró-Roig and Soares [MS09] to arbitrary projective varieties over fields of characteristic 0. Marchesi and Arrondo [Mar12; AM15] initiated the systematic study of Steiner bundles on the Grassmann variety $\operatorname{Gr}_d(\Bbbk^r)$ of d-dimensional subspaces of $\Bbbk^r$ for $1 \leq d < r$. They have proven that the universal vector bundle $\mathcal{U}_{(r,d)}$ and the structure sheaf $\mathcal{O}_{\operatorname{Gr}_d(\Bbbk^r)}$ on $\operatorname{Gr}_d(\Bbbk^r)$ form a so-called

special exceptional pair[1] (see 8.2.4 for the definition) under the assumption that the characteristic of the ground field $\Bbbk$ is 0.

In this thesis, we focus our investigation on the connection between Kronecker representations of K_r and Steiner bundles on $\mathrm{Gr}_d(A_r) \cong \mathrm{Gr}_d(\Bbbk^r)$, where $A_r \coloneqq \bigoplus_{i=1}^r \Bbbk\gamma_i$ denotes the *arrow space* of K_r. We consider a representation $M \in \mathrm{rep}(K_r)$ as a triple $M = (M_1, M_2, \psi_M)$ consisting of finite dimensional $\Bbbk$-vector spaces M_1, M_2 and a $\Bbbk$-linear map $\psi_M \colon A_r \otimes_\Bbbk M_1 \longrightarrow M_2$, called *structure map* of M. By showing that $(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)})$ is a special exceptional pair (see 8.2.4) with

$$A_r^* \cong \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)})$$

for every algebraically closed field, we are able to build on the functorial approach of Jardim and Prata [JP15], and construct a right exact functor

$$\widetilde{\Theta}_d \colon \mathrm{rep}(K_r) \longrightarrow \mathrm{Coh}(\mathrm{Gr}_d(A_r))$$

that assigns to each Kronecker representation $M \in \mathrm{rep}(K_r)$ a coherent sheaf on $\mathrm{Gr}_d(A_r)$ such that $\widetilde{\Theta}_d(M)$ is a vector bundle if and only if

$$\mathrm{rk}(\psi_{M,\mathfrak{v}}) \coloneqq \mathrm{rk}(\psi_M|_{\mathfrak{v}\otimes_\Bbbk M_1})$$

is independent of $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$. This insight naturally leads to the full subcategory of $\mathrm{rep}(K_r)$ consisting of representations such that $\psi_{M,\mathfrak{v}}$ is injective for all $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$, in other words representations with trivial *d-th rank variety*

$$\mathcal{R}(K_r, d)_M \coloneqq \{\mathfrak{v} \in \mathrm{Gr}_d(A_r) \mid \mathrm{rk}(\psi_{M,\mathfrak{v}}) < d \dim_\Bbbk M_1\} \subseteq \mathrm{Gr}_d(A_r),$$

whose definition is analogous to that investigated in [CFP12]. One of the main results of this work, explaining the relationship between representations of the Kronecker quiver K_r and the category of Steiner bundles on $\mathrm{Gr}_d(A_r)$, can be formulated as follows and is proven in Chapter 8.

Theorem A.

(1) A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathrm{Gr}_d(A_r))$ is a Steiner bundle if and only if there is $M \in \mathrm{rep}(K_r)$ with $\mathcal{R}(K_r, d)_M = \varnothing$ such that $\mathcal{F} \cong \widetilde{\Theta}_d(M)$.

(2) Upon restriction, the functor $\widetilde{\Theta}_d$ induces an exact equivalence between the full subcategory of $\mathrm{rep}(K_r)$ consisting of representations with trivial rank variety and the full subcategory of Steiner bundles $\mathrm{StVect}(\mathrm{Gr}_d(A_r)) \subseteq \mathrm{Vect}(\mathrm{Gr}_d(A_r))$.

(3) Let $M \in \mathrm{rep}(K_r)$ be a representation with $\mathcal{R}(K_r, d)_M = \varnothing$. Then $\mathrm{rk}(\widetilde{\Theta}_d(M)) = \dim_\Bbbk M_2 - d \dim_\Bbbk M_1$ and the first Chern class of $\widetilde{\Theta}_d(M)$ is given (upon identification with the first special Schubert cycle) by $\dim_\Bbbk M_1$.

The pair $(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(\Bbbk^r)})$ being special exceptional enables us to establish a link between the Euler characteristic and the Euler-Ringel form, leading to the classification of exceptional Steiner bundles on $\mathrm{Gr}_d(A_r)$. For $d = 1$, i.e. $\mathrm{Gr}_d(A_r) = \mathbb{P}(A_r)$, this extends the results of [Bra05, (2.1)], where the bundles were presumed to be generic. More precisely, we prove the following:

[1] The choice of terminology derives from Rudakov's more general notion of an exceptional pair, cf. [Rud90]. Pairs that meet the above conditions are referred to as 'strongly exceptional' in [Mar12]. The authors of [BF24] have introduced the terminology *special exceptional pair* to avoid confusion.

Corollary. *Let $M, N \in \operatorname{rep}(K_r)$ be representations with trivial rank variety. The following statements hold.*

(1) We have $\dim_{\Bbbk} \operatorname{Ext}^i_{K_r}(M,N) = \dim_{\Bbbk} \operatorname{Ext}^i_{\operatorname{Gr}_d(A_r)}(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(N))$ for all $i \in \mathbb{N}_0$.

(2) We have $\chi(\mathsf{Hom}_{\operatorname{Gr}_d(A_r)}(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(N))) = \langle \underline{\dim}\, M, \underline{\dim}\, N\rangle_r$, where χ is the Euler characteristic, $\mathsf{Hom}_{\operatorname{Gr}_d(A_r)}(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(N))$ is the sheaf of $\mathcal{O}_{\operatorname{Gr}_d(A_r)}$-homomorphisms and $\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle_r$ denotes the Euler-Ringel form on K_r of the dimension vectors $\underline{\dim}\, M$ and $\underline{\dim}\, N$.

(3) A Steiner bundle $\mathcal{F} \in \operatorname{StVect}(\operatorname{Gr}_d(A_r))$ is indecomposable and exceptional if and only if there exists an indecomposable preprojective representation $P \in \operatorname{rep}(K_r)$ such that $\mathcal{F} \cong \widetilde{\Theta}_d(P)$.

In order to study the category of representations with trivial rank variety, and thus the category of Steiner bundles, with modern representation theory methods, we establish in Chapters 1 to 4 equivalent descriptions of representations with trivial rank variety that involve shift functors and restrictions to $\operatorname{rep}(K_d)$:

The shift functors on $\operatorname{rep}(K_r)$, denoted by $\sigma_{K_r}, \sigma_{K_r}^{-1}\colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_r)$, correspond to the BGP-reflection functors (see [ASS06, §VII.5]) but take into account that the opposite quiver of K_r is isomorphic to K_r, i.e. $D_{K_r} \circ \sigma_{K_r} \cong \sigma_{K_r}^{-1} \circ D_{K_r}$, where $D_{K_r}\colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_r)$ denotes the standard duality on $\operatorname{rep}(K_r)$. Moreover, we let $\operatorname{Inj}_{\Bbbk}(A_d, A_r)$ be the space of $\Bbbk$-linear injective maps $A_d \longrightarrow A_r$. Given $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ and $M \in \operatorname{rep}(K_r)$ we denote by

$$\alpha^*(M) \coloneqq (M_1, M_2, \psi_{\alpha^*}(M) \coloneqq \psi_M \circ (\alpha \otimes \mathrm{id}_{M_1})) \in \operatorname{rep}(K_d)$$

the *restriction* of M (along α). Our findings may be summarized as follows:

Theorem B. *Let $1 \leq d < r$ and $M \in \operatorname{rep}(K_r)$ be a representation. The following statements are equivalent.*

(1) We have $\mathcal{R}(K_r, d)_M = \varnothing$.

(2) The restriction $\alpha^*(M) \in \operatorname{rep}(K_d)$ is projective for every $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$.

(3) We have $\bigcap_{a \in \mathfrak{u}} \ker \psi_{\sigma_{K_r}(M)}(a \otimes -) = \{0\}$ for all $\mathfrak{u} \in \operatorname{Gr}_{r-d}(A_r)$.

This result is the gateway for our applications of the shift functors $\sigma_{K_r}, \sigma_{K_r}^{-1}$ and Auslander-Reiten theory to the category of Steiner bundles via the exact equivalence of categories

$$\widetilde{\Theta}_d\colon \operatorname{rep}_{\operatorname{proj}}(K_r, d) \longrightarrow \operatorname{StVect}(\operatorname{Gr}_d(A_r))\ ;\ M \mapsto \widetilde{\Theta}_d(M),$$

where

$$\begin{aligned}\operatorname{rep}_{\operatorname{proj}}(K_r, d) \coloneqq &\{M \in \operatorname{rep}(K_r) \mid \forall \alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)\colon \alpha^*(M) \in \operatorname{rep}(K_d) \text{ is projective}\}\\ = &\{M \in \operatorname{rep}(K_r) \mid \forall \mathfrak{u} \in \operatorname{Gr}_{r-d}(A_r)\colon \bigcap_{a \in \mathfrak{u}} \ker \psi_{\sigma_{K_r}(M)}(a \otimes -) = \{0\}\}\\ = &\{M \in \operatorname{rep}(K_r) \mid \mathcal{V}(K_r, d)_M = \varnothing\}\end{aligned}$$

is the category of *relative d-projective* representations. As a direct consequence, we obtain:

Corollary.

(1) The category $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ *is a torsion-free class closed under direct summands,* $\sigma_{K_r}^{-1}$ *and the Auslander-Reiten translation* $\tau_{K_r}^{-1} \cong \sigma_{K_r}^{-1} \circ \sigma_{K_r}^{-1}$.

(2) The category $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ *contains all preprojective representations and no non-zero preinjective representation.*

Combing this result with findings from [Bis23] we are able to conclude that for a wild Kronecker quiver, i.e. $r \geq 3$, every regular component gives rise to an abundance of Steiner bundles on $\mathrm{Gr}_d(A_r)$ that all arrange nicely in Auslander-Reiten components:

Theorem C. *Let* $r \geq 3$ *and* $\mathcal{C}$ *be a regular component of the Auslander-Reiten component. For each* $1 \leq d \leq r-1$, *there exists a quasi-simple representation* $M_{\mathcal{C},d} \in \mathcal{C}$ *such that* $\mathcal{C} \cap \operatorname{rep}_{\mathrm{proj}}(K_r, d) = (M_{\mathcal{C},d} \to)$, *where* $(M_{\mathcal{C},d} \to)$ *is the cone of successors of* M *in* $\mathcal{C}$.

We also provide another description of $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ by showing that relative d-projective representations can be characterized as being right Hom-orthogonal to a certain algebraic family of representations that we call "test representations": Characterizing a category using such test representations is a well-established technique that has been effectively employed several times in the study of path algebras (cf. [HU91; Wor13a; Bis20]). We present a new construction for such families that unifies the ad hoc constructions in loc. cit. and produces a family of test representations for every indecomposable preprojective K_d-representation $P_n(d) \in \operatorname{rep}(K_d)$, where $P_0(d)$ is the simple projective representation with dimension vector $(0, 1)$ and $P_{\ell+1}(d) \coloneqq \sigma_{K_d}^{-1}(P_\ell(d))$ for all $\ell \in \mathbb{N}$. The main ingredient in the construction are adjoint pairs of functors between $\operatorname{rep}(K_d)$ and $\operatorname{rep}(K_r)$ for $1 \leq d < r$ that are defined as compositions of restriction, inflation (see Section 2.1 for the definitions), and shift functors:

Theorem D.

(1) The functor $\sigma_{K_r}^{-1} \circ \inf \colon \operatorname{rep}(K_d) \longrightarrow \operatorname{rep}(K_r)$ *is left adjoint to* $\sigma_{K_d} \circ \operatorname{res} \colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_d)$[2].

(2) The functor $\sigma_{K_d}^{-1} \circ \operatorname{res} \colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_d)$ *is left adjoint to* $\sigma_{K_r} \circ \inf \colon \operatorname{rep}(K_d) \longrightarrow \operatorname{rep}(K_r)$.

Given $g \in \mathrm{GL}(A_r) = \mathrm{Inj}_{\Bbbk}(A_r, A_r)$ and $M \in \operatorname{rep}(K_r)$, we denote by $g.M \in \operatorname{rep}(K_r)$ the representation $g.M \coloneqq (g^{-1})^*(M)$. For $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ and $g \in \mathrm{GL}(A_r)$ such that $g|_{A_d} = \alpha$, we define

$$P_n^-(\alpha) \coloneqq g.(\sigma_{K_r}^{-1} \circ \inf)(P_n(d)) \text{ and } P_n^+(\alpha) \coloneqq g.(\sigma_{K_r} \circ \inf)(P_n(d)).$$

These representations satisfy $P_n^-(\alpha) \cong P_n^-(\beta)$ and $P_n^+(\alpha) \cong P_n^+(\beta)$ for all $\alpha, \beta \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ with $\operatorname{im}\alpha = \operatorname{im}\beta$ which allows us to define

$$P_n^{\pm}(\operatorname{im}\alpha) \coloneqq P_n^{\pm}(\alpha)$$

for every $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$. We obtain two families of test representations $(P_n^-(\mathfrak{v}))_{\mathfrak{v} \in \mathrm{Gr}_d(A_r)}$, $(P_n^+(\mathfrak{v}))_{\mathfrak{v} \in \mathrm{Gr}_d(A_r)}$ for every $n \in \mathbb{N}$. Applying the adjoint pairs of functors to these families of representations yields the following result.

[2] Actually, we prove a more general result for an arbitrary acyclic quiver Q.

Corollary. *Let $M \in \operatorname{rep}(K_r)$ be a representation, $n \in \mathbb{N}_0$ and $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$. The following statements are equivalent.*

(1) The representation $\alpha^*(M) \in \operatorname{rep}(K_d)$ is in $\operatorname{add}(P_n(d), P_{n+1}(d))$.

(2) We have $\operatorname{Hom}_{K_r}(P^-_{n+1}(\alpha), M) = 0 = \operatorname{Hom}_{K_r}(M, P^+_n(\alpha))$.

In Chapter 5, we employ the various characterizations of $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ along with an adapted version of Reineke's result [Rei24] on general Kronecker representations to show that being relative d-projective is an open property in the representation space

$$\operatorname{rep}(K_r; V_1, V_2) := \operatorname{Hom}_{\Bbbk}(A_r \otimes_{\Bbbk} V_1, V_2) \cong \operatorname{Hom}_{\Bbbk}(V_1, V_2)^r$$

and to determine the conditions under which the subset

$$\operatorname{rep}_{\mathrm{proj}}(K_r, d) \cap \operatorname{Hom}_{\Bbbk}(A_r \otimes_{\Bbbk} V_1, V_2)$$

is non-empty for a pair (V_1, V_2) of finite dimensional $\Bbbk$-vector spaces. Additionally, we examine restrictions of representations $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ to the tame quiver K_2 via elements $\alpha \in \operatorname{Inj}_{\Bbbk}(A_2, A_r)$ in more detail. We arrive at:

Theorem E. *Let (V_1, V_2) be a pair of finite dimensional $\Bbbk$-vector spaces with $V_1 \oplus V_2 \neq \{0\}$.*

(1) We have $\operatorname{rep}_{\mathrm{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2) \neq \varnothing$ if and only if

$$\dim_{\Bbbk} V_2 - d \dim_{\Bbbk} V_1 \geq \min\{d(r-d), \dim_{\Bbbk} V_1 (r-d)\}.$$

(2) Let $n \in \mathbb{N}$. There exists a representation $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ such that $\operatorname{End}_{K_r}(M) = \Bbbk$ and

$$\alpha^*(M) \cong [(n+2)(n+1)(r-2)]P_n(2) \oplus [n(n+1)(r-2)]P_{n+1}(2)$$

for every $\alpha \in \operatorname{Inj}_{\Bbbk}(A_2, A_r)$.

In Chapter 6 we demonstrate that the properties of the category $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ not only provide new insights into the categories of Steiner bundles $\operatorname{StVect}(\operatorname{Gr}_d(A_r))$, but also facilitate the study of elementary representations:

By definition, elementary representations are the simple objects in the full subcategory of regular representations in $\operatorname{rep}(K_r)$ for $r \geq 3$ and the analog of quasi-simple regular representations in the context of tame quivers. Elementary representations for wild quivers were introduced by Crawley-Boevey and first systematically studied in [KL96] and [Luk92]. Ringel has proven in [Rin16] that for $r = 3$, given two elementary representations $E_1, E_2 \in \operatorname{rep}(K_r)$ with the same dimension vector, there exists $g \in \operatorname{GL}(A_3)$ such that $g.E_1 \cong E_2$. Furthermore, he classified all dimension vectors which allow an elementary representation.

We extend his classification to all $r \geq 3$. Moreover, we prove:

Theorem F. *Let $r \geq 4$. There exists a family of elementary representations $(E_i)_{i \in \mathbb{N}}$ such that $g.E_i \ncong E_j$ and $\underline{\dim}\, E_i = \underline{\dim}\, E_j$ for all $i \neq j \in \mathbb{N}$ and $g \in \operatorname{GL}(A_r)$.*

This shows that for $r \geq 4$ there exist pairs of vector spaces (V_1, V_2) such that the algebraic group $\mathrm{GL}(A_r) \times \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ no longer acts with finitely many orbits on the subset $\mathcal{E}(V_1, V_2) \subseteq \mathrm{rep}(K_r; V_1, V_2)$ of elementary representations. Consequently, determining elementary representations for $r \geq 4$ poses greater challenges compared to the case $r = 3$, although both categories are wild.

In Chapters 7 and 8 we recall well-known results from the theory of vector bundles on $\mathrm{Gr}_d(A_r)$ and give the proof of Theorem A. This allows us to transfer our findings from Chapters 2 to 5 into the category of Steiner bundles. As an immediate consequence, we are able to employ Theorem E and obtain the following result, which was proven in [AM15] for fields of characteristic 0 using Porteous' formula.

Corollary. *Let $1 \leq d < r$ and $(c, s) \in \mathbb{N}_0 \times \mathbb{N}$. There exists a Steiner bundle $\mathcal{F} \in \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ of rank s and with first Chern class c if and only if $s \geq \min\{c(r-d), d(r-d)\}$.*

Chapter 9 is devoted to the study of the behavior of Steiner bundles on $\mathbb{P}(A_r)$ under linear pullbacks to $\mathbb{P}(A_d)$ for $1 \leq d < r$. Restricting vector bundles is a well-used technique in algebraic geometry, since restrictions of a given vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}(A_r))$ often allow conclusions about the bundle itself. For example, by Horrocks' splitting principle (see [OSS80, (2.3.1), (2.3.2)]) a vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}(A_r))$ splits (as a direct sum of line bundles) if and only if a restriction of $\mathcal{F}$ to some plane $\mathbb{P}(A_3)$ splits.

By Grothendieck's Theorem the restriction of every Steiner bundle $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ to a line $\mathbb{P}(A_2) \cong L \subseteq \mathbb{P}(A_r)$ splits and $\mathcal{F}$ being globally generated gives us a decomposition

$$\mathcal{F}|_L \cong \bigoplus_{i \in \mathbb{N}_0} b_i(L)\mathcal{O}_L(i).$$

Moreover, there exists a uniquely determined sequence $(b_i(\mathcal{F}))_{i \in \mathbb{N}_0}$ such that

$$O_{\mathcal{F}} \coloneqq \{L \subseteq \mathbb{P}(A_r) \mid L \text{ line}, \forall i \in \mathbb{N}_0 \colon b_i(L) = b_i(\mathcal{F})\}$$

is a non-empty open subset of $\mathbb{P}(A_r)$. The vector bundle $\mathcal{F}$ is called *uniform*, provided $O_{\mathcal{F}} = \{L \subseteq \mathbb{P}(A_r) \mid L \text{ line}\}$, and *almost-uniform* if the set of *jumping lines* $\mathcal{J}_{\mathcal{F}} \coloneqq \{L \subseteq \mathbb{P}(A_r) \mid L \text{ line}\} \setminus O_{\mathcal{F}}$ is finite and non-empty. Given a Steiner bundle $\mathcal{F}$, we call

$$\mathrm{supp}(\mathcal{F}) \coloneqq \{i \in \mathbb{N}_0 \mid b_i(\mathcal{F}) \neq 0\}$$

the *support* of $\mathcal{F}$. Recently, Marchesi and Miró-Roig initiated the systematic study of uniform Steiner bundles on $\mathbb{P}(A_r)$ in [MM21]. By showing that for every $\Bbbk$-linear map $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$ the induced pullback functor $\hat{\alpha}^* \colon \mathrm{Coh}(\mathbb{P}(A_r)) \longrightarrow \mathrm{Coh}(\mathbb{P}(A_2))$ satisfies

$$\widetilde{\Theta}_{K_2,1}(\alpha^*(M)) \cong \hat{\alpha}^*(\widetilde{\Theta}_{K_r,1}(M))$$

for every representation $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ we are able to build on their work and show the existence of uniform Steiner bundles that have, in contrast to the findings in [MM21], a non-connected splitting type. This result is obtained using Theorem C and Theorem E.

Theorem G. *Let $n \in \mathbb{N}_0$.*

(1) There exists a simple and uniform Steiner bundle such that $\operatorname{supp}(\mathcal{F}) = \{n, n+1\}$.

(2) For $n \geq 2$ there exists an indecomposable and uniform Steiner bundle $\mathcal{F}$ such that $\operatorname{supp}(\mathcal{F}) = \{0, 1, n, n+1\}$. *In particular, there is no upper bound for the size of gaps for indecomposable uniform Steiner bundles.*

Moreover, we use a construction of Bongartz [Bon81] to prove that for every finite set X of lines in $\mathbb{P}(A_r)$, the category consisting of almost-uniform Steiner bundles whose set of jumping lines is X corresponds, via the equivalence of categories $\operatorname{rep}_{\mathrm{proj}}(K_r, 1) \cong \operatorname{StVect}(\mathbb{P}(A_r))$, to a wild subcategory of $\operatorname{rep}_{\mathrm{proj}}(K_r, 1)$.

In the last chapter we investigate stable Steiner bundles. By adapting the work of Karpov [Kar93] we show that a stable Steiner bundle $\widetilde{\Theta}_1(M) \in \operatorname{StVect}(\mathbb{P}(A_r))$ can only arise from a stable representation $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ with respect to the slope

$$\mu(M) \coloneqq \frac{\dim_{\Bbbk} M_1}{\dim_{\Bbbk} M_2 - \dim_{\Bbbk} M_1}.$$

Specializing to $r = 3$ we use methods by Drézet [Dré87] to give a sufficient condition for the stability of a bundle corresponding to a stable Kronecker representation. In particular, we prove:

Theorem H. *Let $M \in \operatorname{rep}_{\mathrm{proj}}(K_3, 1)$ be a representation. The following statements hold.*

(1) If $\dim_{\Bbbk} M_2 > 2 \dim_{\Bbbk} M_1$ and M is stable, so is $\widetilde{\Theta}_1(M)$.

(2) If M is elementary, then $\widetilde{\Theta}_1(M)$ is stable.

(3) Let $\mathcal{C} \subseteq \Gamma(K_3)$ be an Auslander-Reiten component containing an elementary representation. Then every Steiner bundle $\mathcal{F} \in \widetilde{\Theta}_1(\mathcal{C} \cap \operatorname{rep}_{\mathrm{proj}}(K_r, 1))$ possesses a Harder-Narasimhan filtration by Steiner bundles, whose filtration factors are stable.

In the appendix, we provide background information on the relationship between vector bundles and locally trivial fibrations, Chern classes for Grassmann varieties, and principal bundles locally trivial in the Zariski topology.

1. Basic results in representation theory

In this chapter we introduce basic definitions from representation theory of hereditary algebras and collect well-known results that we will use throughout the thesis. We refer the reader to [ARS95], [ASS06], and [Ker94] for further details and unexplained terminology. This chapter is based on [Bis24, §1] and [BF24, §1.1].

1.1. Notation

Throughout this work, $\Bbbk$ denotes an algebraically closed field and, if not stated otherwise, $\Bbbk$ is of arbitrary characteristic. By a $\Bbbk$-algebra we mean an associative, finite dimensional, unitary, and connected $\Bbbk$-algebra. For a $\Bbbk$-algebra A we denote by $\operatorname{mod} A$ the category of finitely generated left A-modules. Given $M, N \in \operatorname{mod} A$ we denote by $\operatorname{Hom}_A(M,N)$ the $\Bbbk$-vector space of A-linear maps from M to N. The $\Bbbk$-linear maps from M to N are denoted by $\operatorname{Hom}_\Bbbk(M,N)$. Moreover, we denote by $\mathbb{N}$ the set of natural numbers $\{1,2,3,\dots\}$ and define $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$.

1.2. Quivers and representations

A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of two sets Q_0, Q_1 with $Q_0 \neq \varnothing$ and maps $s, t \colon Q_1 \longrightarrow Q_0$. An element in Q_0 is called *vertex* and an element in Q_1 is called *arrow*. If α is an arrow, we say that α *starts in* $s(\alpha)$ and *ends in* $t(\alpha)$, and just write $\alpha \colon s(\alpha) \to t(\alpha)$. If $\alpha \colon x \to y$ is an arrow, then by definition $s(\alpha) = x$ and $t(\alpha) = y$. Given a quiver Q, we define the *underlying graph* $\overline{Q} = (Q_0, Q_1, e)$ of Q, where e is the map $e \colon Q_1 \longrightarrow \mathcal{P}^{Q_0}, \alpha \mapsto \{s(\alpha), t(\alpha)\}$ and $\mathcal{P}^{Q_0}$ is the set of all subsets of Q_0. That means that we forget the orientation of arrows. For a vertex $x \in Q_0$ let

$$x^+ \coloneqq \{y \in Q_0 \mid \exists \alpha \colon x \to y \in Q_1\} \text{ and } x^- \coloneqq \{y \in Q_0 \mid \exists \alpha \colon y \to x \in Q_1\}.$$

We define the *neighborhood* $n(x)$ of x as $n(x) \coloneqq x^+ \cup x^-$. For $x, y \in Q_0$ a *path from* x *to* y *of length* $n \in \mathbb{N}$ is a sequence of arrows $\alpha_n \cdots \alpha_1$ with $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i < n$.

The trivial path in x is denoted by ϵ_x and is by definition the unique path of length 0 starting and ending in x.

A path $\alpha_n \cdots \alpha_1$ of length $n \geq 1$ is called *cycle* if $t(\alpha_n) = s(\alpha_1)$. The quiver Q is called *acyclic* if Q has no cycles and is called *connected* if the underlying graph $\overline{Q}$ is connected.

Convention: If not mentioned otherwise, we assume throughout this thesis that Q is

- *finite*, i.e. Q_0 and Q_1 are finite,
- acyclic and
- connected.

Definition. Let Q be a quiver.

(1) A *(finite dimensional) representation* $M = ((M_x)_{x\in Q_0}, (M(\alpha))_{\alpha\in Q_1})$ over Q is a collection consisting of finite dimensional $\Bbbk$-vector spaces M_x and $\Bbbk$-linear maps $M(\alpha)\colon M_{s(\alpha)} \longrightarrow M_{t(\alpha)}$ for every vertex $x \in Q_0$ and every arrow $\alpha \in Q_1$.

(2) A *morphism* $f\colon M \longrightarrow N$ between representations is a collection of linear maps $(f_x)_{x\in Q_0}$ such that for each arrow $\alpha \in Q_1$ there is a commutative diagram

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{M(\alpha)} & M_{t(\alpha)} \\ \downarrow{\scriptstyle f_{s(\alpha)}} & & \downarrow{\scriptstyle f_{t(\alpha)}} \\ N_{s(\alpha)} & \xrightarrow{N(\alpha)} & N_{t(\alpha)}. \end{array}$$

We denote by $\mathrm{Hom}_Q(M, N)$ the corresponding space of morphisms.

(3) The category of finite dimensional representations over Q is denoted by $\mathrm{rep}(Q)$.

Since Q is an acyclic, connected, and finite quiver, the corresponding *path algebra* $\Bbbk Q$ of Q with idempotents e_x ($x \in Q_0$) is a finite-dimensional, basic, and connected $\Bbbk$-algebra. Since the categories $\mathrm{mod}\,\Bbbk Q$ and $\mathrm{rep}(Q)$ are equivalent (see for example [ASS06, (III.1.6)]), we will switch freely between representations of Q and modules of $\Bbbk Q$. Recall the definition of the dimension function

$$\underline{\dim}\colon\ \mathrm{rep}(Q) \longrightarrow \mathbb{N}^{Q_0}\ ;\ M \mapsto (\dim_{\Bbbk} M_x)_{x\in Q_0}.$$

If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence, then $\underline{\dim}\, A + \underline{\dim}\, C = \underline{\dim}\, B$. The quiver Q defines a (non-symmetric) bilinear form $\langle -, -\rangle_Q\colon \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}$ given by

$$((x_i), (y_j)) \mapsto \sum_{i\in Q_0} x_i y_i - \sum_{\alpha\in Q_1} x_{s(\alpha)} y_{t(\alpha)},$$

which coincides with the *Euler-Ringel form* (see [Rin76]) on the Grothendieck group $K_0(\Bbbk Q) \cong \mathbb{Z}^{Q_0}$, i.e. for $M, N \in \mathrm{rep}(Q)$ we have

$$\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle_Q = \dim_{\Bbbk} \mathrm{Hom}_Q(M, N) - \dim_{\Bbbk} \mathrm{Ext}^1_Q(M, N).$$

By $q_Q\colon \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}, x \mapsto \langle x, x\rangle_Q$ we denote the corresponding quadratic form, also called *Tits form*.

Recall that the category $\mathrm{rep}(Q) \cong \mathrm{mod}\,\Bbbk Q$ is *hereditary*, i.e. subrepresentations of projective representations are projective, which is equivalent to $\mathrm{Ext}^2_Q(M, N) = 0$ for all $M, N \in \mathrm{rep}(Q)$.

1.3. The representation type of an algebra

In this section we recall the Trichotomy Theorem for the representation type of finite dimensional path algebras.

Definition. A $\Bbbk$-algebra A has

(1) *finite representation type* if there are only finitely many isomorphism classes of indecomposable A-modules.

(2) *tame representation type* if A does not have finite representation type and for every $n \in \mathbb{N}$, there is a finite set of $A - \Bbbk[T]$-bimodules M_i which are free as right $\Bbbk[T]$-modules such that all but finitely many indecomposable A-modules of dimension n are of the form $M_i \otimes_{\Bbbk[T]} M$ for some i, and for some indecomposable $\Bbbk[T]$-module M.

(3) *wild representation type* if there is a finitely generated $A - \Bbbk\langle X, Y\rangle$-bimodule M which is free as a right $\Bbbk\langle X, Y\rangle$-module such that the functor

$$M \otimes_{\Bbbk\langle X,Y\rangle} - : \ \operatorname{fin} \Bbbk\langle X, Y\rangle \longrightarrow \operatorname{mod} A$$

preserves indecomposable modules and isomorphism classes, where $\operatorname{fin} \Bbbk\langle X, Y\rangle$ denotes the category of finite dimensional $\Bbbk\langle X, Y\rangle$-modules.

The following result goes back to Drozd and is known as the *Trichotomy Theorem*.

Theorem 1.3.1. *A $\Bbbk$-algebra A has finite, tame, or wild representation type, and these types are mutually exclusive.*

For a quiver Q the representation type of $\Bbbk Q$ is determined by the underlying graph $\overline{Q}$:

Theorem 1.3.2. *Let Q be a quiver. The following statements hold.*

(1) $\Bbbk Q$ has finite representation type if and only if $\overline{Q}$ is a simply-laced Dynkin diagram.

(2) $\Bbbk Q$ has tame representation type if and only if $\overline{Q}$ is a simply-laced Euclidean diagram.

(3) In any other case, $\Bbbk Q$ has wild representation type.

Figure 1.1 shows the simply-laced Dynkin diagrams and simply-laced Euclidean diagrams. The simply-laced Dynkin diagrams are the graphs of type A_n, D_n, E_6, E_7, E_8 and shown on the left-hand side. Recall that for $Z \in \{A, D, E\}$ the Dynkin diagram Z_n has n vertices and the Euclidean diagram $\tilde{Z}_n$ has $n + 1$ vertices.

1.4. Auslander-Reiten theory

Let Q be a quiver and $\Bbbk Q$ be the corresponding path algebra. Let $M \in \operatorname{mod} \Bbbk Q$. Then $\operatorname{Ext}^1_{\Bbbk Q}(M, \Bbbk Q)$ is a right $\Bbbk Q$-module, so that

$$\tau_{\Bbbk Q}(M) \coloneqq \operatorname{Ext}^1_{\Bbbk Q}(M, \Bbbk Q)^*$$[1]

[1] For $V \in \operatorname{mod} \Bbbk$, we denote by $V^* \coloneqq \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$ the $\Bbbk$-dual.

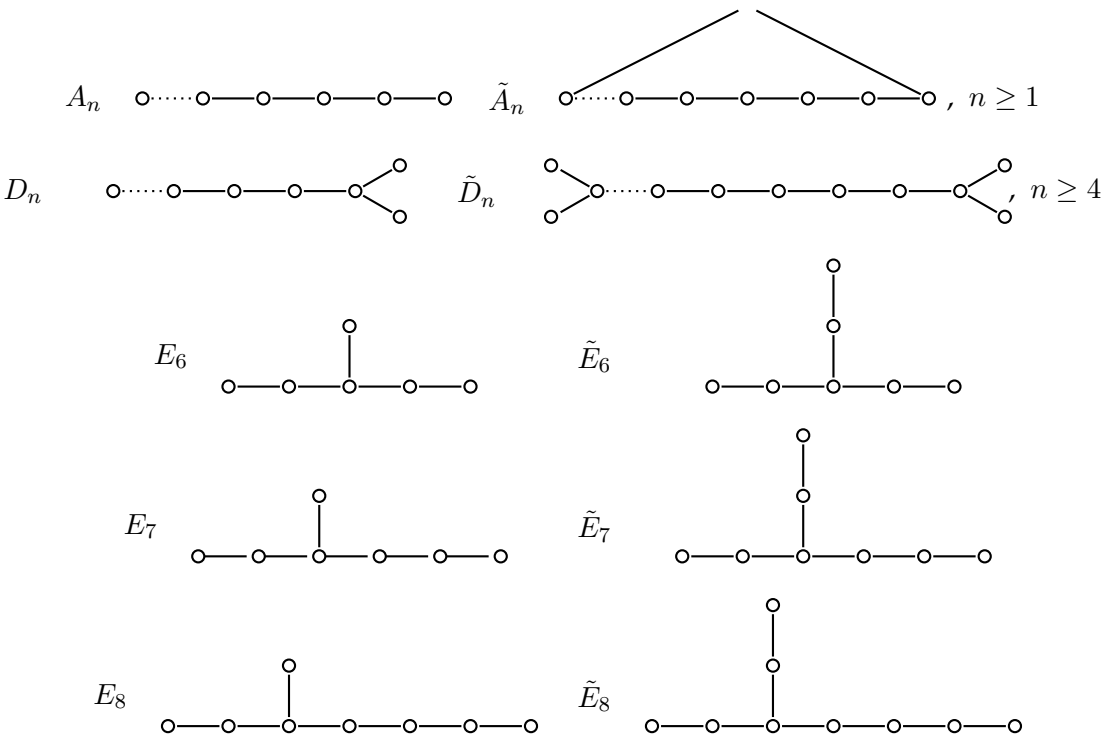


Figure 1.1.: List of simply-laced Dynkin diagrams and simply-laced Euclidean diagrams.

is a left $\Bbbk Q$-module. There results a (covariant) endofunctor

$$\tau_{\Bbbk Q} : \operatorname{mod} \Bbbk Q \longrightarrow \operatorname{mod} \Bbbk Q$$

which, by virtue of $\Bbbk Q$ being hereditary, is left exact. By the same token, $\tau_{\Bbbk Q}$ is exact when restricted to the full subcategory of those modules which have no non-zero projective direct summands.

The functor $\tau_{\Bbbk Q}$ induces an endofunctor on $\operatorname{rep}(Q)$, the so-called *Auslander-Reiten translation*, which we will denote by

$$\tau_Q \colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(Q).$$

Similarly, we consider the functor

$$\tau_{\Bbbk Q}^{-1} : \operatorname{mod} \Bbbk Q \longrightarrow \operatorname{mod} \Bbbk Q \ ; \ M \mapsto \operatorname{Ext}^1_{\Bbbk Q}(M^*, \Bbbk Q)$$

and obtain a functor

$$\tau_Q^{-1} \colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(Q).$$

An indecomposable representation $M \in \operatorname{rep}(Q)$ is called *preprojective (preinjective)*, provided $\tau_Q^n(M) = 0$ (resp. $\tau_Q^{-1}(M) = 0$) for some $n \in \mathbb{N}$. All other indecomposable representations are called *regular*. If Q is a tame or a wild quiver (i.e. the path algebra $\Bbbk Q$ is tame or wild, respectively), the three classes preprojective, preinjective and regular are mutually exclusive.
An arbitrary non-zero representation $M \in \operatorname{rep}(Q)$ is called *preprojective*, *preinjective* or *regular*, provided all its indecomposable direct summands are preprojective, preinjective or regular, respectively. By definition, the zero representation is preprojective, preinjective and regular. Given

an indecomposable representation M that is not projective, there exists a (up to isomorphism of short exact sequences) unique *almost split sequence* or *Auslander-Reiten sequence* (see [ASS06, (IV.1.11)] for the definition)

$$0 \longrightarrow \tau_Q(M) \longrightarrow E \longrightarrow M \longrightarrow 0.$$

By the same token, given N indecomposable and not injective, there exists a unique almost split sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow \tau_Q^{-1}(N) \longrightarrow 0.$$

Almost split sequences give rise to the *Auslander-Reiten quiver* $\Gamma(Q)$ of Q. The vertices are by definition the isomorphism classes of indecomposable representations in $\mathrm{rep}(Q)$. We draw (exactly) n arrows from $[M]$ to $[N]$, if there exists an almost split sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow \tau_Q^{-1}(M) \longrightarrow 0$$

such that N appears with (exactly) multiplicity n as a direct summand in E, or there is an almost split sequence

$$0 \longrightarrow \tau_{K_r}(N) \longrightarrow E \longrightarrow N \longrightarrow 0$$

such that M appears with (exactly) multiplicity n as a direct summand in E. We draw a dashed arrow from $[M]$ to $[N]$ if M is not projective and $\tau_Q(M) \cong N$.

If $\Bbbk Q$ is of tame or wild representation type, the Auslander-Reiten quiver $\Gamma(Q)$ has two Auslander-Reiten components (maximal connected components of the translation quiver $\Gamma(Q)$) $\mathcal{P}_Q, \mathcal{I}_Q$ that contain precisely the preprojective and preinjective indecomposable representations, respectively. All other indecomposable representations lie in so-called *regular* components. If $\Bbbk Q$ is of wild representation type, Ringel [Rin78] has proven that all regular components are of type $\mathbb{Z}A_\infty$. Figure 1.2 shows the shape of such a component. Note that these components are only bounded at the "bottom".

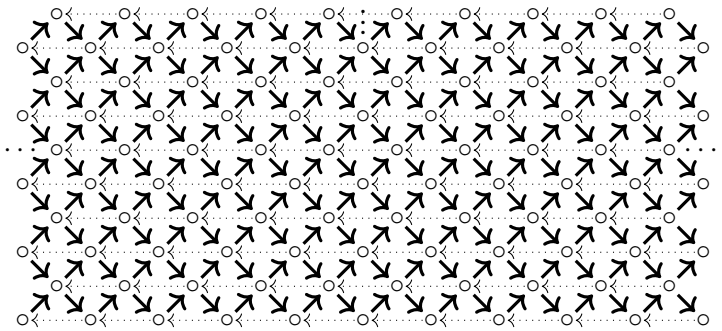


Figure 1.2.: Regular component of type $\mathbb{Z}A_\infty$.

A representation M in a component $\mathcal{C}$ of type $\mathbb{Z}A_\infty$ is called *quasi-simple*, provided M has precisely one direct predecessor. The quasi-simple representations are therefore located at the "bottom" of the component. Recall that an arrow $[M] \longrightarrow [N]$ in the component corresponds to a so-called *irreducible morphism* from M to N. This irreducible morphism is injective if the corresponding arrow is uprising and surjective otherwise. For each $M \in \mathcal{C}$, there is therefore a unique directed path

$$M_{[1]} \to M_{[2]} \to \cdots \to M_{[n-1]} \to M_{[n]} = M$$

in $\mathcal{C}$ such that

(a) for every $i \in \{1, \dots, n-1\}$ the irreducible morphism $M_{[i]} \longrightarrow M_{[i+1]}$ is injective, and

(b) for every $i \in \{1, \dots, n\}$ the factor module $M_{[i]}/M_{[i-1]} \cong \tau_{K_r}^{-i+1}(M_{[1]})$ ($M_{[0]} \coloneqq 0$) is quasi-simple.

In that case $\mathrm{ql}(M) \coloneqq n$ is called the *quasi-length* of M, and we refer to $M_{[1]}$ and $M/M_{[n-1]}$ as the *quasi-socle* and *quasi-top* of M, respectively. The filtration

$$\{0\} \subsetneq M_{[1]} \subsetneq \cdots \subsetneq M_{[n-1]} \subsetneq M$$

is called the *quasi-composition series* of M with *quasi-composition factors*

$$\{\tau_{K_r}^{-i}(M_{[1]}) \mid 0 \leq i \leq \mathrm{ql}(M) - 1\}.$$

1.5. Representations of Kronecker quivers

In this section we specialize the theory introduced in the previous sections to the family of Kronecker quivers. Let $r \in \mathbb{N}$. We denote by K_r the (generalized) *Kronecker quiver*

$$K_r = \qquad 1 \;\overset{\gamma_1}{\underset{\gamma_r}{\rightrightarrows}}\; 2$$

with two vertices 1 and 2, r arrows $\gamma_i \colon 1 \longrightarrow 2$ and *arrow space* $A_r \coloneqq \bigoplus_{i=1}^{r} \Bbbk\gamma_i$.

Moreover, we denote by $\mathcal{K}_r$ the category whose objects are triples

$$M = (M_1, M_2, \psi_M \colon A_r \otimes_{\Bbbk} M_1 \longrightarrow M_2),$$

where M_1, M_2 are finite dimensional $\Bbbk$-vector spaces and ψ_M is a $\Bbbk$-linear map, called the *structure map* of M.

A morphism $f \colon M \longrightarrow N$ in the category $\mathcal{K}_r$ is a pair of $\Bbbk$-linear maps $f_i \colon M_i \longrightarrow N_i$, $i \in \{1, 2\}$ such that the diagram

$$\begin{array}{ccc} A_r \otimes_{\Bbbk} M_1 & \xrightarrow{\psi_M} & M_2 \\ \downarrow{\scriptstyle \mathrm{id}_{A_r} \otimes f_1} & & \downarrow{\scriptstyle f_2} \\ A_r \otimes_{\Bbbk} N_1 & \xrightarrow{\psi_N} & N_2 \end{array}$$

commutes. We have an equivalence of categories

$$\mathrm{rep}(K_r) \longrightarrow \mathcal{K}_r \;\; ; \;\; (M_1, M_2, (M(\gamma_i))_{1 \leq i \leq r}) \mapsto (M_1, M_2, \psi_M)$$

that is the identity on morphisms and on the level objects is given by $\psi_M(\gamma_i \otimes m) = M(\gamma_i)(m)$ for all $i \in \{1, \dots, r\}$ and all $m \in M_1$. From now on we identify $\mathcal{K}_r$ and $\mathrm{rep}(K_r)$, and denote by a_M the $\Bbbk$-linear map

$$a_M \colon M_1 \longrightarrow M_2 \; ; \; m \mapsto \psi_M(a \otimes m)$$

for all $a \in A_r$ and $M \in \operatorname{rep}(K_r)$.

The bilinear form for K_r is given by

$$\langle -,- \rangle_r \coloneqq \langle -,- \rangle_{K_r} \colon \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}\,;\ (x,y) \mapsto x_1y_1 + x_2y_2 - rx_1y_2$$

with corresponding Tits quadratic form $q_r \coloneqq q_{K_r} \colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}\,;\ x \mapsto \langle x,x\rangle_r = x_1^2 + x_2^2 - rx_1x_2$. For all $M, N \in \operatorname{rep}(K_r)$ we have

$$\begin{aligned}&\dim_{\Bbbk} M_1 \dim_{\Bbbk} N_1 + \dim_{\Bbbk} M_2 \dim_{\Bbbk} N_2 - r \dim_{\Bbbk} M_1 \dim_{\Bbbk} N_2\\ =&\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle_r = \dim_{\Bbbk} \operatorname{Hom}_{K_r}(M,N) - \dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(M,N).\end{aligned}$$

Moreover, we denote by $D_{K_r}\colon\ \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_r)$ the standard duality, defined by $D_{K_r}(M) = (M_2^*, M_1^*, \psi_{D_{K_r}(M)})$, where

$$\psi_{D_{K_r}(M)}(a \otimes h) \coloneqq h \circ a_M \colon M_1 \longrightarrow \Bbbk$$

for all $a \in A_r$ and $h \in M_2^*$.

The category $\operatorname{rep}(K_r)$ has two simple objects (representations) $S(1), S(2)$ with dimension vectors $(1,0)$ and $(0,1)$, respectively. The object $S(1)$ is injective, while $S(2)$ is projective. For $i \in \{1,2\}$ we denote by $\operatorname{rep}_i(K_r)$ the full subcategory of $\operatorname{rep}(K_r)$, whose objects do not have any direct summands isomorphic to the simple representation $S(i)$ and set $\operatorname{rep}_{1,2}(K_r) \coloneqq \operatorname{rep}_1(K_r) \cap \operatorname{rep}_2(K_r)$.

General theory [ARS95, (VIII.2.2)] implies that

$$\underline{\dim}\, \tau_{K_r}(M) = \Phi_r . \underline{\dim}\, M$$

for every non-projective indecomposable representation $M \in \operatorname{rep}(K_r)$,[2] where

$$\Phi_r = \begin{pmatrix} r^2-1 & -r \\ r & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

is the *Coxeter matrix* of the quiver K_r. By the same token,

$$\underline{\dim}\, \tau_{K_r}^{-1}(M) = \Phi_r^{-1} . \underline{\dim}\, M$$

for every non-injective indecomposable representation $M \in \operatorname{rep}(K_r)$.

We define $P_0(r) \coloneqq S(2)$ and denote by $P_1(r)$ the projective cover of $S(1)$ with dimension vector $(1,r)$ and structure map

$$\psi_{P_1(r)} \colon A_r \otimes_{\Bbbk} \Bbbk \longrightarrow A_r\ ;\ \gamma_i \otimes \lambda \mapsto \lambda\gamma_i.$$

Let $r = 1$, then the underlying graph of K_1 is the Dynkin diagram A_2, so $\operatorname{rep}(K_r)$ contains up to isomorphism only finitely many non-isomorphic indecomposable representations and they are given $P_0(1)$, $P_1(1)$ and $S(1)$. Figure 1.3 shows the Auslander-Reiten quiver of $\operatorname{rep}(K_1)$.

Let $r = 2$ ($r \geq 3$), then $\Bbbk K_r$ is tame (wild) and the indecomposable representations fall into the three distinct non-empty classes. The preprojective and preinjective indecomposable representations of K_r are well understood: We define

$$P_{2\ell}(r) \coloneqq \tau_{K_r}^{-\ell}(P_0(r))$$

[2] When multiplying matrices with vectors, we will always implicitly assume that the vector has the correct shape and thus dispense with transposing vectors.

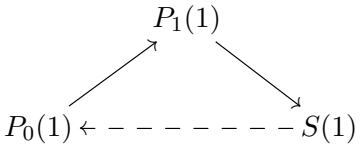


Figure 1.3.: Auslander-Reiten quiver $\Gamma(K_1)$.

and

$$P_{2\ell+1}(r) \coloneqq \tau_{K_r}^{-\ell}(P_1(r))$$

for all $\ell \in \mathbb{N}_0$. The representations $P_i(r)$ form a complete list of representatives of the isomorphism classes of indecomposable preprojective Kronecker representations and for every $i \in \mathbb{N}_0$ there is an almost split sequence

$$0 \longrightarrow P_i(r) \longrightarrow rP_{i+1}(r) \longrightarrow P_{i+2}(r) \longrightarrow 0.$$

By the same token, a complete list of representatives of the isomorphism classes of indecomposable preinjective Kronecker representations is given by $I_i(r) \coloneqq D_{K_r}(P_i(r))$, $i \in \mathbb{N}_0$ with almost split sequences

$$0 \longrightarrow I_{i+2}(r) \longrightarrow rI_{i+1}(r) \longrightarrow I_i(r) \longrightarrow 0.$$

Note that $I_0(r) = S(1)$ and $I_1(r)$ is the injective hull of $P_0(r) = S(2)$. Figure 1.4 shows the shape of the Auslander-Reiten quiver of K_r for $r \geq 2$.

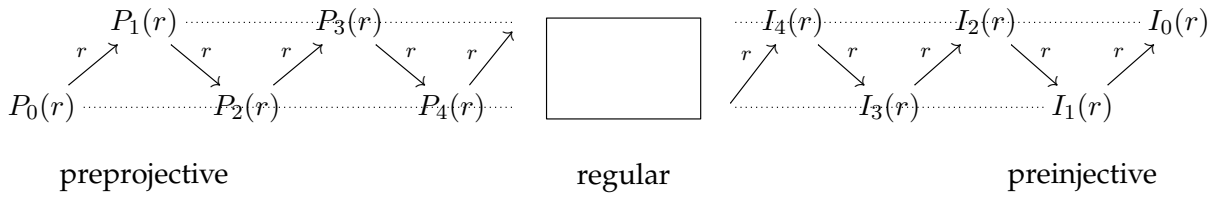


Figure 1.4.: Auslander-Reiten quiver $\Gamma(K_r)$ for $r \geq 2$.

Recall from [ASS06, (VIII.2.7)] that every indecomposable preprojective (preinjective) representation $M \in \operatorname{rep}(K_r)$ is a *brick*, i.e. $\operatorname{End}_{K_r}(M) = \operatorname{id}_M \Bbbk$. We write $\underline{\dim}\, P_i(r) = (a_i(r), a_{i+1}(r))$ and note that $a_0(r) = 0, a_1(r) = 1$, and

$$(*) \quad a_{i+2}(r) = ra_{i+1}(r) - a_i(r)$$

for all $i \in \mathbb{N}_0$. We let $\delta\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2\ ;\ (a,b) \mapsto (b,a)$ be the twist function on $\mathbb{Z}^2$. Then we have

$$\underline{\dim}\, P_i(r) = (a_i(r), a_{i+1}(r)) = \delta(\underline{\dim}\, I_i(r)).$$

We define

$$L_r \coloneqq \frac{r + \sqrt{r^2 - 4}}{2},$$

and note that $(*)$ implies that

$$a_i(r) = \begin{cases} \frac{(L_r)^i - (\frac{1}{L_r})^i}{\sqrt{r^2-4}} & r \geq 3 \\ i & r = 2 \end{cases} \quad \text{for all } i \in \mathbb{N}_0.$$

Moreover, L_r and $\frac{1}{L_r}$ are the roots of the polynomial $f_r \coloneqq X^2 - rX + 1 \in \mathbb{Z}[X]$. Therefore, they satisfy the equations $\frac{1}{L_r} = r - L_r$ and $\frac{1}{L_r} = \frac{r-\sqrt{r^2-4}}{2}$. Moreover, we have $r - 1 < L_r < r$ for $r \geq 3$. We recall a simplified version of Kac's Theorem (see [Kac82, Thm.B] and [Rin76, Thm.3]) and an immediate consequence thereof that suffice for our purposes.

Theorem 1.5.1. *(Kac's Theorem for K_r) Let $r \geq 2$ and $\delta \in \mathbb{N}_0^2 \setminus \{0\}$.*

(1) If $\delta = \underline{\dim}\, M$ for some indecomposable $M \in \operatorname{rep}(K_r)$, then $q_r(\delta) \leq 1$.

(2) If $q_r(\delta) = 1$, then there is a, up to isomorphism, unique indecomposable representation $M \in \operatorname{rep}(K_r)$ such that $\underline{\dim}\, M = \delta$. The representation M is preprojective or preinjective and preprojective if and only if $\delta_1 \leq \delta_2$.

(3) If $q_r(\delta) \leq 0$, then there are infinitely many pairwise non-isomorphic indecomposable representations with dimension vector δ, each being regular.

As a direct consequence, we obtain the following useful result.

Corollary 1.5.2. *Let $r \geq 3$ and $M \in \operatorname{rep}(K_r)$ be indecomposable. The following statements hold.*

(1) M is preprojective if and only if $\dim_{\Bbbk} M_1 < \frac{1}{L_r} \dim_{\Bbbk} M_2$.

(2) M is regular if and only if $\frac{1}{L_r} \dim_{\Bbbk} M_2 < \dim_{\Bbbk} M_1 < L_r \dim_{\Bbbk} M_2$.

(3) M is preinjective if and only if $L_r \dim_{\Bbbk} M_2 < \dim_{\Bbbk} M_1$.

Proof. Let $(x, y) \in \mathbb{N}_0^2$. Direct computation shows that $q_r(x, y) = 0$ holds if and only if $x = yL_r$ or $y = xL_r$. Since $L_r = \frac{r+\sqrt{r^2-4}}{2}$ is irrational, this implies that $q_r(x, y) = 0$ is equivalent to $(x, y) = (0, 0)$.

We have $\underline{\dim}\, M \neq (0, 0)$ and therefore $q_r(\underline{\dim}\, M) < 0$ or $q_r(\underline{\dim}\, M) = 1$ by Theorem 1.5.1. Direct computation shows that $q_r(\underline{\dim}\, M) < 0$ is equivalent to $\frac{1}{L_r} \dim_{\Bbbk} M_2 < \dim_{\Bbbk} M_1 < L_r \dim_{\Bbbk} M_2$.

Hence $q_r(\underline{\dim}\, M) > 0$ ($\Leftrightarrow q_r(\underline{\dim}\, M) = 1$) is equivalent to $\dim_{\Bbbk} M_1 < \frac{1}{L_r} \dim_{\Bbbk} M_2$ or $\dim_{\Bbbk} M_1 > L_r \dim_{\Bbbk} M_2$. Since $\dim_{\Bbbk} M_1 < \frac{1}{L_r} \dim_{\Bbbk} M_2$ implies $\dim_{\Bbbk} M_1 < \dim_{\Bbbk} M_2$ and $\dim_{\Bbbk} M_1 > L_r \dim_{\Bbbk} M_2$ implies $\dim_{\Bbbk} M_1 > \dim_{\Bbbk} M_2$, the statement follows from Theorem 1.5.1. □

2. Restriction, inflation and reflection functors

Let Q be a quiver and $x \in Q_0$ be a *sink*. By definition we have $x^+ = \varnothing$. We denote by Q' the quiver obtained from Q by reversing all arrows ending in x. This turns x into a *source* in Q', that is, $x^- = \varnothing$ in Q'. We denote by

$$\sigma_Q \colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(Q')$$

and

$$\sigma_{Q'}^{-1} \colon \operatorname{rep}(Q') \longrightarrow \operatorname{rep}(Q)$$

the classical reflection functors at the sink $x \in Q_0$ and the source $x \in Q_0'$, respectively (see [ASS06, (VII.5.7)] for the definition). These functors play a fundamental role in the proof of Gabriel's theorem and it is well-known that $\sigma_{Q'}^{-1}$ is left adjoint to σ_Q. In the following we prove a generalization of this result. In addition to reversing all arrows ending in $x \in Q_0$, we also allow the deletion of arbitrary arrows $x \to y$ in Q_0' starting in $x \in Q_0'$ as long as x and y remain neighbors after deletion.

2.1. Restriction, inflation and reflection

Throughout we fix a natural number $n \in \mathbb{N}$ and consider a quiver Q with vertex set $\{0, 1, \dots, n\} \subseteq Q_0 \subseteq \mathbb{N}_0$ such that 0 is a sink and $0^- = \{1, \dots, n\}$ are the neighbors of 0 in Q. For $1 \leq i \leq n$ we let $m_i := |\{\alpha \in Q_1 \mid \alpha \colon i \to 0\}| \in \mathbb{N}$ and denote by

$$\gamma_{i,j} \colon i \to 0, \quad 1 \leq j \leq m_i$$

the arrows from i to 0. For each $1 \leq i \leq n$ we fix a natural number $1 \leq s_i \leq m_i$.

We denote by $\operatorname{res}(Q)$ the connected quiver obtained by removing the arrows $\gamma_{i,j}$ for all $i \in \{1, \dots, n\}$ and $j > s_i$.

We denote by $\operatorname{res}(Q)'$ the quiver obtained from $\operatorname{res}(Q)$ by reversing all arrows ending in 0. These new arrows will be denoted by $\gamma'_{i,j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq s_i$. All other names remain unchanged.

Moreover, we denote by Q' the quiver obtained from Q by reversing all arrows ending in 0. These new arrows will be denoted by $\gamma'_{i,j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$. All other names remain unchanged.

Finally, we denote by $\operatorname{res}(Q')$ the quiver obtained from Q' by removing the arrows $\gamma'_{i,j}$ for all $i \in \{1, \dots, n\}$ and $j > s_i$.

Remark 2.1.1. By definition we have $\operatorname{res}(Q)' = \operatorname{res}(Q')$, i.e. it does not matter if we first remove or reverse the arrows. Hence we have a commutative diagram as shown in Figure 2.1.

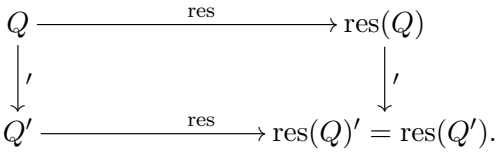


Figure 2.1.: Illustration of the involved quivers.

Example 2.1.2. *We consider an example such that* $0^- = \{1, 2, 3\}$ *with* $m_1 = 4, m_2 = 2$ *and* $m_3 = 1$*. Now we fix* $s_1 = 2$*,* $s_2 = 1$*, and* $s_3 = 1$ *(note that* $s_3 = 1$ *is the only choice). The neighborhoods of* 0 *in* Q *and* $\mathrm{res}(Q') = \mathrm{res}(Q)'$ *are illustrated in Figure 2.2.*



Figure 2.2.: Neighborhoods of 0 in Q and $\mathrm{res}(Q') = \mathrm{res}(Q)'$ without labeling of the arrows.

Given $M \in \mathrm{rep}(Q)$, we denote by $\mathrm{res}(M) \in \mathrm{rep}(\mathrm{res}(Q))$ the *restriction* to $\mathrm{res}(Q)$. By definition we have $\mathrm{res}(M)_x = M_x$ for all $x \in \mathrm{res}(Q)_0 = Q_0$ and $\mathrm{res}(M)(\gamma) = M(\gamma)$ for $\gamma \in \mathrm{res}(Q)_1$.

Given $X \in \mathrm{rep}(\mathrm{res}(Q'))$, we denote by $\inf(X)$ the *inflation* to Q'. By definition we have $\inf(X)_x = X_x$ for all $x \in \mathrm{res}(Q')_0 = Q'_0$, $\inf(X)(\gamma) = X(\gamma)$ for $\gamma \in \mathrm{res}(Q')_1$, and $\inf(X)(\delta) = 0$ for all $\delta \in Q'_1 \setminus \mathrm{res}(Q')_1$. These assignments extend in a canonical way to functors

$$\mathrm{res}\colon \mathrm{rep}(Q) \longrightarrow \mathrm{rep}(\mathrm{res}(Q))$$

and

$$\inf\colon \mathrm{rep}(\mathrm{res}(Q')) \longrightarrow \mathrm{rep}(Q').$$

Since the proof of Theorem 2.2.2 is quite technical, we recall in the following the definition of the reflection functors in our special case to make it easier for the reader to follow the arguments.

2.1.1. Reflection at the sink 0 in res(Q)

We denote by $\sigma_{\mathrm{res}(Q)}\colon \mathrm{rep}(\mathrm{res}(Q)) \longrightarrow \mathrm{rep}(\mathrm{res}(Q)')$ the reflection functor at the sink 0 in $\mathrm{res}(Q)$.

Let $N \in \mathrm{rep}(\mathrm{res}(Q))$. By definition $\sigma_{\mathrm{res}(Q)}(N)_\ell = N_\ell$ for all $\ell \in Q_0 \setminus \{0\}$ and $\sigma_{\mathrm{res}(Q)}(N)_0 = \ker \chi_N$, where

$$\chi_N\colon \bigoplus_{i=1}^{n} N_i^{s_i} \longrightarrow N_0\ ;\ (y_1, \ldots, y_n) \mapsto \sum_{i=1}^{n} \sum_{j=1}^{s_i} N(\gamma_{i,j})(y_{i,j}).$$

Here we write $y_i = (y_{i,1}, \dots, y_{i,s_i})$ for all $i \in \{1, \dots, n\}$. Moreover, for $i \in \{1, \dots, n\}$ and $1 \leq j \leq s_i$, the map $\sigma_{\mathrm{res}(Q)}(N)(\gamma'_{i,j})\colon \ker \chi_N \longrightarrow N_i$ is the embedding of $\ker \chi_N$ into $\bigoplus_{\ell=1}^n N_\ell^{s_\ell}$ followed by the projection onto the j-th component of $N_i^{s_i}$. All other linear maps remain the same. Given $f\colon N \longrightarrow N' \in \mathrm{Hom}_{\mathrm{res}(Q)}(N, N')$, we have a morphism

$$\sigma_{\mathrm{res}(Q)}(f)\colon \sigma_{\mathrm{res}(Q)}(N) \longrightarrow \sigma_{\mathrm{res}(Q)}(N') \in \mathrm{Hom}_{\mathrm{res}(Q)'}(\sigma_{\mathrm{res}(Q)}(N), \sigma_{\mathrm{res}(Q)}(N'))$$

defined by $\sigma_{\mathrm{res}(Q)}(f)_\ell \coloneqq f_\ell$ for all $0 \neq \ell \in Q_0$ and $\sigma_{\mathrm{res}(Q)}(f)_0$ is the restriction of

$$\oplus_{i=1}^n f_i^{s_i}\colon \bigoplus_{i=1}^n N_i^{s_i} \longrightarrow \bigoplus_{i=1}^n (N'_i)^{s_i}$$

to $\ker \chi_N \subseteq \bigoplus_{i=1}^n N_i^{s_i}$, i.e. the unique $\Bbbk$-linear map making the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \ker \chi_N & \longrightarrow & \bigoplus_{i=1}^n N_i^{s_i} \xrightarrow{\chi_N} N_0 \\ & & \downarrow{\scriptstyle \sigma_{\mathrm{res}(Q)}(f)_0} & & \downarrow{\scriptstyle \oplus_{i=1}^n f_i^{s_i}} \qquad \downarrow{\scriptstyle f_0} \\ 0 & \longrightarrow & \ker \chi_{N'} & \longrightarrow & \bigoplus_{i=1}^n (N'_i)^{s_i} \xrightarrow{\chi_{N'}} N'_0 \end{array}$$

commute.

2.1.2. Reflection at the source 0 in Q'

We denote by $\sigma_{Q'}^{-1}\colon \mathrm{rep}(Q') \longrightarrow \mathrm{rep}(Q)$ the reflection functor at the source 0 in Q'. Let $Y \in \mathrm{rep}(Q')$. By definition $\sigma_{Q'}^{-1}(Y)_\ell = Y_\ell$ for all $\ell \neq 0$ and $\sigma_{Q'}^{-1}(Y)_0 = \mathrm{coker}\, \eta_Y$, where

$$\eta_Y\colon Y_0 \longrightarrow \bigoplus_{i=1}^n Y_i^{m_i}\ ;\ y \mapsto (Y(\gamma'_{i,j})(y))_{1 \leq i \leq n, 1 \leq j \leq m_i}.$$

Moreover, for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m_i\}$, the map $\sigma_{Q'}^{-1}(Y)(\gamma_{i,j})\colon Y_i \longrightarrow \mathrm{coker}\, \eta_Y$ is the embedding of Y_i into the j-th component of $Y_i^{m_i} \subseteq \bigoplus_{\ell=1}^n Y_\ell^{m_\ell}$ followed by the projection onto the cokernel. All other linear maps remain the same. Given $f\colon Y \longrightarrow Y' \in \mathrm{Hom}_{Q'}(Y, Y')$, we obtain a morphism

$$\sigma_{Q'}^{-1}(f)\colon \sigma_{Q'}^{-1}(Y) \longrightarrow \sigma_{Q'}^{-1}(Y') \in \mathrm{Hom}_Q(\sigma_{Q'}^{-1}(Y), \sigma_{Q'}^{-1}(Y'))$$

given by $\sigma_{Q'}^{-1}(f)_\ell = f_\ell$ for all $0 \neq \ell \in Q_0$ and $\sigma_{Q'}^{-1}(f)_0$ is the unique $\Bbbk$-linear map making the diagram

$$\begin{array}{ccccccc} Y_0 & \xrightarrow{\eta_Y} & \bigoplus_{i=1}^n Y_i^{m_i} & \longrightarrow & \mathrm{coker}\, \eta_Y & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \oplus_{i=1}^n f_i^{m_i}} & & \downarrow{\scriptstyle \sigma_{Q'}^{-1}(f)_0} & & \\ Y'_0 & \xrightarrow{\eta_{Y'}} & \bigoplus_{i=1}^n (Y'_i)^{m_i} & \longrightarrow & \mathrm{coker}\, \eta_{Y'} & \longrightarrow & 0 \end{array}$$

commute.

2.2. The adjoint pair

The aim of this section is to prove that the functors $\sigma_{Q'}^{-1} \circ \inf\colon \operatorname{rep}(\operatorname{res}(Q')) \longrightarrow \operatorname{rep}(Q)$ and $\sigma_{\operatorname{res}(Q)} \circ \operatorname{res}\colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(\operatorname{res}(Q'))$ form an adjoint pair. Before we give the proof, we consider the following example to show that it does not suffice to consider $\inf$ and res.

Example 2.2.1. *Let $1 \leq d < r$, $Q = K_r$ and $\operatorname{res}(Q) = K_d$, i.e. we obtain K_d from K_r by forgetting the arrows γ_i for $d < i \leq r$. We consider the inflation and restriction functor $\inf\colon \operatorname{rep}(K_d) \longrightarrow \operatorname{rep}(K_r)$, $\operatorname{res}\colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_d)$. We have $\operatorname{res}(P_1(r)) \cong P_1(d) \oplus (r-d)P_0(d)$ and*

$$1 = \dim_{\Bbbk} \operatorname{res}(P_1(d))_1 = \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_1(d), \operatorname{res}(P_1(r))).$$

The representation $\inf(P_1(d)) \in \operatorname{rep}(K_r)$ is indecomposable and regular (see Theorem 1.5.1). Since there are no non-zero morphisms from regular to projective representations (see [ASS06, (VIII.2.13)]), we conclude

$$\operatorname{Hom}_{K_r}(\inf(P_1(d)), P_1(r)) = 0 \neq \operatorname{Hom}_{K_d}(P_1(d), \operatorname{res}(P_1(r))).$$

By the same token, we have

$$\operatorname{Hom}_{K_r}(I_1(r), \inf(I_1(d))) = 0 \neq \operatorname{Hom}_{K_d}(\operatorname{res}(I_1(r)), I_1(d)).$$

This shows that we can not hope for natural isomorphisms of the form

$$\operatorname{Hom}_Q(\inf(X), M) \cong \operatorname{Hom}_{\operatorname{res}(Q)}(X, \operatorname{res}(M)) \text{ and } \operatorname{Hom}_Q(M, \inf(X)) \cong \operatorname{Hom}_{\operatorname{res}(Q)}(\operatorname{res}(M), X)$$

for a quiver Q as considered in Section 2.1.

Theorem 2.2.2. *The functor*

$$\sigma_{Q'}^{-1} \circ \inf\colon \operatorname{rep}(\operatorname{res}(Q')) \longrightarrow \operatorname{rep}(Q)$$

is left adjoint to

$$\sigma_{\operatorname{res}(Q)} \circ \operatorname{res}\colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(\operatorname{res}(Q')).$$

Proof. Let $X \in \operatorname{rep}(\operatorname{res}(Q'))$ and $M \in \operatorname{rep}(Q)$. We start by defining a $\Bbbk$-linear map

$$\tau_{X,M}\colon \operatorname{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), M) \longrightarrow \operatorname{Hom}_{\operatorname{res}(Q')}(X, (\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(M)).$$

Let $f \in \operatorname{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), M)$. We define $\tau_{X,M}(f)_\ell = f_\ell$ for all $\ell \in Q_0 \setminus \{0\} = \operatorname{res}(Q')_0 \setminus \{0\}$. We consider the maps

$$\oplus_{i=1}^n f_i^{s_i}\colon \bigoplus_{i=1}^n X_i^{s_i} \longrightarrow \bigoplus_{i=1}^n M_i^{s_i}$$

and

$$\nu_X\colon X_0 \longrightarrow \bigoplus_{i=1}^n X_i^{s_i}\ ;\ x \mapsto (X(\gamma'_{i,j})(x))_{1 \leq i \leq n, 1 \leq j \leq s_i}.$$

For $x \in X_0$, we have

$$(\oplus_{i=1}^n f_i^{s_i} \circ \nu_X)(x) \in \bigoplus_{i=1}^n M_i^{s_i},$$

$\operatorname{res}(M) \in \operatorname{rep}(\operatorname{res}(Q))$ and

$$\begin{aligned}\chi_{\operatorname{res}(M)}((\oplus_{i=1}^{n} f_i^{s_i} \circ \nu_X)(x)) &= \sum_{i=1}^{n}\sum_{j=1}^{s_i}[M(\gamma_{i,j}) \circ f_i \circ X(\gamma'_{i,j})](x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{s_i}[f_0 \circ \sigma_{Q'}^{-1}(\inf(X))(\gamma_{i,j}) \circ X(\gamma'_{i,j})](x) \\ &= f_0(\sum_{i=1}^{n}\sum_{j=1}^{s_i}[\sigma_{Q'}^{-1}(\inf(X))(\gamma_{i,j}) \circ X(\gamma'_{i,j})](x)) \\ &= f_0(\sum_{i=1}^{n}\sum_{j=1}^{m_i}[\sigma_{Q'}^{-1}(\inf(X))(\gamma_{i,j}) \circ \inf(X)(\gamma'_{i,j})](x)),\end{aligned}$$

since $\inf(X)(\gamma'_{i,j}) = 0$ for all $j > s_i$ and $\inf(X)(\gamma'_{i,j}) = X(\gamma'_{i,j})$ for $1 \leq j \leq s_i$. We have $Y \coloneqq \inf(X) \in \operatorname{rep}(Q')$ and the definition (see Section 2.1.2) gives us

$$\begin{aligned}\chi_{\operatorname{res}(M)}((\oplus_{i=1}^{n} f_i^{s_i} \circ \nu_X)(x)) &= f_0[\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sigma_{Q'}^{-1}(Y)(\gamma_{i,j})(Y(\gamma'_{i,j})(x))] \\ &= f_0(\eta_Y(x) + \operatorname{im}\eta_Y) = f_0(0) = 0.\end{aligned}$$

In conclusion, we obtain a $\Bbbk$-linear map

$$\tau_{X,M}(f)_0 \coloneqq \oplus_{i=1}^{n} f_i^{s_i} \circ \nu_X \colon X_0 \longrightarrow \ker \chi_{\operatorname{res}(M)} = (\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(M)_0.$$

We proceed in steps:

(i) $\tau_{X,M}(f) \in \operatorname{Hom}_{\operatorname{res}(Q')}(X, (\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(M))$: By definition it suffices to show that

$$(\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(M)(\gamma'_{i,j}) \circ \tau_{X,M}(f)_0 = \tau_{X,M}(f)_i \circ X(\gamma'_{i,j})$$

for all $i \in \{1, \ldots, n\}$, $1 \leq j \leq s_i$. Let $1 \leq a \leq n$, $1 \leq b \leq s_a$ and $x \in X_0$. We have

$$\begin{aligned}[(\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(M)(\gamma'_{a,b}) \circ \tau_{X,M}(f)_0](x) &= \sigma_{\operatorname{res}(Q)}(\operatorname{res}(M))(\gamma'_{a,b})((f_i(X(\gamma'_{i,j})(x)))_{1\leq i\leq n, 1\leq j\leq s_i}) \\ &= f_a(X(\gamma'_{a,b})(x)) = [f_a \circ X(\gamma'_{a,b})](x) \\ &= [\tau_{X,M}(f)_a \circ X(\gamma'_{a,b})](x).\end{aligned}$$

(ii) $\tau_{X,M}$ is injective: Assume that $\tau_{X,M}(f) = 0$. By definition $f_\ell = 0$ for all $\ell \in Q_0 \setminus \{0\}$. It remains to show that

$$f_0 \colon \operatorname{coker}\eta_{\inf(X)} \longrightarrow M_0$$

is the zero map. Let $z = x + \operatorname{im}\eta_{\inf(X)} \in \operatorname{coker}\eta_{\inf(X)}$ with $x = (x_1, \ldots, x_n) \in \bigoplus_{i=1}^{n} X_i^{m_i}$. We have

$$z = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\sigma_{Q'}^{-1}(\inf(X))(\gamma_{i,j})(x_{i,j})$$

and conclude

$$f_0(z) = f_0(\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sigma_{Q'}^{-1}(\inf(X))(\gamma_{i,j})(x_{i,j})) = \sum_{i=1}^{n}\sum_{j=1}^{m_i}[f_0\circ\sigma_{Q'}^{-1}(\inf(X))(\gamma_{i,j})](x_{i,j})$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m_i}[M(\gamma_{i,j})\circ f_i](x_{i,j}) = 0,$$

since $f_i = 0$ for all $i \in Q_0 \setminus \{0\}$.

(iii) $\tau_{X,M}$ is surjective: We define

$$\zeta\colon \bigoplus_{i=1}^{n} M_i^{m_i} \longrightarrow M_0\ ;\ (m_1,\ldots,m_n) \mapsto \sum_{i=1}^{n}\sum_{j=1}^{m_i} M(\gamma_{i,j})(m_{i,j}).$$

Let $g \in \mathrm{Hom}_{\mathrm{res}(Q')}(X, (\sigma_{\mathrm{res}(Q)}\circ\mathrm{res})(M))$ and $x \in \inf(X)_0 = X_0$. Since $\inf(X)(\gamma'_{i,j}) = 0$ for $s_i < j$ and $\inf(X)(\gamma'_{i,j}) = X(\gamma'_{i,j})$ for $1 \le i \le s_i$ we have

$$[\zeta\circ\oplus_{i=1}^{n} g_i^{m_i}\circ\eta_{\inf(X)}](x) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} M(\gamma_{i,j})(g_i(\inf(X)(\gamma'_{i,j})(x)))$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{s_i} M(\gamma_{i,j})(g_i(\inf(X)(\gamma'_{i,j})(x))) = \sum_{i=1}^{n}\sum_{j=1}^{s_i} M(\gamma_{i,j})(g_i(X(\gamma'_{i,j})(x)))$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{s_i}[M(\gamma_{i,j})\circ g_i\circ X(\gamma'_{i,j})](x) = \sum_{i=1}^{n}\sum_{j=1}^{s_i}[M(\gamma_{i,j})\circ(\sigma_{\mathrm{res}(Q)}\circ\mathrm{res})(M)(\gamma'_{i,j})\circ g_0](x).$$

We write $g_0(x) \in \sigma_{\mathrm{res}(Q)}(\mathrm{res}(M))_0 = \ker\chi_{\mathrm{res}(M)} \subseteq \bigoplus_{i=1}^{n} M_i^{s_i}$ as $g_0(x) = (m_1,\ldots,m_n)$. By definition we have

$$[\sigma_{\mathrm{res}(Q)}(\mathrm{res}(M))(\gamma'_{i,j})\circ g_0](x) = \sigma_{\mathrm{res}(Q)}(\mathrm{res}(M))(\gamma'_{i,j})(g_0(x)) = m_{i,j}$$

and conclude

$$[\zeta\circ\oplus_{i=1}^{n} g_i^{m_i}\circ\eta_{\inf(X)}](x) = \sum_{i=1}^{n}\sum_{j=1}^{s_i} M(\gamma_{i,j})(m_{i,j}) = \chi_{\mathrm{res}(M)}(g_0(x)) = 0.$$

Hence there exists a unique $\Bbbk$-linear map $h_0\colon \operatorname{coker}\eta_{\inf(X)} \longrightarrow M_0$ making the diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\eta_{\inf(X)}} & \bigoplus_{\ell=1}^{n} X_\ell^{m_\ell} & \xrightarrow{\pi} & \operatorname{coker}\eta_{\inf(X)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \oplus_{\ell=1}^{n} g_\ell^{m_\ell}} & & \vdots{\scriptstyle h_0} & & \\ & & \bigoplus_{\ell=1}^{n} M_\ell^{m_\ell} & \xrightarrow{\zeta} & M_0 & & \end{array}$$

commute. Let $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,m_i\}$ and $\iota\colon X_i \longrightarrow \bigoplus_{\ell=1}^{n} X_\ell^{m_\ell}$ be the embedding of X_i into the j-th component of $X_i^{m_i} \subseteq \bigoplus_{\ell=1}^{n} X_\ell^{m_\ell}$. Then $\pi\circ\iota = (\sigma_{Q'}^{-1}\circ\inf)(X)(\gamma_{i,j})$ by definition. Hence the commutative diagram yields

$$(*)\quad M(\gamma_{i,j})\circ g_i = h_0\circ(\sigma_{Q'}^{-1}\circ\inf)(X)(\gamma_{i,j}).$$

We define $h_i := g_i$ for all $0 \neq i \in Q_0$. Then $(*)$ implies $h := (h_i)_{i \in Q_0} \in \mathrm{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), M)$. We consider the morphism

$$f := \tau_{X,M}(h) - g \in \mathrm{Hom}_{\mathrm{res}(Q')}(X, (\sigma_{\mathrm{res}(Q)} \circ \mathrm{res})(M))$$

and note that $f_i = 0$ for all $i \neq 0$. Let $x \in X_0$, then $f_0(x) \in (\sigma_{\mathrm{res}(Q)} \circ \mathrm{res})(M)_0 = \ker \chi_{\mathrm{res}(M)} \subseteq \bigoplus_{i=1}^n M_i^{s_i}$. By definition (see Section 2.1.1), we have

$$[(\sigma_{\mathrm{res}(Q)} \circ \mathrm{res})(M)(\gamma'_{i,j})(f_0(x_0))]_{1 \leq i \leq n, 1 \leq j \leq s_i} = f_0(x_0).$$

Since

$$[\sigma_{\mathrm{res}(Q)}(\mathrm{res}(M))(\gamma'_{i,j}) \circ f_0](x_0) = [f_i \circ X(\gamma'_{i,j})](x_0) = 0$$

for all $i \in \{1, \ldots, n\}$ and $1 \leq j \leq s_i$, we conclude $f_0(x) = 0$ and $f_0 = 0$. This means $\tau_{X,M}(h) = g$.

(iv) τ is natural in the first component: We let $X, Y \in \mathrm{rep}(\mathrm{res}(Q'))$, $M \in \mathrm{rep}(Q)$ be representations and $f \in \mathrm{Hom}_{\mathrm{res}(Q')}(X, Y)$ and $g \in \mathrm{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(Y), M)$ be morphisms. For $0 \neq \ell \in Q_0$ we have

$$\begin{aligned}
[\tau_{X,M} \circ \mathrm{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(f), M)](g)_\ell &= \tau_{X,M}(g \circ (\sigma_{Q'}^{-1} \circ \inf)(f))_\ell = (g \circ (\sigma_{Q'}^{-1} \circ \inf)(f))_\ell \\
&= g_\ell \circ f_\ell = (\tau_{Y,M}(g) \circ f)_\ell \\
&= [\mathrm{Hom}_{\mathrm{res}(Q')}(f, (\sigma_{\mathrm{res}(Q)} \circ \mathrm{res})(M)) \circ \tau_{Y,M}](g)_\ell.
\end{aligned}$$

Note that we have a commutative diagram

$$\begin{array}{ccc}
X_0 & \xrightarrow{\nu_X} & \bigoplus_{i=1}^n X_i^{s_i} \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \oplus_{i=1}^n f_i^{s_i}} \\
Y_0 & \xrightarrow{\nu_Y} & \bigoplus_{i=1}^n Y_i^{s_i}
\end{array}$$

since $f \in \mathrm{Hom}_{\mathrm{res}(Q')}(X, Y)$. We conclude

$$\begin{aligned}
[\mathrm{Hom}_{\mathrm{res}(Q')}(f, (\sigma_{\mathrm{res}(Q)} \circ \mathrm{res})(M)) \circ \tau_{Y,M}](g)_0 &= (\tau_{Y,M}(g) \circ f)_0 \\
&= \tau_{Y,M}(g)_0 \circ f_0 = \oplus_{i=1}^n g^{s_i} \circ \nu_Y \circ f_0 \\
&= \oplus_{i=1}^n g^{s_i} \circ \oplus_{i=1}^n f_i^{s_i} \circ \nu_X \\
&= \oplus_{i=1}^n g_i^{s_i} \circ \oplus_{i=1}^n (\sigma_{Q'}^{-1} \circ \inf)(f)_i^{s_i} \circ \nu_X \\
&= \oplus_{i=1}^n (g \circ (\sigma_{Q'}^{-1} \circ \inf)(f))_i^{s_i} \circ \nu_X \\
&= \tau_{X,M}(g \circ (\sigma_{Q'}^{-1} \circ \inf)(f))_0 \\
&= [\tau_{X,M} \circ \mathrm{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(f), M)](g)_0.
\end{aligned}$$

Hence we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(Y), M) & \xrightarrow{\tau_{Y,M}} & \mathrm{Hom}_{\mathrm{res}(Q')}(Y, (\sigma_{\mathrm{res}(Q)} \circ \mathrm{res})(M)) \\
\downarrow{\scriptstyle \mathrm{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(f), M)} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathrm{res}(Q')}(f, (\sigma_{\mathrm{res}(Q)} \circ \mathrm{res})(M))} \\
\mathrm{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), M) & \xrightarrow{\tau_{X,M}} & \mathrm{Hom}_{\mathrm{res}(Q')}(X, (\sigma_{\mathrm{res}(Q)} \circ \mathrm{res})(M)).
\end{array}$$

(v) τ is natural in the second component: We let $X \in \operatorname{rep}(\operatorname{res}(Q'))$, $M, N \in \operatorname{rep}(Q)$, $f \in \operatorname{Hom}_Q(M,N)$ and $g \in \operatorname{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), M)$. For $0 \neq \ell \in Q_0$ we have

$$\begin{aligned}[\tau_{X,N} \circ \operatorname{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), f)](g)_\ell &= \tau_{X,N}(f \circ g)_\ell = f_\ell \circ g_\ell \\ &= [(\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(f) \circ \tau_{X,M}(g)]_\ell \\ &= [\operatorname{Hom}_{\operatorname{res}(Q')}(X, (\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(f)) \circ \tau_{X,M}](g)_\ell.\end{aligned}$$

Moreover,

$$\begin{aligned}&[\tau_{X,N} \circ \operatorname{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), f)](g)_0 = \tau_{X,N}(f \circ g)_0 = \oplus_{i=1}^n (f_i \circ g_i)^{s_i} \circ \nu_X \\ =& \oplus_{i=1}^n \operatorname{res}(f)_i^{s_i}|_{\ker \chi_{\operatorname{res}(M)}} \circ (\oplus_{i=1}^n g_i^{s_i} \circ \nu_X) = [\sigma_{\operatorname{res}(Q)}(\operatorname{res}(f)) \circ \tau_{X,M}(g)]_0 \\ =& [\operatorname{Hom}_{\operatorname{res}(Q')}(X, (\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(f)) \circ \tau_{X,M}](g)_0\end{aligned}$$

Hence we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), M) & \xrightarrow{\tau_{X,M}} & \operatorname{Hom}_{\operatorname{res}(Q')}(X, (\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(M)) \\ \downarrow{\scriptstyle \operatorname{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), f)} & & \downarrow{\scriptstyle \operatorname{Hom}_{\operatorname{res}(Q')}(X, (\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(f))} \\ \operatorname{Hom}_Q((\sigma_{Q'}^{-1} \circ \inf)(X), N) & \xrightarrow{\tau_{X,N}} & \operatorname{Hom}_{\operatorname{res}(Q')}(X, (\sigma_{\operatorname{res}(Q)} \circ \operatorname{res})(N)). \end{array}$$

□

Remark 2.2.3. If we consider $\operatorname{res}(Q) = Q$, i.e. we do not remove any arrows, we obtain the classical result that $\sigma_{Q'}^{-1}$ is left adjoint to σ_Q. (cf. [ASS06, (VII.5.7)]).

For completeness, we also recall the dual statement. Let Q be a finite connected quiver with vertex set $\{0, 1, \ldots, n\} \subseteq Q_0 \subseteq \mathbb{N}_0$ such that 0 is a source and the elements in $\{1, \ldots, n\}$ are the neighbors of 0 in Q. For $1 \leq i \leq n$ we let $m_i := |\{\alpha \in Q_1 \mid \alpha \colon 0 \to i\}| \in \mathbb{N}$ and denote by

$$\gamma_{i,j} \colon 0 \to i, \quad 1 \leq j \leq m_i$$

the arrows from 0 to i. For each $1 \leq i \leq n$ we fix a natural number $1 \leq s_i \leq m_i$. After proper modification of the notation introduced in Section 2.1, similar arguments give the following result.

Theorem 2.2.4. *The functor*

$$\sigma_{\operatorname{res}(Q)}^{-1} \circ \operatorname{res} \colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(\operatorname{res}(Q'))$$

is left adjoint to

$$\sigma_{Q'} \circ \inf \colon \operatorname{rep}(\operatorname{res}(Q')) \longrightarrow \operatorname{rep}(Q).$$

Remark 2.2.5. We give a detailed proof of Theorem 2.2.4 for the special case $Q = K_r$ in Section 2.4.

2.3. Shift functors for Kronecker quivers

In the sequel the *shift functors* $\sigma_{K_r}, \sigma_{K_r}^{-1} : \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_r)$ will be of major importance. These functors correspond to the reflection functors defined in Sections 2.1.1 and 2.1.2 but take into account that the opposite quiver of K_r is isomorphic to K_r, i.e. $D_{K_r} \circ \sigma_{K_r} \cong \sigma_{K_r}^{-1} \circ D_{K_r}$. We recall the definitions of σ_{K_r} and $\sigma_{K_r}^{-1}$ in terms of the corresponding structure maps. Given a representation $M \in \operatorname{rep}(K_r)$, the structure map

$$\psi_{\sigma_{K_r}(M)} \colon A_r \otimes_{\Bbbk} \ker \psi_M \longrightarrow M_1$$

of $\sigma_{K_r}(M) \in \operatorname{rep}(K_r)$ is the restriction of the $\Bbbk$-linear map

$$A_r \otimes_{\Bbbk} (A_r \otimes_{\Bbbk} M_1) \longrightarrow M_1 \; ; \; \gamma_i \otimes (\gamma_j \otimes m) \mapsto \delta_{ij} m.$$

If $f \in \operatorname{Hom}_{K_r}(M, N)$, then $\sigma_{K_r}(f)_1 : \sigma_{K_r}(M)_1 \longrightarrow \sigma_{K_r}(N)_1$ is the restriction of

$$\mathrm{id}_{A_r} \otimes f_1 \colon A_r \otimes_{\Bbbk} M_1 \longrightarrow A_r \otimes_{\Bbbk} N_1$$

to $\ker \psi_M$ while $\sigma_{K_r}(f)_2 \coloneqq f_1$.

The representation $\sigma_{K_r}^{-1}(M)$ is given by $(M_2, \operatorname{coker} \eta_M)$, where

$$\eta_M \colon M_1 \longrightarrow A_r \otimes_{\Bbbk} M_2 \; ; \; m \mapsto \sum_{i=1}^{r} \gamma_i \otimes \psi_M(\gamma_i \otimes m)$$

with structure map

$$\psi_{\sigma_{K_r}^{-1}(M)} \colon A_r \otimes_{\Bbbk} M_2 \longrightarrow \operatorname{coker} \eta_M \; ; \; a \otimes m \mapsto a \otimes m + \operatorname{im} \eta_M.$$

If $f \in \operatorname{Hom}_{K_r}(M, N)$, then $\sigma_{K_r}^{-1}(f)_1 = f_2$ and $\sigma_{K_r}^{-1}(f)_2 : \sigma_{K_r}(M)_2 \longrightarrow \sigma_{K_r}(N)_1$ is the unique $\Bbbk$-linear map making the diagram

$$\begin{array}{ccc} A_r \otimes_{\Bbbk} M_2 & \xrightarrow{\psi_{\sigma_{K_r}^{-1}(M)}} & \operatorname{coker} \eta_M \\ {\scriptstyle \mathrm{id}_{A_r} \otimes f_2}\downarrow & & \downarrow{\scriptstyle \sigma_{K_r}^{-1}(f)_2} \\ A_r \otimes_{\Bbbk} N_2 & \xrightarrow{\psi_{\sigma_{K_r}^{-1}(N)}} & \operatorname{coker} \eta_N \end{array}$$

commute. As $(\sigma_{K_r}^{-1}, \sigma_{K_r})$ is an adjoint pair, σ_{K_r} is left exact, while $\sigma_{K_r}^{-1}$ is right exact. By [ASS06, (VII.5.3)] the functor σ_{K_r} induces an equivalence

$$\sigma_{K_r} : \operatorname{rep}_2(K_r) \longrightarrow \operatorname{rep}_1(K_r)^{1}.$$

By the same token, $\sigma_{K_r}^{-1} \colon \operatorname{rep}_1(K_r) \longrightarrow \operatorname{rep}_2(K_r)$ is a quasi-inverse of σ_{K_r}. We also note that

$$\underline{\dim}\, \sigma_{K_r}(M) = (r \dim_{\Bbbk} M_1 - \dim_{\Bbbk} M_2, \dim_{\Bbbk} M_1)$$

for M in $\operatorname{rep}_2(K_r)$, while $\sigma_{K_r}(S(2)) = 0$. In conjunction with the left exactness of σ_{K_r} this implies that $\sigma_{K_r} \colon \operatorname{rep}_2(K_r) \longrightarrow \operatorname{rep}_1(K_r)$ is exact. By the same token, $\sigma_{K_r}^{-1} \colon \operatorname{rep}_1(K_r) \longrightarrow \operatorname{rep}_2(K_r)$ is exact. The map

$$\sigma_r \colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \; ; \; (x, y) \mapsto (rx - y, x)$$

[1] Recall the definition of $\operatorname{rep}_i(K_r)$ from Section 1.5.

is invertible and satisfies

$$\underline{\dim}\,\sigma_{K_r}(M) = \sigma_r(\underline{\dim}\,M) \text{ and } \underline{\dim}\,\sigma_{K_r}^{-1}(N) = \sigma_r^{-1}(\underline{\dim}\,N)$$

for all $M \in \operatorname{rep}_2(K_r)$ and $N \in \operatorname{rep}_1(K_r)$. Finally, we recall (see [ASS06, (VII.5.8)]) that $\sigma_{K_r} \circ \sigma_{K_r}$ is just the Auslander-Reiten translation τ_{K_r}. By the same token, we have $\sigma_{K_r}^{-1} \circ \sigma_{K_r}^{-1} \cong \tau_{K_r}^{-1}$.

2.4. The adjoint pairs for Kronecker quivers

We fix $1 \leq d < r$. Let $\iota\colon A_d \longrightarrow A_r$ be the canonical embedding sending γ_i to γ_i for all $1 \leq i \leq d$ and consider the restriction $\operatorname{res}\colon K_r \longrightarrow K_d$ that forgets the arrows γ_i for $d < i \leq r$. We have

$$\operatorname{res}\colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_d)\ ;\ M \mapsto \operatorname{res}(M)$$

with

$$\psi_{\operatorname{res}(M)} = \psi_M \circ (\iota \otimes \mathrm{id}_{M_1})$$

and

$$\inf\colon \operatorname{rep}(K_d) \longrightarrow \operatorname{rep}(K_r)\ ;\ X \mapsto \inf(X)$$

with

$$\psi_{\inf(X)}(\gamma_i \otimes x) = \begin{cases} \psi_X(\gamma_i \otimes x), & 1 \leq i \leq d \\ 0, & i > d. \end{cases}$$

The diagram from Figure 2.1 reads as

$$\begin{array}{ccc} K_r & \xrightarrow{\quad\operatorname{res}\quad} & \operatorname{res}(K_r) = K_d \\ \downarrow{\scriptstyle \prime} & & \downarrow{\scriptstyle \prime} \\ (K_r)' = K_r & \xrightarrow{\quad\operatorname{res}\quad} & \operatorname{res}(K_r)' = (K_d)' = K_d, \end{array}$$

while Theorem 2.2.2 simplifies as follows.

Theorem 2.4.1. *The functor*

$$\sigma_{K_r}^{-1} \circ \inf\colon \operatorname{rep}(K_d) \longrightarrow \operatorname{rep}(K_r)$$

is left adjoint to

$$\sigma_{K_d} \circ \operatorname{res}\colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_d).$$

Since we only sketched the proof of Theorem 2.2.4 for a general quiver Q, we provide detailed arguments for K_r.

Theorem 2.4.2. *The functor*

$$\sigma_{K_d}^{-1} \circ \operatorname{res}\colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_d)$$

is left adjoint to

$$\sigma_{K_r} \circ \inf\colon \operatorname{rep}(K_d) \longrightarrow \operatorname{rep}(K_r)$$

Proof. By Theorem 2.4.1 we have an adjoint pair $(\sigma_{K_r}^{-1} \circ \inf, \sigma_{K_d} \circ \mathrm{res})$. Let $Y \in \mathrm{rep}(K_d)$ and $N \in \mathrm{rep}(K_r)$. We write $Y \cong D_{K_d}(X)$ and $N \cong D_{K_r}(M)$ with $X \in \mathrm{rep}(K_d)$ and $M \in \mathrm{rep}(K_r)$. Since we have natural equivalences $D_{K_r} \circ \inf \cong \inf \circ D_{K_d}$, $D_{K_d} \circ \mathrm{res} \cong \mathrm{res} \circ D_{K_r}$, $D_{K_r} \circ \sigma_{K_r} \cong \sigma_{K_r}^{-1} \circ D_{K_r}$ and $D_{K_d} \circ \sigma_{K_d} \cong \sigma_{K_d}^{-1} \circ D_{K_d}$, we obtain a sequence of natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{K_r}(N, (\sigma_{K_r} \circ \inf)(Y)) &\cong \mathrm{Hom}_{K_r}((D_{K_r} \circ \sigma_{K_r} \circ \inf)(Y), D_{K_r}(N)) \\
&\cong \mathrm{Hom}_{K_r}((\sigma_{K_r}^{-1} \circ \inf)(X), M) \\
&\cong \mathrm{Hom}_{K_d}(X, (\sigma_{K_d} \circ \mathrm{res})(M)) \\
&\cong \mathrm{Hom}_{K_d}((D_{K_d} \circ \sigma_{K_d} \circ \mathrm{res})(M), D_{K_d}(X)) \\
&\cong \mathrm{Hom}_{K_d}((\sigma_{K_d}^{-1} \circ \mathrm{res})(N), Y).
\end{aligned}$$

□

Definition. Let $1 \leq d < r$ and $X \in \mathrm{rep}(K_d)$. We define K_r-representations

$$X_{d,r}^- := (\sigma_{K_r}^{-1} \circ \inf)(X) \quad \text{and} \quad X_{d,r}^+ := (\sigma_{K_r} \circ \inf)(X).$$

In the following we collect important properties of $X_{d,r}^-$ and $X_{d,r}^+$.

Lemma 2.4.3. *(1) The functor* $\inf\colon \mathrm{rep}(K_d) \longrightarrow \mathrm{rep}(K_r)$ *is full, faithful, exact, and the essential image is closed under injective and surjective morphisms in* $\mathrm{rep}(K_r)$.

(2) For $X \in \mathrm{rep}(K_d)$ *the representation* $\inf(X)$ *is indecomposable if and only if* X *is indecomposable.*

(3) Let $X \in \mathrm{rep}(K_d)$ *be indecomposable. If* $\inf(X)$ *is not regular, then* $X \cong I_0(d)$ *or* $X \cong P_0(d)$.

(4) For $X \in \mathrm{rep}_{1,2}(K_d)$ *we have* $X_{d,r}^+ \cong \tau_{K_r}(X_{d,r}^-)$.

Proof. (1) Its clear that $\inf$ is full, faithful and exact. Let $f\colon U \longrightarrow \inf(X)$ be an injective morphism with $U \in \mathrm{rep}(K_r)$ and $X \in \mathrm{rep}(K_d)$. Let $i > d$ and $u \in U_1$, then

$$(f_2 \circ \psi_U)(\gamma_i \otimes u) = \psi_{\inf(X)}(\gamma_i \otimes f_1(u)) = 0.$$

Injectivity of f_2 implies $\psi_U(\gamma_i \otimes -) = 0$ and $U = \inf(\mathrm{res}(U))$. The arguments for $f\colon \inf(X) \longrightarrow U$ surjective are similar.

(2) Follows from (1).

(3) By (2) the representation $\inf(X)$ is indecomposable. Note that $\underline{\dim} \inf(X) = \underline{\dim} X$ and therefore

$$\begin{aligned}
q_r(\underline{\dim} \inf(X)) = q_r(\underline{\dim} X) &= q_d(\underline{\dim} X) - (r-d) \dim_{\Bbbk} X_1 \dim_{\Bbbk} X_2 \\
&\leq 1 - (r-d) \dim_{\Bbbk} X_1 \dim_{\Bbbk} X_2.
\end{aligned}$$

Hence $q_r(\underline{\dim} \inf(X)) < 1$ for $\dim_{\Bbbk} X_1 \dim_{\Bbbk} X_2 \neq 0$ and $\inf(X)$ is regular by Kac's Theorem. Note that $\dim_{\Bbbk} X_1 \dim_{\Bbbk} X_2 \neq 0$ if and only if $X \ncong I_0(d), P_0(d)$.

(4) Since $\sigma_{K_r}^{-1}, \sigma_{K_r}$ and $\inf$ commute with finite direct sums, we may assume that X is indecomposable and not simple. By (3) the representation $\inf(X)$ is regular. Hence we have $\sigma_{K_r}(\sigma_{K_r}^{-1}(\inf(X))) \cong \inf(X)$ and

$$\tau_{K_r}(X_{d,r}^-) \cong \sigma_{K_r}^2(\sigma_{K_r}^{-1}(\inf(X))) \cong (\sigma_{K_r} \circ \inf)(X) = X_{d,r}^+.$$

□

We combine Theorem 2.4.1 and Theorem 2.4.2 and obtain:

Corollary 2.4.4. *Let $X \in \operatorname{rep}(K_d)$ and $M \in \operatorname{rep}(K_r)$. We have isomorphisms of $\Bbbk$-vector spaces*

$$\operatorname{Hom}_{K_r}(X_{d,r}^-, M) \cong \operatorname{Hom}_{K_d}(X, (\sigma_{K_d} \circ \operatorname{res})(M)),$$

and

$$\operatorname{Hom}_{K_r}(M, X_{d,r}^+) \cong \operatorname{Hom}_{K_d}((\sigma_{K_d}^{-1} \circ \operatorname{res})(M), X).$$

2.5. Restrictions of Kronecker representations

Aim of this section is to generalize Corollary 2.4.4 to arbitrary restrictions. We let

$$\operatorname{Inj}_{\Bbbk}(A_d, A_r) \coloneqq \{\alpha \in \operatorname{Hom}_{\Bbbk}(A_d, A_r) \mid \ker \alpha = \{0\}\}$$

and for $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ we denote by $\psi_{\alpha^*(M)} \in \operatorname{rep}(K_d)$ the *restriction of M along α* with structure map

$$\psi_{\alpha^*(M)} \coloneqq \psi_M \circ (\alpha \otimes \mathrm{id}_{M_1})\colon A_d \otimes_{\Bbbk} M_1 \longrightarrow M_2.$$

Given $g \in \mathrm{GL}(A_r)$ and $M \in \operatorname{rep}(K_r)$, we define the representation $g.M \in \operatorname{rep}(K_r)$ given by

$$\psi_{g.M} \coloneqq \psi_M \circ (g^{-1} \otimes \mathrm{id}_{M_1})\colon A_r \otimes_{\Bbbk} M_1 \longrightarrow M_2.$$

Moreover, we define $g.f \coloneqq f$ for every morphism $f \in \operatorname{Hom}_{K_r}(M, N)$. The group acts in this way on $\operatorname{rep}(K_r)$ via auto-equivalences. Let $M \in \operatorname{rep}(K_r)$ be indecomposable. If M is not projective, we have an almost split sequence

$$0 \longrightarrow \tau_{K_r}(M) \longrightarrow E \longrightarrow M \longrightarrow 0$$

and obtain for each $g \in \mathrm{GL}(A_r)$ an almost split sequence

$$0 \longrightarrow g.\tau_{K_r}(M) \longrightarrow g.E \longrightarrow g.M \longrightarrow 0.$$

Since there is a unique almost split sequence ending in $g.M$, we conclude $\tau_{K_r}(g.M) \cong g.\tau_{K_r}(M)$. Note that the isomorphism holds for trivial reasons if M is projective. This observation allows us to prove the following result.

Lemma 2.5.1. *Let $X \in \operatorname{rep}(K_d)$ and $g \in \mathrm{GL}(A_r)$ be such that $g|_{A_d} \coloneqq g \circ \iota = \iota$ holds, then $g.X_{d,r}^- \cong X_{d,r}^-$ and $g.X_{d,r}^+ \cong X_{d,r}^+$.*

Proof. We have

$$\psi_{X^-_{d,r}} = \psi_{\sigma^{-1}_{K_r}(\inf(X))} \colon A_r \otimes_{\Bbbk} X_2 \longrightarrow \operatorname{coker} \eta_{\inf(X)} \; ; \; a \otimes x \mapsto a \otimes x + \operatorname{im} \eta_{\inf(X)}$$

with

$$\eta_{\inf(X)} \colon X_1 \longrightarrow A_r \otimes_{\Bbbk} X_2 \; ; \; x \mapsto \sum_{i=1}^{r} \gamma_i \otimes \psi_{\inf(X)}(\gamma_i \otimes x) = \sum_{i=1}^{d} \gamma_i \otimes \psi_X(\gamma_i \otimes x)$$

and

$$\psi_{g.X^-_{d,r}} = \psi_{X^-_{d,r}} \circ (g^{-1} \otimes \mathrm{id}_{X_2}).$$

Since $g \circ \iota = \iota$, we have $g^{-1}(\gamma_i) = \gamma_i$ for all $i \in \{1, \dots, d\}$ and conclude for $x \in X_1$,

$$\psi_{g.X^-_{d,r}}(\eta_{\inf(X)}(x)) = [\psi_{X^-_{d,r}} \circ (g^{-1} \otimes \mathrm{id}_{X_2})](\sum_{i=1}^{d} \gamma_i \otimes \psi_X(\gamma_i \otimes x)) = \psi_{X^-_{d,r}}(\sum_{i=1}^{d} \gamma_i \otimes \psi_X(\gamma_i \otimes x)) = 0.$$

This shows that there exists a unique $\Bbbk$-linear map h_2 making the diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\eta_{\inf(X)}} & A_r \otimes_{\Bbbk} X_2 & \xrightarrow{\psi_{X^-_{d,r}}} & \operatorname{coker} \eta_{\inf(X)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_{A_r} \otimes \mathrm{id}_{X_2}} & & \vdots{\scriptstyle h_2} & & \\ & & A_r \otimes_{\Bbbk} X_2 & \xrightarrow{\psi_{g.X^-_{d,r}}} & \operatorname{coker} \eta_{\inf(X)} & & \end{array}$$

commute. Note that h_2 is an isomorphism. Hence (id_{X_2}, h_2) is the desired isomorphism and $g.X^-_{d,r} \cong X^-_{d,r}$.

Now we prove the second isomorphism. We write $X = Y \oplus P \oplus I$ with $Y \in \operatorname{rep}_{1,2}(K_d), P \in \operatorname{add}(P_0(d))$, $I \in \operatorname{add}(I_0(d))$ and prove the isomorphism for each direct summand. We have $\inf(P_0(d)) = P_0(r)$ and $\inf(I_0(d)) = I_0(r)$ and conclude $P_0(d)^+_{d,r} = 0$ and $I_0(d)^+_{d,r} \cong I_1(r)$. Since $g.I_1(r) = I_1(r)$ (see Example 2.6.1), the statement follows for P and I. Twofold application of Lemma 2.4.3(4) in conjunction with $g.Y^-_{d,r} \cong Y^-_{d,r}$ yields

$$g.Y^+_{d,r} \cong g.(\tau_{K_r}(Y^-_{d,r})) \cong \tau_{K_r}(g.Y^-_{d,r}) = \tau_{K_r}(Y^-_{d,r}) \cong Y^+_{d,r}.$$

□

Let $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ and $g, h \in \operatorname{GL}(A_r)$ be such that $g|_{A_d} = \alpha = h|_{A_d}$. Then $(h^{-1} \circ g)|_{A_d} = \iota$ and Lemma 2.5.1 implies

$$(h^{-1} \circ g).X^{\pm}_{d,r} \cong X^{\pm}_{d,r}$$

and $g.X^{\pm}_{d,r} \cong h.X^{\pm}_{d,r}$. Therefore, the following definition is well-defined up to isomorphism of representations.

Definition. Let $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ and $X \in \operatorname{rep}(K_d)$. Given $g \in \operatorname{GL}(A_r)$ such that $g|_{A_d} = \alpha$, we define

$$X^-(\alpha) \coloneqq g.X^-_{d,r} \text{ and } X^+(\alpha) \coloneqq g.X^+_{d,r}.$$

Remark 2.5.2.

(1) Note that $X^-_{d,r} \cong X^-(\iota)$ and $X^+_{d,r} \cong X^+(\iota)$.

(2) Let $X \in \mathrm{rep}_{1,2}(K_r)$, $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ and $g \in \mathrm{GL}(A_r)$ be such that $g|_{A_d} = \alpha$. Lemma 2.4.3 implies

$$\tau_{K_r}(X^-(\alpha)) = \tau_{K_r}(g.X^-_{d,r}) \cong g.\tau_{K_r}(X^-_{d,r}) \cong g.X^+_{d,r} \cong X^+(\alpha).$$

In the following result we collect the important properties of X^- and X^+. If a statement is true for both X^- and X^+, we simply write $X^\pm$.

Lemma 2.5.3. *Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ and $X, Y \in \mathrm{rep}(K_d)$.*

(1) For $h \in \mathrm{GL}(A_r)$ we have $h.X^\pm(\alpha) \cong X^\pm(h \circ \alpha)$.

(2) We have $(X \oplus Y)^\pm(\alpha) \cong X^\pm(\alpha) \oplus Y^\pm(\alpha)$.

(3) If X is indecomposable and $X \ncong I_0(d), P_0(d)$, then $X^\pm(\alpha)$ is regular indecomposable.

(4) If $\inf(X)$ and $\inf(Y)$ are regular, then $\mathrm{Hom}_{K_r}(X^\pm(\alpha), Y^\pm(\alpha)) \cong \mathrm{Hom}_{K_d}(X, Y)$.

Proof. We fix $g \in \mathrm{GL}(A_r)$ such that $g|_{A_d} = \alpha$.

(1) Let $h \in \mathrm{GL}(A_r)$. We have $(h \circ g)|_{A_d} = h \circ \alpha$ and conclude

$$h.X^\pm(\alpha) \cong h.(g.X^\pm_{d,r}) = (h \circ g).X^\pm_{d,r} \cong X^\pm(h \circ \alpha).$$

(2) Clear.

(3) By Lemma 2.4.3 the representation $X^\pm_{d,r}$ is regular and indecomposable. Hence the same is true for $g.X^\pm_{d,r} \cong X^\pm(\alpha)$.

(4) For $Z \in \{X, Y\}$ we have $Z^\pm(\alpha) \cong g.Z^\pm_{d,r}$ and $\inf(Z)$ being regular gives us $\sigma^{-1}_{K_r}(Z^+_{d,r}) \cong \inf(Z)$. Since $\sigma^{-1}_{K_r}$ induces an equivalence on the category of regular representations, we conclude with $\inf\colon \mathrm{rep}(K_d) \longrightarrow \mathrm{rep}(K_r)$ being full and faithful

$$\begin{aligned}\mathrm{Hom}_{K_r}(X^+(\alpha), Y^+(\alpha)) &\cong \mathrm{Hom}_{K_r}(g.X^+_{d,r}, g.Y^+_{d,r}) = \mathrm{Hom}_{K_r}(X^+_{d,r}, Y^+_{d,r}) \\ &\cong \mathrm{Hom}_{K_r}(\sigma^{-1}_{K_r}(X^+_{d,r}), \sigma^{-1}_{K_r}(Y^+_{d,r})) \cong \mathrm{Hom}_{K_r}(\inf(X), \inf(Y)) \\ &\cong \mathrm{Hom}_{K_d}(X, Y).\end{aligned}$$

The isomorphism $\mathrm{Hom}_{K_r}(X^-(\alpha), Y^-(\alpha)) \cong \mathrm{Hom}_{K_d}(X, Y)$ follows in the same fashion.

□

Proposition 2.5.4. *Let $M \in \mathrm{rep}(K_r)$, $X \in \mathrm{rep}(K_d)$ and $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$.*

(1) We have isomorphisms of $\Bbbk$-vector spaces

$$\mathrm{Hom}_{K_r}(X^-(\alpha), M) \cong \mathrm{Hom}_{K_d}(X, (\sigma_{K_d} \circ \alpha^*)(M)) \cong \mathrm{Hom}_{K_d}(\sigma^{-1}_{K_d}(X), \alpha^*(M)).$$

(2) *We have isomorphisms of $\Bbbk$-vector spaces*

$$\mathrm{Hom}_{K_r}(M, X^+(\alpha)) \cong \mathrm{Hom}_{K_d}((\sigma_{K_d}^{-1} \circ \alpha^*)(M), X) \cong \mathrm{Hom}_{K_d}(\alpha^*(M), \sigma_{K_d}(X)).$$

Proof. We fix $g \in \mathrm{GL}(A_r)$ such that $g|_{A_d} = \alpha$ and note that

$$\begin{aligned}\psi_{\mathrm{res}(g^{-1}.M)} &= \psi_{g^{-1}.M} \circ (\iota \otimes \mathrm{id}_{M_1}) = \psi_M \circ (g \otimes \mathrm{id}_{M_1}) \circ (\iota \otimes \mathrm{id}_{M_1}) \\ &= \psi_M \circ (g|_{A_d} \otimes \mathrm{id}_{M_1}) = \psi_{\alpha^*(M)}.\end{aligned}$$

(1) We have

$$\begin{aligned}\mathrm{Hom}_{K_r}(X^-(\alpha), M) &\cong \mathrm{Hom}_{K_r}(g.X^-_{d,r}, M) \cong \mathrm{Hom}_{K_r}(X^-_{d,r}, g^{-1}.M) \\ &\overset{2.4.4}{\cong} \mathrm{Hom}_{K_d}(X, (\sigma_{K_d} \circ \mathrm{res})(g^{-1}.M)) \\ &\cong \mathrm{Hom}_{K_d}(X, (\sigma_{K_d} \circ \alpha^*)(M)).\end{aligned}$$

The second isomorphism holds since $(\sigma_{K_d}^{-1}, \sigma_{K_d})$ is an adjoint pair.

(2) We have

$$\begin{aligned}\mathrm{Hom}_{K_r}(M, X^+(\alpha)) &\cong \mathrm{Hom}_{K_r}(g^{-1}.M, X^+_{d,r}) \overset{2.4.4}{\cong} \mathrm{Hom}_{K_d}((\sigma_{K_d}^{-1} \circ \mathrm{res})(g^{-1}.M), X) \\ &\cong \mathrm{Hom}_{K_d}((\sigma_{K_d}^{-1} \circ \alpha^*)(M), X).\end{aligned}$$

The second isomorphism holds since $(\sigma_{K_d}^{-1}, \sigma_{K_d})$ is an adjoint pair.

□

2.6. Homogeneous Kronecker representations

Given $M \in \mathrm{rep}(K_r)$, the stabilizer

$$\mathrm{GL}(A_r)_M := \{g \in \mathrm{GL}(A_r) \mid g.M \cong M\}$$

is a closed subgroup of $\mathrm{GL}(A_r)$ (cf. [Far11, (2.1)]). We say that M is *homogeneous*, provided $\mathrm{GL}(A_r)_M = \mathrm{GL}(A_r)$.

Example 2.6.1. *Let $M \in \mathrm{rep}(K_r)$ be an indecomposable representation that is preprojective or preinjective. Let $g \in \mathrm{GL}(A_r)$, then $g.M$ is indecomposable and*

$$q_r(\underline{\dim}\, g.M) = q_r(\underline{\dim}\, M) = 1$$

in conjunction with Kac's Theorem implies $g.M \cong M$. Hence M is homogeneous.

The aim of this section is to show that for a homogeneous representation $X \in \mathrm{rep}(K_d)$ and $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$, the isomorphism class of $X^{\pm}(\alpha)$ only depends on the image of α, i.e. $X^{\pm}(\alpha) \cong X^{\pm}(\beta)$ for every $\beta \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ with $\mathrm{im}\,\alpha = \mathrm{im}\,\beta$.

Definition. Let $s \in \mathbb{N}$, $\mathcal{B} = (\gamma_1, \dots, \gamma_s)$ be the standard basis of A_s and $f \in \mathrm{End}_{\Bbbk}(A_s)$. We denote by $f^{\mathrm{tr}} \in \mathrm{End}_{\Bbbk}(A_s)$ the unique endomorphism that satisfies the equation $\mathrm{Mat}_{\mathcal{B}}(f^{\mathrm{tr}}) = \mathrm{Mat}_{\mathcal{B}}(f)^{\mathrm{tr}}$.

Lemma 2.6.2. *Let $g \in \mathrm{GL}(A_r)$ and $M \in \mathrm{rep}(K_r)$. We have*

$$\sigma_{K_r}^{-1}(g.M) \cong (g^{-1})^{\mathrm{tr}}.\sigma_{K_r}^{-1}(M) \text{ and } \sigma_{K_r}(g.M) \cong (g^{-1})^{\mathrm{tr}}.\sigma_{K_r}(M).$$

Proof. We only prove the first isomorphism. For $i \in \{1, \dots, r\}$ we write $g^{\mathrm{tr}}(\gamma_i) = \sum_{j=1}^{r} \lambda_{ji}\gamma_j$ and $g^{-1}(\gamma_i) = \sum_{k=1}^{r} \mu_{ki}\gamma_k$ and note that

$$(*) \quad \delta_{jk} = (M_{\mathcal{B}}(g^{-1})M_{\mathcal{B}}(g))_{jk} = \sum_{i=1}^{r} M_{\mathcal{B}}(g^{-1})_{ji}M_{\mathcal{B}}(g)_{ik} = \sum_{i=1}^{r} \mu_{ji}(M_{\mathcal{B}}(g^{\mathrm{tr}})^{\mathrm{tr}})_{ik} = \sum_{i=1}^{r} \mu_{ji}\lambda_{ki}$$

for all $j, k \in \{1, \dots, n\}$, where δ denotes the Kronecker delta. For $m \in M_1$ we have

$$\begin{aligned}
&(g^{\mathrm{tr}} \otimes \mathrm{id}_{M_2})(\eta_{g.M}(m)) \\
=&\sum_{i=1}^{r} g^{\mathrm{tr}}(\gamma_i) \otimes \psi_M(g^{-1}(\gamma_i) \otimes m) = \sum_{k=1}^{r}\sum_{j=1}^{r}(\sum_{i=1}^{r} \lambda_{ji}\mu_{ki})\gamma_j \otimes \psi_M(\gamma_k \otimes m) \\
\overset{(*)}{=}&\sum_{k=1}^{r}\sum_{j=1}^{r} \delta_{jk}\gamma_j \otimes \psi_M(\gamma_k \otimes m) = \sum_{k=1}^{r} \gamma_k \otimes \psi_M(\gamma_k \otimes m) = \eta_M(m).
\end{aligned}$$

This induces a $\Bbbk$-linear isomorphism

$$f\colon \operatorname{coker} \eta_{g.M} \mapsto \operatorname{coker} \eta_M \ ; \ a \otimes m + \operatorname{im} \eta_{g.M} \mapsto g^{\mathrm{tr}}(a) \otimes m + \operatorname{im} \eta_M.$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc}
A_r \otimes_{\Bbbk} M_2 & \xrightarrow{\psi_{\sigma_{K_r}^{-1}(g.M)}} & \operatorname{coker} \eta_{g.M} \\
\downarrow{\scriptstyle \mathrm{id}_{A_r} \otimes \mathrm{id}_{M_2}} & & \downarrow{\scriptstyle f} \\
A_r \otimes_{\Bbbk} M_2 & \xrightarrow{\psi_{(g^{-1})^{\mathrm{tr}}.\sigma_{K_r}^{-1}(M)}} & \operatorname{coker} \eta_M.
\end{array}$$

□

Proposition 2.6.3. *Let $X \in \mathrm{rep}(K_d)$ be homogeneous. Then $X^{\pm}(\alpha)$ only depends on $\operatorname{im} \alpha$.*

Proof. Let $\alpha, \alpha' \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ such that $\mathfrak{v} := \operatorname{im} \alpha = \operatorname{im} \alpha'$. We define $A_d^{\perp} := \bigoplus_{i>d}^{r} \Bbbk\gamma_i$ and fix a $\Bbbk$-complement $\mathfrak{u} \subseteq A_r$ of $\mathfrak{v}$ and an isomorphism $k\colon A_d^{\perp} \longrightarrow \mathfrak{u}$. Let $\pi\colon A_r = \mathfrak{v} \oplus \mathfrak{u} \longrightarrow \mathfrak{v}$ be the canonical projection and $\beta \in \{\alpha, \alpha'\}$. Then $\pi \circ \beta \in A_d \longrightarrow \mathfrak{v}$ is an isomorphism and we define

$$g_{\beta} := \begin{pmatrix} \pi \circ \beta & 0 \\ 0 & k \end{pmatrix} : A_d \oplus A_d^{\perp} \longrightarrow \mathfrak{v} \oplus \mathfrak{u} \in \mathrm{GL}(A_r).$$

By definition we have $g_\beta|_{A_d} = \beta$ and therefore $X^\pm(\beta) \cong g_\beta.X^\pm_{d,r}$. We have to show that $X^\pm(\alpha) \cong X^\pm(\alpha')$ which is equivalent to $[g_{\alpha'}^{-1} \circ g_\alpha].X^\pm_{d,r} = X^\pm_{d,r}$. Let $h \coloneqq (\pi \circ \alpha')^{-1} \circ (\pi \circ \alpha) \in \mathrm{GL}(A_d)$, then

$$g \coloneqq g_{\alpha'}^{-1} \circ g_\alpha = \begin{pmatrix} (\pi \circ \alpha')^{-1} \circ (\pi \circ \alpha) & 0 \\ 0 & \mathrm{id}_{A_d^\perp} \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & \mathrm{id}_{A_d^\perp} \end{pmatrix}.$$

Hence

$$g^{\mathrm{tr}} = \begin{pmatrix} h^{\mathrm{tr}} & 0 \\ 0 & \mathrm{id}_{A_d^\perp} \end{pmatrix}.$$

Since $X \in \mathrm{rep}(K_d)$ is homogeneous, there is an isomorphism $f \in \mathrm{Hom}_{K_d}(h^{-\mathrm{tr}}.X, X)$. We claim that the diagram

$$\begin{array}{ccc} A_r \otimes_{\Bbbk} X_1 & \xrightarrow{\psi_{\inf(X)} \circ (g^{\mathrm{tr}} \otimes \mathrm{id}_{X_1})} & X_2 \\ \downarrow{\scriptstyle \mathrm{id}_{A_r} \otimes f_1} & & \downarrow{\scriptstyle f_2} \\ A_r \otimes_{\Bbbk} X_1 & \xrightarrow[\psi_{\inf(X)}]{} & X_2 \end{array}$$

commutes. Let $x \in X_1$ and $i \in \{1, \dots, d\}$, then $h^{\mathrm{tr}}(\gamma_i) \in A_d$ and

$$\begin{aligned} [f_2 \circ \psi_{\inf(X)} \circ (g^{\mathrm{tr}} \otimes \mathrm{id}_{X_1})](\gamma_i \otimes x) &= [f_2 \circ \psi_{\inf(X)}](h^{\mathrm{tr}}(\gamma_i) \otimes x) = (f_2 \circ \psi_X)(h^{\mathrm{tr}}(\gamma_i) \otimes x) \\ &= [f_2 \circ \psi_X \circ (h^{\mathrm{tr}} \otimes \mathrm{id}_{X_1})](\gamma_i \otimes x) \\ &= [f_2 \circ \psi_{h^{-\mathrm{tr}}.X}](\gamma_i \otimes x) \\ &= [\psi_X \circ (\mathrm{id}_{A_d} \otimes f_1)](\gamma_i \otimes x) \\ &= [\psi_{\inf(X)} \circ (\mathrm{id}_{A_r} \otimes f_1)](\gamma_i \otimes x). \end{aligned}$$

For $i > d$, we have

$$\begin{aligned} [f_2 \circ \psi_{\inf(X)} \circ (g^{\mathrm{tr}} \otimes \mathrm{id}_{X_1})](\gamma_i \otimes x) &= [f_2 \circ \psi_{\inf(X)}](\gamma_i \otimes x) = f_2(0) = 0 = \psi_{\inf(X)}(\gamma_i \otimes f_1(x)) \\ &= [\psi_{\inf(X)} \circ (\mathrm{id}_{A_r} \otimes f_1)](\gamma_i \otimes x). \end{aligned}$$

Hence the diagram commutes and $g^{-\mathrm{tr}}.\inf(X) \cong \inf(X)$. Finally, we conclude with Lemma 2.6.2

$$g.X^-_{d,r} = g.((\sigma_{K_r}^{-1} \circ \inf)(X)) \cong \sigma_{K_r}^{-1}(g^{-\mathrm{tr}}.\inf(X)) \cong \sigma_{K_r}^{-1}(\inf(X)) = X^-_{d,r},$$

and

$$g.X^+_{d,r} = g.((\sigma_{K_r} \circ \inf)(X)) \cong \sigma_{K_r}(g^{-\mathrm{tr}}.\inf(X)) \cong \sigma_{K_r}(\inf(X)) = X^+_{d,r}.$$

□

By Proposition 2.6.3 the following definition is well-defined up to isomorphism of representations. We denote by

$$\mathrm{Gr}_d(A_r) \coloneqq \{\mathfrak{v} \subseteq A_r \mid \mathfrak{v} \subseteq A_r \text{ } d\text{-dimensional subspace}\}$$

the *Grassmannian* of d-dimensional subspaces of the arrow space A_r.

Definition. Let $X \in \operatorname{rep}(K_r)$ be homogeneous and $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$. We define

$$X^+(\mathfrak{v}) \coloneqq X^+(\alpha) \text{ and } X^-(\mathfrak{v}) \coloneqq X^-(\alpha)$$

for any $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ such that $\operatorname{im}\alpha = \mathfrak{v}$.

Remark 2.6.4. Let $X \in \operatorname{rep}(K_d)$ be homogeneous. The assumption $\mathfrak{v} \neq \mathfrak{u}$ does not guarantee that $X^{\pm}(\mathfrak{v}) \ncong X^{\pm}(\mathfrak{u})$ since $P_0(d)^-(\mathfrak{v}) \cong P_1(r)$, $P_0(d)^+(\mathfrak{v}) = 0 = I_0(d)^-(\mathfrak{v})$ and $I_0(d)^+(\mathfrak{v}) \cong I_1(r)$ for all $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$. The following result shows that these are basically the only exceptions.

Proposition 2.6.5. *Let $X \in \operatorname{rep}(K_d)$ be homogeneous, not semisimple, and $\alpha, \beta \in \operatorname{Gr}_d(A_r)$ be such that $X^{\pm}(\alpha) \cong X^{\pm}(\beta)$. Then $\operatorname{im}\alpha = \operatorname{im}\beta$.*

Proof. Let $\alpha, \beta \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ such that $X^-(\alpha) \cong X^-(\beta)$. In the following we suppress the exponent "-", i.e. we have $X(\alpha) \cong X(\beta)$. We write $X = aP_0(d) \oplus bI_0(d) \oplus Y$ with $0 \neq Y \in \operatorname{rep}_{1,2}(K_d)$. We have

$$aP_1(r) \oplus Y(\alpha) \cong X(\alpha) \cong X(\beta) = aP_1(r) \oplus Y(\beta)$$

and conclude with Krull-Remark-Schmidt that $Y(\alpha) \cong Y(\beta)$. Hence we may assume that $X \in \operatorname{rep}_{1,2}(K_d)$. First we consider the special case $X(\alpha) \cong X(\iota)$. We need to show that $\operatorname{im}\alpha = \operatorname{im}\iota = A_d$ and proceed in steps.

(i) Let $g \in \operatorname{GL}(A_r)$ such that $g \circ \iota = \alpha$, then $A_d \cap g^{\mathrm{tr}}(A_d^{\perp}) = \{0\}$: We have

$$X(\iota) \cong X(\alpha) \cong X(g \circ \iota) \overset{2.5.3(1)}{\cong} g.X(\iota).$$

Since $X \in \operatorname{rep}_{1,2}(K_r)$, we conclude with Lemma 2.4.3(3) that $\inf(X) \in \operatorname{rep}(K_r)$ is regular. Hence $\inf(X) \cong (\sigma_{K_r} \circ \sigma_{K_r}^{-1})(\inf(X)) \cong \sigma_{K_r}(X(\iota))$ and

$$\inf(X) \cong \sigma_{K_r}(X(\iota)) \cong \sigma_{K_r}(g.X(\iota)) \overset{2.6.2}{\cong} g^{-\mathrm{tr}}.\sigma_{K_r}(X(\iota)) \cong g^{-\mathrm{tr}}.\inf(X).$$

Let $f\colon \inf(X) \longrightarrow g^{-\mathrm{tr}}.\inf(X)$ be an isomorphism. We assume that $A_d \cap g^{\mathrm{tr}}(A_d^{\perp}) \neq \{0\}$ and find $0 \neq a \in A_d^{\perp}$ such that $0 \neq g^{\mathrm{tr}}(a) \in A_d$. We conclude for $x \in \inf(X)_1 = X_1$ that

$$\begin{aligned} 0 &= (f_2 \circ \psi_{\inf(X)})(a \otimes x) = (\psi_{g^{-\mathrm{tr}}.\inf(X)} \circ (\mathrm{id}_{A_r} \otimes f_1))(a \otimes x) \\ &= \psi_{\inf(X)} \circ (g^{\mathrm{tr}}(a) \otimes f_1(x)) = \psi_X(g^{\mathrm{tr}}(a) \otimes f_1(x)). \end{aligned}$$

Since $f_1\colon X_1 \longrightarrow X_1$ is surjective, we conclude

$$(\dagger) \quad \psi_X(g^{\mathrm{tr}}(a) \otimes x) = 0$$

for all $x \in X_1$. Since X is homogeneous, this implies $\psi_X = 0$. This is a contradiction since $0 \neq X \in \operatorname{rep}_{1,2}(K_d)$.

(ii) Let $g \in \operatorname{GL}(A_r)$ such that $g \circ \iota = \alpha$, then $A_d \cap g(A_d^{\perp}) = \{0\}$: We write

$$M_{\mathcal{B}}(g^{tr}) = (\lambda_{ji})_{1 \leq j,i \leq n},$$

so for $1 \leq i \leq r$ we have $g^{\mathrm{tr}}(\gamma_i) = \sum_{j=1}^{r} \lambda_{ji}\gamma_j$. The assumption that the lower right hand block of $M_{\mathcal{B}}(g^{\mathrm{tr}})$ of size $(r-d)\times(r-d)$ is not invertible, yields a non-trivial linear combination

$$(*) \quad \mu_{d+1}\begin{pmatrix}\lambda_{d+1,d+1}\\ \lambda_{d+2,d+1}\\ \vdots\\ \lambda_{r,d+1}\end{pmatrix} + \mu_{d+2}\begin{pmatrix}\lambda_{d+1,d+2}\\ \lambda_{d+2,d+2}\\ \vdots\\ \lambda_{r,d+2}\end{pmatrix} + \cdots + \mu_r\begin{pmatrix}\lambda_{d+1,r}\\ \lambda_{d+2,r}\\ \vdots\\ \lambda_{r,r}\end{pmatrix} = \begin{pmatrix}0\\0\\ \vdots\\ 0\end{pmatrix}.$$

We have $0 \neq \sum_{i=d+1}^{r} \mu_i\gamma_i \in A_d^{\perp}$ and $(*)$ implies that $g^{\mathrm{tr}}(\sum_{i=d+1}^{r} \mu_i\gamma_i)$ is a non-zero element in A_d. Hence $A_d \cap g^{\mathrm{tr}}(A_d^{\perp}) \neq \{0\}$, in contradiction to (i). Therefore, the lower right hand block of $M_{\mathcal{B}}(g) = M_{\mathcal{B}}(g^{\mathrm{tr}})^{\mathrm{tr}}$ that we denote by $(c_{ji})_{d+1\leq j,i\leq r}$ is also invertible.

Finally, we assume that $A_d \cap g(A_d^{\perp}) \neq \{0\}$ and find $0 \neq a = \sum_{i=d+1}^{r} \eta_i\gamma_i$ such that $g(a) \in A_d$. We have

$$g(a) = \sum_{i=d+1}^{r} \eta_i g(\gamma_i)$$

and conclude

$$\begin{pmatrix}0\\0\\ \vdots\\ 0\end{pmatrix} = \eta_{d+1}\begin{pmatrix}c_{d+1,d+1}\\ c_{d+2,d+1}\\ \vdots\\ c_{r,d+1}\end{pmatrix} + \eta_{d+2}\begin{pmatrix}c_{d+1,d+2}\\ c_{d+2,d+2}\\ \vdots\\ c_{r,d+2}\end{pmatrix} + \cdots + \eta_r\begin{pmatrix}c_{d+1,r}\\ c_{d+2,r}\\ \vdots\\ c_{r,r}\end{pmatrix},$$

a contradiction.

(iii) We have $\operatorname{im}\alpha = A_d = \operatorname{im}\iota$: Let $a \in A_r \setminus \operatorname{im}\alpha$ and $a \in \mathfrak{u}$ be a $\Bbbk$-complement of $\operatorname{im}\alpha$ in A_r. We find $g \in \mathrm{GL}(A_r)$ such that $g\circ\iota = \alpha$ and $g|_{A_d^{\perp}} \longrightarrow \mathfrak{u}$ is an isomorphism. By (ii) we have $A_d \cap \mathfrak{u} = A_d \cap g(A_d^{\perp}) = \{0\}$ and conclude $a \notin A_d$. Since $\dim_{\Bbbk}\mathfrak{u} = \dim_{\Bbbk} A_d$, we conclude $\operatorname{im}\alpha = A_d$.

Now we prove the general case, i.e. we $X(\alpha) \cong X(\beta)$. We let $g_\alpha, g_\beta \in \mathrm{GL}(A_r)$ such that $g_\alpha\circ\iota = \alpha$ and $g_\beta\circ\iota = \beta$. We have $g_\alpha.X_{d,r} \cong X(\alpha) \cong X(\beta) \cong g_\beta.X_{d,r}$ and conclude

$$X(g_\beta^{-1}\circ g_\alpha\circ\iota) \overset{2.5.3(1)}{\cong} (g_\beta^{-1}\circ g_\alpha).X(\iota) = (g_\beta^{-1}\circ g_\alpha).X_{d,r} \cong X_{d,r} \cong X(\iota).$$

Now (iii) implies $\operatorname{im}(g_\beta^{-1}\circ g_\alpha\circ\iota) = \operatorname{im}\iota$ and therefore

$$\operatorname{im}(\alpha) = \operatorname{im}(g_\alpha\circ\iota) = \operatorname{im}(g_\beta\circ\iota) = \operatorname{im}\beta.$$

This completes the proof in case $X^-(\alpha) \cong X^-(\beta)$.

Now we assume $X^+(\alpha) \cong X^+(\beta)$. As before, we conclude that we can assume that $X \in \mathrm{rep}_{1,2}(K_d)$. Then Remark 2.5.2 implies

$$X^-(\alpha) \cong \tau_{K_r}^{-1}(X^+(\alpha)) \cong \tau_{K_r}^{-1}(X^+(\beta)) \cong X^-(\beta).$$

The first case implies $\operatorname{im}\alpha \cong \operatorname{im}\beta$. □

Remark 2.6.6. The statements also hold for every non-semisimple representation $X \in \operatorname{rep}(K_d)$ such that $\psi_X|_{\mathfrak{w}\otimes_{\Bbbk} X_1} \neq 0$ for every 1-dimensional subspace $\mathfrak{w}$ of A_d: This assumption guarantees that in the proof of (i) the equation (†) leads to a contradiction.

Theorem 2.6.7. *Let $X \in \operatorname{rep}(K_r)$ be a homogeneous representation that is not semisimple. The maps*

$$\operatorname{Gr}_d(A_r) \longrightarrow [\operatorname{rep}(K_r)]; \mathfrak{v} \mapsto [X^-(\mathfrak{v})],$$

$$\operatorname{Gr}_d(A_r) \longrightarrow [\operatorname{rep}(K_r)]; \mathfrak{v} \mapsto [X^+(\mathfrak{v})]$$

are injective, where we denote by $[\operatorname{rep}(K_r)]$ the set of all isomorphism classes of K_r-representations.

Proof. Follows immediately from Proposition 2.6.5. □

2.7. Application to preprojective representations

In the following we will consider $1 \leq d < r$ and $n \in \mathbb{N}_0$. Since $P_n(d) \in \operatorname{rep}(K_d)$ is homogeneous (see Example 2.6.1), we may define

$$P_n^+(\mathfrak{v}) \coloneqq P_n(d)^+(\mathfrak{v}) \text{ and } P_n^-(\mathfrak{v}) \coloneqq P_n(d)^-(\mathfrak{v})$$

for all $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$.

In the following Lemma we collect basic properties of preprojective indecomposable Kronecker representations.

Lemma 2.7.1. *Let $n, m \in \mathbb{N}_0$.*

(1) We have $\sigma_{K_r}^{-1}(P_n(r)) \cong P_{n+1}(r)$.

(2) We have $a_{n+2}(r) = ra_{n+1}(r) - a_n(r)$.

(3) We have

$$\dim_{\Bbbk} \operatorname{Hom}_{K_r}(P_n(r), P_m(r)) = \begin{cases} 0 & \text{falls } n > m \\ a_{m-n+1}(r) & \text{falls } n \leq m. \end{cases}$$

Proof. (1) By definition $P_{2\ell}(r) = \tau_{K_r}^{-\ell}(P_0(r))$ and $P_{2\ell+1}(r) = \tau_{K_r}^{-\ell}(P_1(r))$ for all $\ell \in \mathbb{N}_0$. The statement follows since $\sigma_{K_r}^{-2} \cong \tau_{K_r}^{-1}$ and $\sigma_{K_r}^{-1}(P_0(r)) \cong P_1(r)$.

(2) We have

$$\begin{aligned}(a_{n+1}(r), a_{n+2}(r)) &= \underline{\dim}\, P_{n+1}(r) = \underline{\dim}\, \sigma_{K_r}^{-1}(P_n(r)) \\ &= \sigma_r^{-1}.(a_n(r), a_{n+1}(r)) = (a_{n+1}(r), ra_{n+1}(r) - a_n(r)).\end{aligned}$$

(3) If $n > m$, then

$$\mathrm{Hom}_{K_r}(P_n(r), P_m(r)) \cong \mathrm{Hom}_{K_r}(\sigma_{K_r}^m(P_n(r)), \sigma_{K_r}^m(P_n(r))) \cong \mathrm{Hom}_{K_r}(P_{n-m}(r), P_0(r)) = 0,$$

since $P_0(r)$ is projective. Now let $n \leq m$, then

$$\mathrm{Hom}_{K_r}(P_n(r), P_m(r)) \cong \mathrm{Hom}_{K_r}(P_0(r), P_{m-n}(r)) \cong (P_{m-n}(r))_2 = a_{m-n+1}(r).$$

□

In view of Lemma 2.7.1, we can reformulate Proposition 2.5.4 for $P_n(d)$ as follows.

Lemma 2.7.2. *Let $1 \leq d < r$, $n \in \mathbb{N}_0$ and $M \in \mathrm{rep}(K_r)$.*[2] *Let $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ and $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ be such that $\mathrm{im}\,\alpha = \mathfrak{v}$. We have*

$$\mathrm{Hom}_{K_r}(P_n^-(\mathfrak{v}), M) \cong \mathrm{Hom}_{K_d}(P_n(d), (\sigma_{K_d} \circ \alpha^*)(M)) \cong \mathrm{Hom}_{K_d}(P_{n+1}(d), \alpha^*(M))$$

and

$$\mathrm{Hom}_{K_r}(M, P_n^+(\mathfrak{v})) \cong \mathrm{Hom}_{K_d}((\sigma_{K_d}^{-1} \circ \alpha^*)(M), P_n(d)) \cong \mathrm{Hom}_{K_d}(\alpha^*(M), P_{n-1}(d)).$$[3]

In the following we study restrictions via elements in $\mathrm{Inj}_{\Bbbk}(A_d, A_r)$ in more detail. The first result is a generalization of [BF24, (1.3.1)].

Lemma 2.7.3. *Let $M \in \mathrm{rep}(K_r)$ be a representation.*

(1) Let $\alpha, \beta \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ be such that $\mathrm{im}\,\alpha = \mathrm{im}\,\beta$ and let $\delta^*(M) = P_\delta \oplus R_\delta \oplus I_\delta$ be the decomposition of $\delta^*(M)$ into preprojective, regular and preinjective summands for $\delta \in \{\alpha, \beta\}$. Then $P_\alpha \cong P_\beta$, and $I_\alpha \cong I_\beta$.

(2) Let $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$. The following statements are equivalent:

(i) There is $\alpha \in \mathrm{im}^{-1}(\mathfrak{v})$ such that $\alpha^*(M)$ is preprojective.

(ii) For all $\alpha \in \mathrm{im}^{-1}(\mathfrak{v})$ the representation $\alpha^*(M)$ is preprojective.

(iii) For all $\alpha \in \mathrm{im}^{-1}(\mathfrak{v})$ the representation $\alpha^*(M)$ is preprojective and $\alpha^*(M) \cong \beta^*(M)$ for all $\beta \in \mathrm{im}^{-1}(\mathfrak{v})$.

Proof. (1) We have $g := \beta^{-1} \circ \alpha \in \mathrm{GL}(A_d)$. Note that $\psi_{g.\alpha^*(M)} = \psi_M \circ (\alpha \circ g^{-1} \otimes \mathrm{id}_{M_1}) = \psi_M \circ (\beta \otimes \mathrm{id}_{M_1})$ and therefore $\beta^*(M) = g.(\alpha^*(M))$. We conclude

$$P_\beta \oplus R_\beta \oplus I_\beta = \beta^*(M) = g.(\alpha^*(M)) \cong g.P_\alpha \oplus g.R_\alpha \oplus g.I_\alpha.$$

Since preprojective and preinjective representations are homogeneous (see Example 2.6.1), we obtain

$$P_\beta \oplus R_\beta \oplus I_\beta = \beta^*(M) \cong g.(\alpha^*(M)) \cong P_\alpha \oplus g.R_\alpha \oplus I_\alpha.$$

Since regular representations are closed under the action of $\mathrm{GL}(A_d)$, the claim follows from Krull-Remak-Schmidt.

[2] For $d = 1$ we only have projective indecomposables $P_0(1)$ and $P_1(1)$. In this case $n \in \mathbb{N}_0$ means $n \in \{0, 1\}$ and $P_2(1) := \sigma_{K_1}^{-1}(P_1(1)) = I_0(1)$.

[3] Here $P_{-1}(d) := \{0\}$.

(2) It suffices to prove (i) $\Rightarrow$ (iii). By (i) there is $\alpha \in \operatorname{im}^{-1}(\mathfrak{v})$ such that $\alpha^*(M)$ is preprojective. Let $\beta \in \operatorname{im}^{-1}(\mathfrak{v})$. Then $\operatorname{im}\alpha = \operatorname{im}\beta$ and (1) implies that $P_\alpha \cong P_\beta$. Since $\underline{\dim}\,\beta^*(M) = \underline{\dim}\,\alpha^*(M) = \underline{\dim}\,P_\alpha = \underline{\dim}\,P_\beta \leq \underline{\dim}\,\beta^*(M)$, we conclude that $P_\beta = \beta^*(M)$. Hence $\beta^*(M)$ is preprojective with

$$\beta^*(M) = P_\beta \cong P_\alpha = \alpha^*(M).$$

□

Definition. Let $M \in \operatorname{rep}(K_r)$ and $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$. For each $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ such that $\operatorname{im}\alpha = \mathfrak{v}$ we have a decomposition

$$\alpha^*(M) = \alpha^*(M)_{\mathrm{pproj}} \oplus \alpha^*(M)_{\mathrm{reg}} \oplus \alpha^*(M)_{\mathrm{pinj}}$$

into preprojective, regular and preinjective summands. We define

$$M|_{\mathfrak{v},\mathrm{pproj}} \coloneqq \alpha^*(M)_{\mathrm{pproj}} \text{ and } M|_{\mathfrak{v},\mathrm{pinj}} \coloneqq \alpha^*(M)_{\mathrm{pinj}},$$

which is well-defined up to isomorphism by Lemma 2.7.3.

We say that $M|_{\mathfrak{v}}$ is *(pre)projective* if $\alpha^*(M)$ is (pre)projective for some $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ with $\operatorname{im}\alpha = \mathfrak{v}$. By Lemma 2.7.3 this is equivalent to $\beta^*(M)$ being preprojective for all $\beta \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ such that $\operatorname{im}\beta = \mathfrak{v}$ and moreover, we have an isomorphism of representations $\alpha^*(M) \cong \beta^*(M)$ in this case.

Proposition 2.7.4. *Let $M \in \operatorname{rep}(K_r)$, $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$, $n \in \mathbb{N}_0$. We write $M|_{\mathfrak{v},\mathrm{pproj}} = \bigoplus_{i\geq 0} b_i(\mathfrak{v})P_i(d)$.*

(1) If $M|_{\mathfrak{v}}$ is preprojective, we have

$$\dim_{\Bbbk} \operatorname{Hom}_{K_r}(P_n^-(\mathfrak{v}), M) = \sum_{i\geq n+1} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_{n+1}(d), P_i(d)).$$

(2) The followings statements are equivalent.

(i) $\operatorname{Hom}_{K_r}(P_n^-(\mathfrak{v}), M) = 0$.

(ii) $M|_{\mathfrak{v}}$ *is preprojective and* $M|_{\mathfrak{v}} \in \operatorname{add}(P_0(d), \ldots, P_n(d))$.

(3) We have $\dim_{\Bbbk} \operatorname{Hom}_{K_r}(M, P_n^+(\mathfrak{v})) = \sum_{i=0}^{n-1} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_i(d), P_{n-1}(d))$[4].

(4) The following statements are equivalent.

(i) $\operatorname{Hom}_{K_r}(M, P_n^+(\mathfrak{v})) = 0$.

(ii) $M|_{\mathfrak{v},\mathrm{pproj}} \in \operatorname{add}(\{P_i(d) \mid i \geq n\})$.

Proof. (1) Let $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ be such that $\operatorname{im}\alpha = \mathfrak{v}$. By assumption we have $\alpha^*(M) \cong M|_{\mathfrak{v},\mathrm{pproj}}$ and conclude with Lemma 2.7.2 that

$$\begin{aligned}
\dim_{\Bbbk} \operatorname{Hom}_{K_r}(P_n^-(\mathfrak{v}), M) &= \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_{n+1}(d), \alpha^*(M)) \\
&= \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_{n+1}(d), \bigoplus_{i\geq 0} b_i(\mathfrak{v})P_i(d)) \\
&\overset{2.7.1}{=} \sum_{i\geq n+1} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_{n+1}(d), P_i(d)).
\end{aligned}$$

[4] Note that $n = 0$ gives us the empty sum.

(2) (i) $\Rightarrow$ (ii). Let $0 \neq X \in \operatorname{rep}(K_d)$ be such that X does not have a preprojective direct summand. Then $\sigma^n_{K_d}(X)$ does not have $P_0(r)$ as a direct summand. In particular, we have $(\sigma^n_{K_d}(X))_1 \neq 0$ and we obtain

$$\begin{aligned}\operatorname{Hom}_{K_d}(P_{n+1}(d), X) &\cong \operatorname{Hom}_{K_d}(\sigma^n_{K_d}(P_{n+1}(d)), \sigma^n_{K_d}(X)) \\ &\cong \operatorname{Hom}_{K_d}(P_1(d), \sigma^n_{K_d}(X)) \cong (\sigma^n_{K_d}(X))_1 \neq 0.\end{aligned}$$

Hence the assumption in conjunction with Lemma 2.7.2 implies that every indecomposable direct summand of $\alpha^*(M)$ is preprojective, i.e. $M|_{\mathfrak{v}}$ is preprojective. Now we apply (1) and conclude

$$0 = \sum_{i \geq n+1} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_{n+1}(d), P_i(d)).$$

Lemma 2.7.1 implies $b_i(\mathfrak{v}) = 0$ for all $i \geq n+1$. Therefore, we have $M|_{\mathfrak{v}} = M|_{\mathfrak{v},\mathrm{pproj}} = \bigoplus_{i=0}^n b_i(\mathfrak{v}) P_i(\mathfrak{v}) \in \operatorname{add}(P_0(d), \ldots, P_n(d))$.

(ii) $\Rightarrow$ (i). We have $M|_{\mathfrak{v}} = M|_{\mathfrak{v},\mathrm{pproj}} = \sum_{i=0}^n b_i(\mathfrak{v}) P_i(d)$. Hence (1) implies

$$\dim_{\Bbbk} \operatorname{Hom}_{K_r}(P_n^-(\mathfrak{v}), M) = \sum_{i \geq n+1} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_{n+1}(d), P_i(d)) = 0.$$

(3) Let $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ such that $\operatorname{im}\alpha = \mathfrak{v}$. We write $\alpha^*(M) = M|_{\mathfrak{v},\mathrm{pproj}} \oplus L$ such that L does not have a preprojective direct summand. We have $\operatorname{Hom}_{K_d}(L, P_{n-1}(d)) = 0$ (see [ASS06, (VIII.2.13)]) and obtain

$$\begin{aligned}\dim_{\Bbbk} \operatorname{Hom}_{K_r}(M, P_n^+(\mathfrak{v})) &\overset{2.7.2}{=} \dim_{\Bbbk} \operatorname{Hom}_{K_d}(\alpha^*(M), P_{n-1}(d)) \\ &= \sum_{i \geq 0} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}(P_i(d), P_{n-1}(d)) \\ &\overset{2.7.1}{=} \sum_{i=0}^{n-1} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_i(d), P_{n-1}(d)).\end{aligned}$$

(4) Follows immediately from (3) and Lemma 2.7.1.

□

Proposition 2.7.5. *Let $n \in \mathbb{N}$ and $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$. The following statements hold.*

(1) $P_n^{\pm}(\mathfrak{v})$ is a regular brick.

(2) The representation $P_n^-(\mathfrak{v})|_{\mathfrak{v}}$ is preprojective and in $\operatorname{add}(P_0(d), \ldots, P_{n+1}(d))$.

(3) $P_n^-(\mathfrak{v})|_{\mathfrak{v}} \cong (r-d)a_{n+1}(d)P_0(d) \oplus P_{n+1}(d)$.

Proof. (1) Follows from Lemma 2.5.3(3)(4) since preprojective indecomposable representations are bricks by [ASS06, (VIII.2.7)]

(2) By Lemma 2.5.3 we have $\operatorname{Hom}_{K_r}(P_{n+1}^-(\mathfrak{v}), P_n^-(\mathfrak{v})) \cong \operatorname{Hom}_{K_d}(P_{n+1}(d), P_n(d)) = 0$ and application of Proposition 2.7.4 gives us

$$P_n^-(\mathfrak{v})|_{\mathfrak{v}} = P_n^-(\mathfrak{v})|_{\mathfrak{v},\mathrm{pproj}} \in \operatorname{add}(P_0(d), \ldots, P_{n+1}(d)).$$

(3) By (2) $P_n^-(\mathfrak{v})|_{\mathfrak{v}}$ is preprojective and we have $P_n^-(\mathfrak{v})|_{\mathfrak{v}} = \sum_{i=0}^{n+1} b_i(\mathfrak{v})P_i(d)$. Application of (1), Lemma 2.7.1 and Lemma 2.7.2 imply

$$1 = \dim_{\Bbbk} \operatorname{End}_{K_r}(P_n^-(\mathfrak{v})) \cong \dim_{\Bbbk} \operatorname{Hom}_{K_d}(P_{n+1}(d), P_n^-(\mathfrak{v})|_{\mathfrak{v}}) = b_{n+1}(\mathfrak{v}).$$

Hence $P_{n+1}(d)$ appears in the direct sum decomposition of $P_n^-(\mathfrak{v})|_{\mathfrak{v}}$ with multiplicity 1. We have

$$\underline{\dim}\, P_n^-(\mathfrak{v}) = \sigma_r^{-1}(a_n(d), a_{n+1}(d)) = (a_{n+1}(d), ra_{n+1}(d) - a_n(d)).$$

Since $\underline{\dim}\, P_{n+1}(d) = (a_{n+1}(d), a_{n+2}(d))$ and $a_{n+2}(d) \overset{2.7.1}{=} da_{n+1}(d) - a_n(d)$, we conclude

$$P_n^-(\mathfrak{v})|_{\mathfrak{v}} \cong (r-d)a_{n+1}(d)P_0(d) \oplus P_{n+1}(d).$$

□

We arrive at the main result of this section:

Theorem 2.7.6. *Let $M \in \operatorname{rep}(K_r)$, $n \in \mathbb{N}_0$ and $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$. The following statements are equivalent.*

(1) $M|_{\mathfrak{v}} = (-a_{n+2}(d)\dim_{\Bbbk} M_1 + a_{n+1}(d)\dim_{\Bbbk} M_2)P_n(d) \oplus (a_{n+1}(d)\dim_{\Bbbk} M_1 - a_n(d)\dim_{\Bbbk} M_2)P_{n+1}(d)$.

(2) $M|_{\mathfrak{v}} \in \operatorname{add}(P_n(d), P_{n+1}(d))$.

(3) $\operatorname{Hom}_{K_r}(P_{n+1}^-(\mathfrak{v}), M) = 0 = \operatorname{Hom}_{K_r}(M, P_n^+(\mathfrak{v}))$.

Proof. (1) ⇒ (2). This is clear.

(2) ⇒ (3). We have $M|_{\mathfrak{v}} = M|_{\mathfrak{v},\mathrm{pproj}} = b_n(\mathfrak{v})P_n(d) \oplus b_{n+1}(\mathfrak{v})P_{n+1}(d)$. Hence $M|_{\mathfrak{v}} \in \operatorname{add}(\{P_i(d) \mid i \geq n\}) \cap \operatorname{add}(\{P_i(d) \mid i \leq n+1\})$ and Proposition 2.7.4(2)+(4) implies

$$\dim_{\Bbbk} \operatorname{Hom}_{K_r}(P_{n+1}^-(\mathfrak{v}), M) = 0 = \dim_{\Bbbk} \operatorname{Hom}_{K_r}(M, P_n^+(\mathfrak{v})).$$

(3) ⇒ (1). We apply Proposition 2.7.4(2) and conclude that $M|_{\mathfrak{v}}$ is preprojective and of the form $M|_{\mathfrak{v}} = \bigoplus_{i=0}^{n+1} b_i(\mathfrak{v})P_i(d)$. Another application of Proposition 2.7.4(4) gives us $b_i(\mathfrak{v}) = 0$ for all $0 \leq i \leq n-1$. Hence $M|_{\mathfrak{v}} \in \operatorname{add}(P_n(d), P_{n+1}(d))$. We write

$$M|_{\mathfrak{v}} = b_n(\mathfrak{v})P_n(d) \oplus b_{n+1}(\mathfrak{v})P_{n+1}(d).$$

Recall that

$$\begin{aligned} 1 &= q_d(\underline{\dim}\, P_n(d)) = q_d(a_n(d), a_{n+1}(d)) \\ &= a_{n+1}(d)^2 + a_n(d)(a_n(d) - da_{n+1}(d)) \overset{2.7.1}{=} a_{n+1}(d)^2 - a_n(d)a_{n+2}(d). \end{aligned}$$

Hence

$$A := \begin{pmatrix} a_n(d) & a_{n+1}(d) \\ a_{n+1}(d) & a_{n+2}(d) \end{pmatrix}$$

is invertible with $A^{-1} = \frac{1}{-\det(A)} \begin{pmatrix} -a_{n+2}(d) & a_{n+1}(d) \\ a_{n+1}(d) & -a_n(d) \end{pmatrix} = \begin{pmatrix} -a_{n+2}(d) & a_{n+1}(d) \\ a_{n+1}(d) & -a_n(d) \end{pmatrix}$. We conclude

$$\begin{pmatrix} b_n(\mathfrak{v}) \\ b_{n+1}(\mathfrak{v}) \end{pmatrix} = A^{-1} \begin{pmatrix} \dim_{\Bbbk} M_1 \\ \dim_{\Bbbk} M_2 \end{pmatrix} = \begin{pmatrix} -a_{n+2}(d)\dim_{\Bbbk} M_1 + a_{n+1}(d)\dim_{\Bbbk} M_2 \\ a_{n+1}(d)\dim_{\Bbbk} M_1 - a_n(d)\dim_{\Bbbk} M_2 \end{pmatrix}.$$

□

3. Rank varieties and relative projective representations

Let $d \in \{1, \dots, r-1\}$. In this chapter we introduce the full subcategory $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ of *relative d-projective* representations of $\mathrm{rep}(K_r)$, that will turn out to be equivalent to the category of Steiner bundles on the Grassmannian $\mathrm{Gr}_d(A_r)$ (see Chapter 8).

We provide several equivalent descriptions of this category. The objects of $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ may be characterized via rank varieties, whose definition is analogous to those investigated in [CFP12]. By contrast to the setting in [CFP12], $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ may also be defined as being the collection of objects of the right Hom-orthogonal category of a family of elementary K_r-modules that arises from the construction considered in Chapter 2.

3.1. Varieties

Since the notion "variety" is not a standardized term in the literature, we introduce the basic concepts below and repeat known results without proof. In doing so, we are guided by [Kem93], where varieties are introduced as "spaces with functions" with additional properties. Missing details and unexplained terminology can be found there and in [Cra20].

Definition. A *space with functions* $(X, \mathcal{O}_X)$ consists of a topological space X and $\Bbbk$-algebras $\mathcal{O}_X(U) \subseteq \mathrm{Map}(U, \Bbbk)$ for all open sets $U \subseteq X$ such that the following properties are satisfied.

(a) Given open subsets $(U_i \subseteq X)_{i \in I}$ and a map $r \in U := \bigcup_{i \in I} U_i \longrightarrow \Bbbk$, we have $r \in \mathcal{O}_X(U)$ if and only if $r|_{U_i} \in \mathcal{O}_X(U_i)$ for all $i \in I$. The elements in $\mathcal{O}_X(U)$ are called *regular maps* on U.

(b) Given $U \subseteq X$ open and $r \in \mathcal{O}_X(U)$, the set $D(r) := \{u \in U \mid r(u) \neq 0\} \subseteq U \subseteq X$ is open and

$$\frac{1}{r} \colon D(r) \longrightarrow \Bbbk \; ; \; u \mapsto \frac{1}{r(u)}$$

is contained in $\mathcal{O}_X(D(r))$.

A *morphism* $\varphi \colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of spaces with functions is a continuous map $\varphi \colon X \longrightarrow Y$ such that for all $U \subseteq Y$ and $r \in \mathcal{O}_Y(U)$ we have $r \circ \varphi \colon \varphi^{-1}(U) \longrightarrow \Bbbk \in \mathcal{O}_X(\varphi^{-1}(U))$.

Remark 3.1.1.

(1) The definition of spaces with functions is taken from [Kem93]. In [GW10] and [Per08] a space with functions does not necessarily satisfy property (b).

(2) Note that a space with functions is a ringed space (see [GW10, (2.29)]) and a morphism of spaces with functions is a morphism of ringed spaces for trivial reasons, if we consider for $\varphi^\flat$ the collection

$$\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\varphi^{-1}(U)) \; ; \; r \mapsto r \circ \varphi$$

for $U \subseteq Y$ open.

Definition.

(1) Let $(X, \mathcal{O}_X)$ be a space with functions and $Y \subseteq X$ be a (not necessarily open) subset. We extend the sections to Y by defining

$$\mathcal{O}_X(Y) \coloneqq \{r \in \mathrm{Map}(Y, \Bbbk) \mid \forall y \in Y \colon \exists y \in U_y \subseteq X \text{ open}, s \in \mathcal{O}_X(U_y) \colon r|_{U_y \cap Y} = s|_{U_y \cap Y}\}.$$

(2) Let $(X, \mathcal{O}_X)$ be a space with functions and $Y \subseteq X$ be a (not necessarily open) subset. We equip Y with the subspace topology and define $\mathcal{O}_Y(U) \coloneqq \mathcal{O}_X(U)$ for $U \subseteq Y$ open. This renders $(Y, \mathcal{O}_Y)$ a space with functions. Usually, we write $(Y, \mathcal{O}_{X|Y})$ to emphasize the ambient space X.

The following Lemma is an important tool to show that a given map is a morphism of spaces with functions. The proof is straightforward (cf. [Gat21, (4.6)]).

Lemma 3.1.2 (Gluing property for morphisms)**.** *Let $\varphi \colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a map between spaces with functions. Assume that there is an open cover $X = \bigcup_{i \in I} U_i$ such that $\varphi_i \coloneqq \varphi|_{U_i} \colon (U_i, \mathcal{O}_{X|U_i}) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of spaces with functions for all $i \in I$. Then φ is a morphism of spaces with functions.*

Let $r \in \mathbb{N}$ and $\mathbb{A}^r \coloneqq \mathbb{A}^r(\Bbbk) \coloneqq \Bbbk^r$. We define a space with functions denoted by $(\mathbb{A}^r, \mathcal{O}_{\mathbb{A}^r})$. The underlying topology on $\mathbb{A}^r$ is the *Zariski topology*, i.e. the closed subsets (called *affine algebraic sets*) are of the form

$$\mathcal{V}(F) \coloneqq \{x \in \mathbb{A}^r \mid \forall f \in F \colon f(x) = 0\} \subseteq \mathbb{A}^r$$

for $F \subseteq \Bbbk[X_1, \ldots, X_r]$. Given $f \in \Bbbk[X_1, \ldots, X_r]$ we define $D(f) \coloneqq \mathbb{A}^r \setminus \mathcal{V}(f)$[1]. By Hilbert's basis theorem the $\Bbbk$-algebra $\Bbbk[X_1, \ldots, X_r]$ is noetherian and we find $f_1, \ldots, f_n$ such that $(f_1, \ldots, f_n) = (F)$. We conclude

$$\mathcal{V}(F) = \mathcal{V}((F)) = \mathcal{V}((f_1, \ldots, f_n)) = \mathcal{V}(\{f_1, \ldots, f_n\}) = \bigcap_{i=1}^n \mathcal{V}(f_i).$$

This shows that $\mathbb{A}^r \setminus \mathcal{V}(F) = \bigcup_{i=1}^n D(f_i)$. Hence $(D(f))_{f \in \Bbbk[X_1, \ldots, X_r]}$ is a basis of the topological space $\mathbb{A}^r$. Given $f, g \in \Bbbk[X_1, \ldots, X_r]$ we denote by $\frac{f}{g}$ the map $\frac{f}{g} \colon D(g) \longrightarrow \Bbbk \; ; \; x \mapsto \frac{f(x)}{g(x)}$. For $U \subseteq \mathbb{A}^r$ open, we define

$$\mathcal{O}_{\mathbb{A}^r}(U) \coloneqq \{r \colon U \longrightarrow \Bbbk \mid \forall x \in U \colon \exists U_x \subseteq U \text{ open}, f, g \in \Bbbk[X_1, \ldots, X_r], x \in U_x \subseteq D(g) \colon r|_{U_x} = \frac{f}{g}|_{U_x}\}.$$

Direct computation shows that $(\mathbb{A}^r, \mathcal{O}_{A_r})$ is a space with functions.

[1] We consider f as a map $f \colon \mathbb{A}^r \longrightarrow \Bbbk$. By Lemma 3.1.4 we have $f \colon \mathbb{A}^r \longrightarrow \Bbbk \in \mathcal{O}_{\mathbb{A}^r}(\mathbb{A}^r)$. Hence the notation $D(f) = \mathbb{A}^r \setminus \mathcal{V}(f)$ is justified.

Definition. Let $r \in \mathbb{N}$, then $(\mathbb{A}^r, \mathcal{O}_{A_r})$ is called *(full) affine space of dimension* r.

Remark 3.1.3. Usually, we consider an r-dimensional $\Bbbk$-vector space V and denote by $S(V^*)$ the symmetric $\Bbbk$-algebra over the dual space $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$. We let $\iota\colon V^* \longrightarrow S(V^*)$ be the inclusion map. Given $v \in V$, we consider the linear evaluation map $\mathrm{ev}_v\colon V^* \longrightarrow \Bbbk\ ;\ f \mapsto f(v)$. The universal property of $S(V^*)$ yields a unique morphism of $\Bbbk$-algebras $\overline{\mathrm{ev}}_v\colon S(V^*) \longrightarrow \Bbbk$ such that $\overline{\mathrm{ev}}_v \circ \iota = \mathrm{ev}_v$. Given $f \in S(V^*)$ and $v \in V$, we let $f(v) \coloneqq \overline{\mathrm{ev}}_v(f)$. The analogous construction as for $\mathbb{A}^r$ yields the sets

$$\mathcal{V}(F) \coloneqq \{x \in V \mid \forall f \in F\colon f(x) = 0\} \subseteq V$$

for $F \subseteq S(V^*)$, $D(f) \coloneqq \{v \in V \mid f(x) \neq 0\}$ for $f \in S(V^*)$ and

$$\mathcal{O}_V(U) \coloneqq \{r\colon U \longrightarrow \Bbbk \mid \forall x \in U\colon \exists U_x \subseteq U \text{ open}, f, g \in S(V^*), x \in U_x \subseteq D(g)\colon r|_{U_x} = \frac{f}{g}|_{U_x}\}.$$

This renders $(V, \mathcal{O}_V)$ a space with functions that we denote by $(\mathbb{A}(V), \mathcal{O}_{\mathbb{A}(V)})$. Each choice of a basis $(v_1, \ldots, v_r)$ of V yields an isomorphism $\Bbbk[X_1, \ldots, X_r] \cong S(V^*)$ of $\Bbbk$-algebras and an isomorphism of spaces with functions $(\mathbb{A}(V), \mathcal{O}_{\mathbb{A}(V)}) \cong (\mathbb{A}^r, \mathcal{O}_{\mathbb{A}^r})$.

Morphisms between spaces with functions are well understood in the following special case.

Lemma 3.1.4. *(cf. [Cra20, (1.1D, 1.2C)]) Let V be a $\Bbbk$-vector space, $\varnothing \neq Y \subseteq \mathbb{A}(V)$ be a set and $(X, \mathcal{O}_X)$ be a space with functions. Let $(v_1, \ldots, v_r)$ be a $\Bbbk$-basis of V and $\varphi\colon X \longrightarrow Y$ be a map. We write $\varphi = \sum_{i=1}^r \varphi_i v_i$ with maps $\varphi_i\colon X \longrightarrow \Bbbk$. The following statements are equivalent.*

(1) The map $\varphi\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_{\mathbb{A}(V)|Y})$ is a morphism of spaces with functions.

(2) We have $\varphi_i \in \mathcal{O}_X(X)$ for all $i \in \{1, \ldots, r\}$.

In particular, the morphisms from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_{\mathbb{A}(V)|Y})$ form a $\Bbbk$-vector space.

Now we can give the definition of a variety.

Definition.

(1) A space with functions $(Y, \mathcal{O}_Y)$ is called an *affine variety* if there exist $n \in \mathbb{N}$ and an affine algebraic set $X \subseteq \mathbb{A}^n$ such that $(Y, \mathcal{O}_Y) \cong (X, \mathcal{O}_{\mathbb{A}^n|X})$.

(2) A space with functions $(X, \mathcal{O}_X)$ is called *separated* if the *diagonal*

$$\Delta_X \coloneqq \{(x, x) \mid x \in X\}$$

is closed in the topological space $X \times X$ of the space with functions $(X \times X, \mathcal{O}_{X \times X})$ (see [Kem93, (3.1.1)]).

(3) A *prevariety* is a space with functions $(X, \mathcal{O}_X)$ that has a finite open cover $X = \bigcup_{i=1}^n U_i$ such that each $(U_i, \mathcal{O}_{X|U_i})$ is an affine variety. A *variety* is a separated prevariety.

(4) Let $(X, \mathcal{O}_X)$ be a variety. A subvariety of $(X, \mathcal{O}_X)$ is a space with functions $(Y, \mathcal{O}_{X|Y})$ for $Y \subseteq X$ locally closed.

We consider a vector space V of dimension $r \in \mathbb{N}_{\geq 2}$ and let $\mathbb{P}(V) \coloneqq \{\Bbbk v \mid v \in V \setminus \{0\}\}$. Given $\mathfrak{v} \in \mathbb{P}(V)$ and $f \in S(V^*)$ we write $f(\mathfrak{v}) = 0$, provided $f(x) = 0$ for all $x \in \mathfrak{v} \setminus \{0\}$. We equip $\mathbb{P}(V)$ with the *Zariski topology*, i.e. the closed subsets (called *projective algebraic sets*) are of the form

$$\mathcal{V}^+(F) \coloneqq \{\mathfrak{v} \in \mathbb{P}(V) \mid \forall f \in F \colon f(\mathfrak{v}) = 0\} \subseteq \mathbb{P}(V),$$

where $F \subseteq S(V^*)$. Given $g \in S(V^*)$ we let $D^+(g) \coloneqq \mathbb{P}(V) \setminus \mathcal{V}^+(g)$. Since the tensor algebra $T(V^*)$ is positively graded and $S(V^*)$ is its quotient by the homogeneous ideal generated by $f \otimes g - g \otimes f$ $(f, g \in V^*)$, $S(V^*)$ is positively graded. We write $S(V^*) = \bigoplus_{d \geq 0} S(V^*)_d$. Let $d \in \mathbb{N}_0$ and $f, g \in S(V^*)_d$ with $g \neq 0$. The element $\frac{f}{g}$ in the localization of $S(V^*)$ at $\{g^n \mid n \in \mathbb{N}_0\}$ defines a map

$$\frac{f}{g} \colon D^+(g) \longrightarrow \Bbbk \; ; \; \mathfrak{v} = \Bbbk x \mapsto \frac{f(x)}{g(x)}.$$

Let $U \subseteq \mathbb{P}(V)$ be open, $r \colon U \longrightarrow \Bbbk$ be a map and $x \in U$. We say that r is *regular* at x, provided there exist $x \in U_x \subseteq U$ open, $d \in \mathbb{N}_0$ and $f, g \in S(V^*)_d$ such that $U_x \subseteq D^+(g)$ and $r|_{U_x} = \frac{f}{g}|_{U_x}$. For $U \subseteq \mathbb{P}(V)$ open, we set

$$\mathcal{O}_{\mathbb{P}(V)}(U) = \{r \in \mathrm{Map}(U, \Bbbk) \mid r \text{ regular at all } x \in U\}.$$

and obtain a space with functions $(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)})$. By [Per08, (III.8.5), (III.9.9)] $\mathbb{P}(V)$ is a variety that is not an affine variety.

Definition. Let $(Y, \mathcal{O}_Y)$ be a space with functions.

(1) We call $(Y, \mathcal{O}_Y)$ a *projective variety* if there exist a finite dimensional vector space V and a projective algebraic set $X \subseteq \mathbb{P}(V)$ such that $(Y, \mathcal{O}_Y) \cong (X, \mathcal{O}_{\mathbb{P}(V)|X})$.

(2) We call $(Y, \mathcal{O}_Y)$ *irreducible* if the underlying topological space Y is irreducible, i.e. all non-empty open subsets $O_1, O_2 \subseteq Y$ satisfy $O_1 \cap O_2 \neq \varnothing$.

Let $r \geq 2$ and V be an r-dimensional vector space and $1 \leq d \leq r$. We denote by

$$\mathrm{Gr}_d(V) \coloneqq \{\mathfrak{v} \subseteq V \mid \mathfrak{v} \text{ subspace}, \dim_{\Bbbk} \mathfrak{v} = d\}$$

the *Grassmannian* of d-dimensional subspaces and by

$$\mathrm{pl}_{V,d} \colon \mathrm{Gr}_d(V) \longrightarrow \mathbb{P}(\bigwedge^d V) \; ; \; \mathfrak{v} \mapsto \Bbbk \left(\bigwedge^d \mathfrak{v} \right)$$

the *Plücker embedding* which is in fact an injective map. Moreover, one can show (see [Gec13, (3.3.8)]) that $Y \coloneqq \mathrm{im}\, \mathrm{pl}_{V,d} \subseteq \mathbb{P}(\bigwedge^d V)$ is a closed subset. We equip $\mathrm{Gr}_d(V)$ with the structure of a space with functions via structure transport, i.e. a subset $X \subseteq \mathrm{Gr}_d(V)$ is closed if and only if $\mathrm{pl}_{V,d}(X)$ is closed in Y (if and only if $\mathrm{pl}_{V,d}(X)$ is closed in $\mathbb{P}(\bigwedge^d V)$). Consider the homeomorphism

$p \coloneqq \mathrm{Gr}_d(V) \longrightarrow Y \ ; \ \mathfrak{v} \mapsto \mathrm{pl}_{V,d}(\mathfrak{v})$. For $U \subseteq \mathrm{Gr}_d(V)$ open the set $O \coloneqq p(U)$ is open in Y and we define

$$\mathcal{O}_{\mathrm{Gr}_d(V)}(U) = \{g \colon U \longrightarrow \Bbbk \mid g \circ p^{-1} \in \mathcal{O}_Y(O)\}.$$

By definition p is an isomorphism of spaces with functions $p \colon (\mathrm{Gr}_d(V), \mathcal{O}_{\mathrm{Gr}_d(V)}) \longrightarrow (Y, \mathcal{O}_Y)$. Hence $(\mathrm{Gr}_d(V), \mathcal{O}_{\mathrm{Gr}_d(V)})$ is a projective variety. Furthermore, $\mathrm{Gr}_d(V)$ is irreducible (see Corollary C.3.3) and has an open cover by affine spaces each of dimension $d(r-d)$ (see [EH16, §3.2.2]), hence $\dim \mathrm{Gr}_d(V) = d(r-d)$.

Definition.

(1) A space with functions $(X, \mathcal{O}_X)$ is called *connected* (*noetherian*), provided the underlying topological space X is connected (noetherian).

(2) A *quasi-affine (quasi-projective) variety* is an open subvariety of an affine (a projective) variety.

In the following result, we collect basic properties that we will use throughout this thesis without further reference.

Theorem 3.1.5. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be spaces with functions.*

(1) Subvarieties are varieties.

(2) Varieties are noetherian.

(3) (Quasi-)affine and (quasi-)projective spaces are varieties.

(4) If X and Y are (quasi-)affine varieties, then $X \times Y$ is a (quasi-)affine variety.

(5) If X and Y are prevarietes, then $X \times Y$ is a prevariety.

(6) If X and Y are varieties, then $X \times Y$ is a variety.

(7) If X and Y are (quasi-)projective varieties, then $X \times Y$ is a (quasi-)projective variety.

(8) Affine space and projective space are irreducible.

3.2. The d-th rank variety

In this section we introduce, in analogy to [CFP12], the rank variety for Kronecker representations.

Definition. Let X be a topological space and $\varphi \colon X \longrightarrow \mathbb{N}_0$ be a map. We say that φ is *upper semicontinuous*, provided $\varphi^{-1}(\mathbb{N}_{\geq n})$ is closed for every $n \in \mathbb{N}_0$, and *lower semicontinuous*, provided $\varphi^{-1}(\mathbb{N}_{\leq n})$ is closed for all $n \in \mathbb{N}_0$.

Example 3.2.1.

(1) *Let V, W be finite dimensional $\Bbbk$-vector spaces. The map*

$$\mathrm{rk}\colon \mathrm{Hom}_{\Bbbk}(V,W) \longrightarrow \mathbb{N}_0\ ;\ f \mapsto \mathrm{rk}(f)$$

is lower semicontinuous (see [Kem93, (1.7.2)]).

(2) *Let $\varphi\colon X \longrightarrow Y$ be a morphism of varieties. The map*

$$\varepsilon_{\varphi}\colon X \longrightarrow \mathbb{N}_0\ ;\ x \mapsto \dim_x \varphi^{-1}(\varphi(x))$$

is upper semicontinuous (see [Kem93, (6.4.5)])[2].

Example 3.2.1(1) implies that $\mathrm{Inj}_{\Bbbk}(A_d, A_r) = \{\alpha \in \mathrm{Hom}_{\Bbbk}(A_d, A_r) \mid \mathrm{rk}(\alpha) > d-1\}$ is a quasi-affine variety for $1 \leq d \leq r$. Moreover, (see Theorem C.3.2) the map

$$\mathsf{im}\colon \mathrm{Inj}_{\Bbbk}(A_d, A_r) \longrightarrow \mathrm{Gr}_d(A_r)\ ;\ \alpha \mapsto \mathrm{im}\,\alpha$$

is a $\mathrm{GL}(A_d)$-principal bundle locally trivial in the Zariski topology with respect to the right $\mathrm{GL}(A_d)$-action $\alpha.h \coloneqq \alpha \circ h$ for all $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$, $h \in \mathrm{GL}(A_d)$.

Let $1 \leq d < r$, $M \in \mathrm{rep}(K_r)$ be a representation and $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$. We define

$$\psi_{M,\mathfrak{v}} \coloneqq \psi_M|_{\mathfrak{v}\otimes_{\Bbbk} M_1}\colon\ \mathfrak{v}\otimes_{\Bbbk} M_1 \longrightarrow M_2.$$

Given $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ such that $\mathrm{im}\,\alpha = \mathfrak{v}$, the map $\alpha^{-1}\otimes \mathrm{id}_{M_1}\colon\ \mathfrak{v}\otimes_{\Bbbk} M_1 \longrightarrow A_d \otimes_{\Bbbk} M_1$ is an isomorphism and therefore

$$\mathrm{rk}(\psi_{\alpha^*(M)}) = \mathrm{rk}(\psi_M \circ (\alpha \otimes \mathrm{id}_{M_1})) = \mathrm{rk}(\psi_M \circ (\alpha \otimes \mathrm{id}_{M_1}) \circ (\alpha^{-1} \otimes \mathrm{id}_{M_1})) = \mathrm{rk}(\psi_{M,\mathfrak{v}}).$$

Hence we have

$$(\dagger) \quad \mathrm{rk}(\psi_{M,\mathfrak{v}}) = \mathrm{rk}(\psi_{\alpha^*(M)}).$$

In particular, $\mathrm{rk}(\psi_{\alpha^*(M)}) = \mathrm{rk}(\psi_{\beta^*(M)})$ for all $\alpha, \beta \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ such that $\mathrm{im}\,\alpha = \mathrm{im}\,\beta$.

Lemma 3.2.2. *Let $M \in \mathrm{rep}(K_r)$ be a representation. The following statements hold.*

(1) *The set*

$$r(K_r, d)_M \coloneqq \{\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r) \mid \mathrm{rk}(\psi_{\alpha^*(M)}) < d\dim_{\Bbbk} M_1\}$$

is a closed subset of $\mathrm{Inj}_{\Bbbk}(A_d, A_r)$ and $\mathrm{GL}(A_d)$-stable.

(2) *The set*

$$\mathcal{R}(K_r, d)_M \coloneqq \{\mathfrak{v} \in \mathrm{Gr}_d(A_r) \mid \mathrm{rk}(\psi_{M,\mathfrak{v}}) < d\dim_{\Bbbk} M_1\}$$

is a closed subset of $\mathrm{Gr}_d(A_r)$.

[2] Here $\dim_x$ denotes the *local dimension* at x, i.e. the maximal dimension of an irreducible component containing x.

Proof. (1) It is clear that the set is $\mathrm{GL}(A_d)$-stable. By Example 3.2.1(1) the map

$$\mathrm{rk}\colon \mathrm{Hom}_{\Bbbk}(A_d \otimes_{\Bbbk} M_1, M_2) \longrightarrow \mathbb{N}_0\ ;\ f \mapsto \mathrm{rk}(f)$$

is lower semicontinuous and $A \coloneqq \mathrm{rk}^{-1}(\mathbb{N}_{>\dim_{\Bbbk} A_d \otimes_{\Bbbk} M_1 - 1})$ is open in $\mathrm{Hom}_{\Bbbk}(A_d \otimes_{\Bbbk} M_1, M_2)$. Hence the same is true for $\varphi^{-1}(A)$, where φ is the $\Bbbk$-linear map

$$\varphi\colon \mathrm{Hom}_{\Bbbk}(A_d, A_r) \longrightarrow \mathrm{Hom}_{\Bbbk}(A_d \otimes_{\Bbbk} M_1, M_2)\ ;\ f \mapsto \psi_M \circ (f \otimes \mathrm{id}_{M_1}).$$

Note that $\varphi^{-1}(A) \subseteq \mathrm{Inj}_{\Bbbk}(A_d, A_r)$. Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$. We have $\alpha \in \varphi^{-1}(A)$ if and only if $\mathrm{rk}(\psi_{\alpha^*(M)}) \geq \dim_{\Bbbk} A_d \otimes_{\Bbbk} M_1 = d \dim_{\Bbbk} M_1$. Hence $r(K_r, d)_M = \mathrm{Inj}_{\Bbbk}(A_d, A_r) \setminus \varphi^{-1}(A)$ is closed.

(2) We consider the orbit map $\pi\colon \mathrm{Inj}_{\Bbbk}(A_d, A_r) \longrightarrow \mathrm{Inj}_{\Bbbk}(A_d, A_r)/\mathrm{GL}(A_d)$. Since $r(K_r, d)_M$ is $\mathrm{GL}(A_d)$-stable, we have

$$\pi^{-1}(\pi(r(K_r, d)_M)) = r(K_r, d)_M,$$

therefore (1) and the definition of the quotient topology (see Appendix C.2) imply that $\pi(r(K_r, d)_M)$ is closed in $\mathrm{Inj}_{\Bbbk}(A_d, A_r)/\mathrm{GL}(A_d)$. Application of the isomorphism of spaces with functions (see Corollary C.3.3)

$$\overline{\mathsf{im}}\colon \mathrm{Inj}_{\Bbbk}(A_d, A_r)/\mathrm{GL}(A_d) \longrightarrow \mathrm{Gr}_d(A_r)\ ;\ \alpha\,\mathrm{GL}(A_d) \mapsto \mathrm{im}\,\alpha$$

shows that $\overline{\mathsf{im}}(\pi(r(K_r, d)_M))$ is closed in $\mathrm{Gr}_d(A_r)$[3]. By (†) we have

$$\overline{\mathsf{im}}(\pi(r(K_r, d)_M)) = \mathsf{im}(r(K_r, d)_M) = \mathcal{R}(K_r, d)_M.$$

□

Definition. Given a representation $M \in \mathrm{rep}(K_r)$, we call the closed subset

$$\mathcal{R}(K_r, d)_M = \{\mathfrak{v} \in \mathrm{Gr}_d(A_r) \mid \mathrm{rk}(\psi_{M,\mathfrak{v}}) < d \dim_{\Bbbk} M_1\} \subseteq \mathrm{Gr}_d(A_r)$$

the *d-th rank variety* of M.

3.3. Relative projective representations

Throughout this section d denotes a natural number with $1 \leq d < r$. In particular, we have $r \geq 2$.

Definition. Given $M \in \mathrm{rep}(K_r)$ we define (cf. Lemma 2.7.3)

$$\begin{aligned}\mathcal{V}(K_r, d)_M \coloneqq &\{\mathfrak{v} \in \mathrm{Gr}_d(A_r) \mid \alpha^*(M) \text{ is not projective for all } \alpha \in \mathrm{im}^{-1}(\mathfrak{v})\}.\\ =&\{\mathfrak{v} \in \mathrm{Gr}_d(A_r) \mid M|_{\mathfrak{v}} \text{ is not projective}\}\end{aligned}$$

and let $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ be the full subcategory of $\mathrm{rep}(K_r)$, whose objects M are given by the property $\mathcal{V}(K_r, d)_M = \varnothing$. We call the representations in $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ *relative d-projective*.

[3] We only need that the map is a homeomorphism of the underlying topological spaces.

Lemma 3.3.1. *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *be an exact sequence in* $\operatorname{rep}(K_r)$.

(1) We have

$$\mathcal{V}(K_r,d)_{M'} \subseteq \mathcal{V}(K_r,d)_M \subseteq \mathcal{V}(K_r,d)_{M'} \cup \mathcal{V}(K_r,d)_{M''}.$$

(2) If the sequence splits, then

$$\mathcal{V}(K_r,d)_M = \mathcal{V}(K_r,d)_{M'} \cup \mathcal{V}(K_r,d)_{M''}.$$

Proof. (1) Since $\operatorname{rep}(K_d)$ is a hereditary category, we have $\mathcal{V}(K_r,d)_{M'} \subseteq \mathcal{V}(K_r,d)_M$. If $\mathfrak{v} \notin \mathcal{V}(K_r,d)_{M'} \cup \mathcal{V}(K_r,d)_{M''}$, $M'|_{\mathfrak{v}}$ and $M''|_{\mathfrak{v}} \in \operatorname{rep}(K_d)$ are projective. Hence the restriction of the given exact sequence splits and $M|_{\mathfrak{v}}$ is projective, i.e. $\mathfrak{v} \notin \mathcal{V}(K_r,d)_M$.

(2) This is a direct consequence of (1).

□

Definition. For $(x,y) \in \mathbb{N}_0^2$ and $d \in \{1,\dots,r\}$ we write

$$\Delta_{(x,y)}(d) \coloneqq y - dx \text{ and } \nabla_{(x,y)}(d) \coloneqq dy - x.$$

For a representation $M \in \operatorname{rep}(K_r)$, or vector spaces $M_1, M_2 \in \operatorname{mod}\Bbbk$, we define

$$\Delta_M(d) \coloneqq \Delta_{(M_1,M_2)}(d) \coloneqq \Delta_{(\dim_\Bbbk M_1, \dim_\Bbbk M_2)}(d) \text{ and}$$

$$\nabla_M(d) \coloneqq \nabla_{(M_1,M_2)}(d) \coloneqq \nabla_{(\dim_\Bbbk M_1, \dim_\Bbbk M_2)}(d).$$

For $d = 1$ we just write $\Delta_{(x,y)}$ and $\nabla_{(x,y)}$.

We will prove in Chapter 8 that the category $\operatorname{rep}_{\mathrm{proj}}(K_r,d)$ is equivalent to the category of Steiner bundles on the Grassmann variety $\operatorname{Gr}_d(A_r)$. In the following we begin with the systematic investigation of the category $\operatorname{rep}_{\mathrm{proj}}(K_r,d)$. This will allow us later on to draw conclusions for the category of Steiner bundles. The next result is one of the main results of [BF24]. As a first application of Lemma 2.7.2 we give a new proof, thereby establishing several equivalent definitions of $\operatorname{rep}_{\mathrm{proj}}(K_r,d)$, which we will use from now on (sometimes without further notice).

Theorem 3.3.2. *Let* $1 \le d < r$. *Let* $M \in \operatorname{rep}(K_r)$, $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$ *and* $\alpha \in \operatorname{Inj}_\Bbbk(A_d, A_r)$ *be such that* $\operatorname{im}\alpha = \mathfrak{v}$.

(1) We have an isomorphism of vector spaces $\operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M) \cong \ker \psi_{\alpha^*(M)}$.

(2) We have

$$\begin{aligned}\operatorname{rk}(\psi_{M,\mathfrak{v}}) &= d\dim_\Bbbk M_1 - \dim_\Bbbk \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M)\\ &= \dim_\Bbbk M_2 - \dim_\Bbbk \operatorname{Ext}^1_{K_r}(P_1^-(\mathfrak{v}), M).\end{aligned}$$

(3) The following statements are equivalent.

(i) $\psi_{\alpha^*(M)}$ *is injective.*

(ii) $\operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M) = 0$.

(iii) $\alpha^*(M) \cong \Delta_M(d)P_0(d) \oplus (\dim_{\Bbbk} M_1)P_1(d)$.

(vi) $\mathfrak{v} \notin \mathcal{V}(K_r, d)_M$, *i.e.* $M|_{\mathfrak{v}}$ *is projective.*

(4) *We have* $\mathcal{V}(K_r, d)_M = \mathcal{R}(K_r, d)_M$.

(5) *We have*

$$\begin{aligned}\operatorname{rep}_{\mathrm{proj}}(K_r, d) &= \{M \in \operatorname{rep}(K_r) \mid \forall\, \mathfrak{v} \in \operatorname{Gr}_d(A_r)\colon\ \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M) = 0\} \\ &= \{M \in \operatorname{rep}(K_r) \mid \forall\, \mathfrak{v} \in \operatorname{Gr}_d(A_r)\colon\ \dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(P_1^-(\mathfrak{v}), M) = \Delta_M(d)\} \\ &= \{M \in \operatorname{rep}(K_r) \mid \forall\, \mathfrak{v} \in \operatorname{Gr}_d(A_r)\colon\ \operatorname{rk}(\psi_{M,\mathfrak{v}}) = d \dim_{\Bbbk} M_1\}.\end{aligned}$$

Proof. (1) By Lemma 2.7.2 we

$$\operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M) \cong \operatorname{Hom}_{K_d}(P_1(d), (\sigma_{K_d} \circ \alpha^*)(M)) \cong (\sigma_{K_d}(\alpha^*(M)))_1 = \ker \psi_{\alpha^*(M)}.$$

(2) By (1) we have

$$\operatorname{rk}(\psi_{M,\mathfrak{v}}) = \dim_{\Bbbk} \mathfrak{v} \otimes_{\Bbbk} M_1 - \ker \psi_{\alpha^*(M)} = d \dim_{\Bbbk} M_1 - \dim_{\Bbbk} \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M).$$

Moreover, we have $\underline{\dim}\, P_1^-(\mathfrak{v}) = (d, rd-1)$ and conclude with the Euler-Ringel form

$$\begin{aligned}\dim_{\Bbbk} \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M) &= \dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(P_1^-(\mathfrak{v}), M) + \langle (d, rd-1), \underline{\dim}\, M\rangle_r \\ &= \dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(P_1^-(\mathfrak{v}), M) + d \dim_{\Bbbk} M_1 - \dim_{\Bbbk} M_2.\end{aligned}$$

(3) We show three equivalences:

(i) $\Leftrightarrow$ (ii). This follows from (2).

(ii)$\Leftrightarrow$ (iii). Let $g \in \operatorname{GL}(A_r)$ be such that $g|_{A_d} = \alpha$. Then

$$P_0^+(\mathfrak{v}) \cong g.\sigma_{K_r}(\inf(P_0(d)) = g.\sigma_{K_r}(P_0(r)) = 0$$

Therefore, the statement follows from Theorem 2.7.6 for $n = 0$.

(iii) $\Leftrightarrow$ (iv). This follows from the definition (see also Lemma 2.7.3).

(4) Follows from (3).

(5) This is a direct consequence of (2) and (3).

□

In Chapter 6 we will study so-called elementary Kronecker representations in detail and discuss their importance in the study of hereditary algebras. In the following, we examine which representations of the form $P_n^{\pm}(\mathfrak{v})$ (with $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$ and $n \in \mathbb{N}_0$) are elementary. We start this investigation by recalling the definition:

Definition. Let $r \geq 2$. A non-zero regular representation $E \in \operatorname{rep}(K_r)$ is called *elementary*, provided there is no short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

with $A, B \in \operatorname{rep}(K_r)$ regular and non-zero.

Remark 3.3.3. In the literature (and also in Chapter 6), elementary representations are usually defined only for wild hereditary algebras. In this section we include the tame case $r = 2$. For $r = 2$ an elementary representation is the same as a quasi-simple representation at the mouth of a homogeneous tube.

Lemma 3.3.4. *The following statements are equivalent for $X \in \operatorname{rep}(K_d)$.*

(i) $\inf(X) \in \operatorname{rep}(K_r)$ *is elementary.*

(ii) $X \cong P_1(d), I_1(d)$ *or* X *is elementary with* $\dim_{\Bbbk} X_i \leq d$ *for all* $i \in \{1, 2\}$[4].

Proof. (i) $\Rightarrow$ (ii). We assume that $X \ncong P_1(d), I_1(d)$. Since $\inf(X)$ is indecomposable, the same is true for X. Let $0 \neq x \in X_1$ and denote by $U \subseteq X$ the indecomposable subrepresentation of X generated by x. We have $U \ncong I_0(d), P_0(d)$ and conclude with Lemma 2.4.3(2)(3) that $\inf(U)$ indecomposable and regular in $\operatorname{rep}(K_r)$. The short exact sequence

$$0 \longrightarrow \inf(U) \longrightarrow \inf(X) \longrightarrow \inf(X/U) \longrightarrow 0$$

in conjunction with Proposition 6.1.1 implies that $\inf(X/U)$ is preinjective. Another application of Lemma 2.4.3(3) shows that $X/U \in \operatorname{add}(I_0(d))$. We conclude $0 = \dim_{\Bbbk}(X/U)_2 = \dim_{\Bbbk} X_2 - \dim_{\Bbbk} U_2$. Moreover, we have $\dim_{\Bbbk} U_2 \leq d \dim_{\Bbbk} U_1 = d$. Hence $\dim_{\Bbbk} X_2 \leq d$. Since $\inf(D_{K_d}(X)) \cong D_{K_r}(\inf(X))$ is also elementary, we conclude $\dim_{\Bbbk} X_1 \leq d$.

(ii) $\Rightarrow$ (i). Assume that X is elementary with $\dim_{\Bbbk} X_i \leq d$ for $i \in \{1, 2\}$. Let $V \subseteq \inf(X)$ be a proper subrepresentation and $0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$ be a short exact sequence in $\operatorname{rep}(K_d)$ such that application of $\inf$ gives the sequence $0 \longrightarrow V \longrightarrow \inf(X) \longrightarrow \inf(X)/V \longrightarrow 0$. Since X is elementary, either A is preprojective or B is preinjective (see Proposition 6.1.1). We consider the case that A is preprojective, the other case is dual. By dimension reasons we have $A \cong xP_0(d)$ for $x \in \{1, \ldots, d\}$ or $A \cong P_1(d)$. If $A \cong xP_0(d)$, then we obtain that V is projective. For $A \cong P_1(d)$, the quotient $\inf(X)/V$ is in $\operatorname{add}(I_0(r))$. Hence $\inf(X)$ is elementary. The cases $X \cong P_1(d)$ and $X \cong I_1(d)$ follow in the same fashion. □

Proposition 3.3.5. *Let $n \in \mathbb{N}_0$ and $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$. The representation $P_n^{\pm}(\mathfrak{v})$ is elementary if and only if $n = 1$.*

Proof. By Lemma 3.3.4 the representation $\inf(P_n(d))$ is elementary if and only $n = 1$. This is equivalent to $(\sigma_{K_r}^{\pm} \circ \inf)(P_n(d))$ being elementary. This does not change under twists by $g \in \operatorname{GL}(A_r)$. Hence $P_n^{\pm}(\mathfrak{v})$ is elementary if and only if $n = 1$. □

[4] By definition $\operatorname{rep}(K_1)$ does not contain elementary representations and $I_1(1) = P_1(1)$. Therefore, statement (ii) for $d = 1$ is equivalent to $X \cong P_1(d)$.

We consider the projective Kronecker representations $(0, A_d) \cong dP_0(d)$ and $(\Bbbk, A_d) \cong P_1(d)$, with structure map $\psi_{P_1(d)}(a \otimes t) = ta$. Given $\mathrm{Inj}_\Bbbk(A_d, A_r)$, we define an injective morphism of representations

$$\overline{\alpha} \colon (0, A_d) \longrightarrow (\Bbbk, A_r)$$

by setting $\overline{\alpha}_1 = 0$ and $\overline{\alpha}_2 \coloneqq \alpha$. The following result shows that our approach generalizes the ad hoc constructions in [BF24, (2.13), (2.14)] and [Bis20, (2.2)].

Corollary 3.3.6. *Let $1 \leq d < r$.*

(1) For $\alpha \in \mathrm{Inj}_\Bbbk(A_d, A_r)$ we have

$$P_1^+(\operatorname{im}\alpha) \cong D_{K_r}(\operatorname{coker}\overline{\alpha}) \text{ and } P_1^-(\operatorname{im}\alpha) \cong D_{K_r}(\tau_{K_r}(\operatorname{coker}\overline{\alpha})).$$

(2) Let $1 < d < r$ and $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$, then $P_1^-(\mathfrak{v}) \in \mathrm{rep}_{\mathrm{proj}}(K_r, d-1)$.

Proof. (1) We define $E \coloneqq D_{K_r}(\tau_{K_r}(\operatorname{coker}\overline{\alpha}))$. By Proposition 2.7.5 we have

$$\alpha^*(P_1^-(\operatorname{im}\alpha)) = (r-d)a_2(d)P_0(d) \oplus P_2(d).$$

We conclude with [BF24, (2.1.5)] that $0 \neq \mathrm{Hom}_{K_r}(E, P_1^-(\operatorname{im}\alpha))$. By Proposition 3.3.5 and [BF24, (2.1.3)] both representations are elementary. Since they also have the same dimension vector $(d, rd-1)$, we conclude with [KL96, (1.4)] that $E \cong P_1^-(\operatorname{im}\alpha)$. Moreover, we have

$$D_{K_r}(\operatorname{coker}\overline{\alpha}) \cong \tau_{K_r}(E) \cong \tau_{K_r}(P_1^-(\operatorname{im}\alpha)) \overset{2.5.2}{\cong} P_1^+(\operatorname{im}\alpha).$$

(2) Let $\mathfrak{u} \in \mathrm{Gr}_{d-1}(A_r)$. In view of Theorem 3.3.2, it suffices to show that

$$0 = \mathrm{Hom}_{K_r}(P_1^-(\mathfrak{u}), P_1^-(\mathfrak{v})).$$

By (1) we find $\alpha \in \mathrm{Inj}_\Bbbk(A_d, A_r)$ and $\beta \in \mathrm{Inj}_\Bbbk(A_{d-1}, A_r)$ such that $P_1^-(\mathfrak{v}) \cong D_{K_r}(\tau_{K_r}(\operatorname{coker}\overline{\alpha}))$ and $P_1^-(\mathfrak{u}) \cong D_{K_r}(\tau_{K_r}(\operatorname{coker}\overline{\beta}))$. Since all involved representations are regular, we conclude

$$\mathrm{Hom}_{K_r}(P_1^-(\mathfrak{u}), P_1^-(\mathfrak{v})) \cong \mathrm{Hom}_{K_r}(\operatorname{coker}\overline{\beta}, \operatorname{coker}\overline{\alpha}).$$

We have $\underline{\dim}\operatorname{coker}\overline{\beta} = (1, r-d+1)$ and $\underline{\dim}\operatorname{coker}\overline{\alpha} = (1, r-d)$. Note that every proper subrepresentation of $\operatorname{coker}\overline{\alpha}$ is projective. Since $\operatorname{coker}\overline{\beta}$ is regular and there are no non-zero morphisms from regular to preprojective representation (see [ASS06, (VIII.2.13)]), we conclude with $\dim_\Bbbk \operatorname{coker}\overline{\beta} > \dim_\Bbbk \operatorname{coker}\overline{\alpha}$ that $\mathrm{Hom}_{K_r}(\operatorname{coker}\overline{\beta}, \operatorname{coker}\overline{\alpha}) = 0$.

□

3.4. Examples for rank varieties

Let $M \in \operatorname{rep}(K_r)$ and $1 \leq d < r$. Recall the definition of the rank variety

$$\begin{aligned}\mathcal{V}(K_r, d)_M &= \{\mathfrak{v} \in \operatorname{Gr}_d(A_r) \mid M|_{\mathfrak{v}} \text{ is not projective}\}.\\ &= \{\mathfrak{v} \in \operatorname{Gr}_d(A_r) \mid \operatorname{rk}(\psi_{M,\mathfrak{v}}) < d \dim_{\Bbbk} M_1\}.\end{aligned}$$

In the following we study the rank variety of a representation for $d = 1$ in more detail.

Following [Rin18] we call the two-dimensional indecomposable Kronecker representations *bristles*. Given $\lambda \in \Bbbk^r \setminus \{0\}$ we denote by $B(\lambda)$ the bristle with structure map

$$\psi_{B(\lambda)} \colon A_r \otimes_{\Bbbk} \Bbbk \longrightarrow \Bbbk \; ; \; \gamma_i \otimes 1 \mapsto \lambda_i.$$

Note that every bristle is isomorphic to $B(\lambda)$ for a suitable $\lambda \neq 0$.

Let $M \in \operatorname{rep}(K_r)$ and $a \in A_r$. Recall that a_M denotes the $\Bbbk$-linear map

$$a_M \colon M_1 \longrightarrow M_2 \; ; \; m \mapsto \psi_M(a \otimes m).$$

Moreover, we write $a.m \coloneqq a_M(m)$ for $m \in M$ if the considered representation M is clear from the context.

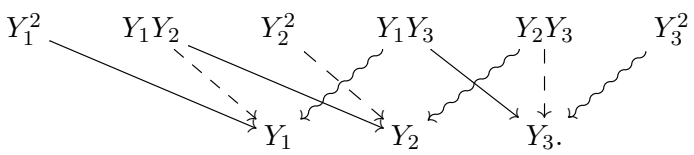
In [Rin17] the author studies the so-called *bristle variety* of a given representations and shows that for every projective variety V there is $n \in \mathbb{N}$ and a representation $M \in \operatorname{rep}(K_n)$ with bristle variety V. An important step in the proof can be rephrased as follows:

Proposition. *Let $\Bbbk[Y_1, \ldots, Y_r]$ be the polynomial ring with r indeterminates $Y_1, \ldots, Y_r$ and let $p \in \Bbbk[Y_1, \ldots, Y_r]$ be a homogeneous polynomial of degree 2. There exists a representation M_p such that $\operatorname{Hom}_{K_r}(B(\lambda), M_p) \neq 0$ if and only if $p(\lambda) = 0$.*

In the following we generalize this statement to all homogeneous polynomials $p \in \Bbbk[Y_1, \ldots, Y_r]_n$ with $n \geq 1$. The main ingredient in [Rin17] is a special representation $C \in \operatorname{rep}(K_r)$ that is called the *canonical bristled representation*. We replace the representation C with so-called *generalized W-representations* introduced in [Wor13a; Wor13b]. We use multiindex notation and write $Y^\mu = Y_1^{\mu_1} \cdots Y_r^{\mu_r}$ for each $\mu \in \mathbb{N}_0^r$. Recall that $\Bbbk[Y_1, \ldots, Y_r]$ admits a natural grading $\Bbbk[Y_1, \ldots, Y_r]_d = \bigoplus_{|\mu|=d} \Bbbk Y^\mu$, where $|\mu| \coloneqq \sum_{i=1}^r \mu_i$. Given $n \in \mathbb{N}$ we denote by $C_n(r) \in \operatorname{rep}(K_r)$ the generalized W-representation introduced in [Wor13b]. We recall a possible way to define $C_n(r)$. By definition $C_n(r)_1 \coloneqq \Bbbk[Y_1, \ldots, Y_r]_n$, $C_n(r)_2 \coloneqq \Bbbk[Y_1, \ldots, Y_r]_{n-1}$ and the arrows $\gamma_1, \ldots, \gamma_r \in A_r$ act in the following way on the basis elements:

$$\gamma_i.Y^\mu = \begin{cases} Y_1^{\mu_1} \cdots Y_{i-1}^{\mu_{i-1}} Y_i^{\mu_i - 1} Y_{i+1}^{\mu_{i+1}} \cdots Y_r^{\mu_r} & \mu_i \geq 1 \\ 0 & \mu_i = 0. \end{cases}$$

The representation $C_2(3) \in \operatorname{rep}(K_3)$ may be visualized as follows:



The representation $C_n(r)$ is denoted by $W^{(r)}_{n+1,2}$ in [Wor13b]. By definition we have $C_1(r) \cong I_1(r)$. Moreover, we have $C_n(2) \cong I_n(2)$ for all $n \geq 1$ in case $r = 2$. The canonical bristle defined in [Rin17] is $C_2(r)$. The next results follow from the definition of $C_n(r)$ and [Wor13b, (2.12), (3.22), (3.24), (4.3)].

Theorem 3.4.1. *Let $n \geq 1$ and $r \geq 2$.*

(1) $\underline{\dim}\, C_n(r) = (\binom{n+r-1}{n}, \binom{n+r-2}{n-1})$.

(2) $C_n(r)$ is a homogeneous brick in $\operatorname{rep}_{\mathrm{inj}}(K_r, 1) \coloneqq D_{K_r}(\operatorname{rep}_{\mathrm{proj}}(K_r, 1))$.

(3) $C_n(r)$ is regular if and only if $r \geq 3$ and $n \geq 2$, and preinjective otherwise.

Following [Rin17] we call a representation $M \in \operatorname{rep}(K_r)$ *fully bristled*, provided M is generated by bristles and any simple subrepresentation of M is contained in a bristle of M. In contrast to the canonical bristle representation $C = C_2(r)$, the representations $C_n(r)$ will in general not be fully bristled, as the following example shows.

Example 3.4.2. *Let $P_0(2) \cong (0, \Bbbk(Y_1^2+Y_2^2)) =: U \subseteq C_3(2) \in \operatorname{rep}(K_2)$. Let $x = \lambda_{(3,0)}Y_1^3+\lambda_{(2,1)}Y_1^2Y_2^1+\lambda_{(1,2)}Y_1^1Y_2^2+\lambda_{(0,3)}Y_2^3 \in C_3(2)_1$ such that $\gamma_1.x, \gamma_2.x \in \Bbbk(Y_1^2+Y_2^2)$. We have $\gamma_1.x = \lambda_{(3,0)}Y_1^2+\lambda_{(2,1)}Y_1^1Y_2^1+\lambda_{(1,2)}Y_2^2$, $\gamma_2.x = \lambda_{(2,1)}Y_1^2+\lambda_{(1,2)}Y_1^1Y_2^1+\lambda_{(0,3)}Y_2^2$ and conclude $\lambda_{(2,1)} = \lambda_{(1,2)} = 0$ and $\lambda_{(0,3)} = \lambda_{(3,0)} = 0$. Hence $x = 0$ and therefore U can not be contained in a bristle.*

This example shows that we can not hope to extend the arguments given in [Rin17] to our situation without proper modification. Before we continue with the explanation of the modifications, we collect basic properties of bristles.

Lemma 3.4.3. *Let $\lambda, \lambda' \in \Bbbk^r \setminus \{0\}$.*

(1) We have $B(\lambda) \cong B(\lambda')$ if and only if $\Bbbk\lambda = \Bbbk\lambda'$.

(2) There exists $g \in \mathrm{GL}(A_r)$ such that $g.B(\lambda) = B(\lambda')$.

(3) Let $B \in \operatorname{rep}(K_r)$ be a bristle, then $D_{K_r}(B) \cong B$.

(4) Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_1, A_r)$ be such that $\alpha(\gamma_1) = \sum_{i=1}^r \lambda_i\gamma_i$. Then $\sigma_{K_r}(P_1^-(\operatorname{im}\alpha)) \cong B(\lambda)$.

Proof. (1) Let $(f_1, f_2)\colon B(\lambda) \longrightarrow B(\lambda')$ be an isomorphism. We find $a_i \in \Bbbk \setminus \{0\}$ such that $f_i = a_i \operatorname{id}_{\Bbbk}$ for $i \in \{1, 2\}$ and get

$$a_2\lambda_j = a_1\lambda'_j$$

for all $j \in \{1, \dots, r\}$. In conclusion, $a_2a_1^{-1}\lambda = \lambda'$ and $\Bbbk\lambda = \Bbbk\lambda'$. Now assume that $\Bbbk\lambda = \Bbbk\lambda'$. We find $0 \neq \mu \in \Bbbk$ such that $\mu\lambda = \lambda'$ and check that $(\operatorname{id}_{\Bbbk}, \mu\operatorname{id}_{\Bbbk})\colon B(\lambda) \longrightarrow B(\lambda')$ is an isomorphism.

(2) Let $A = (a_{ij}) \in \mathrm{GL}_r(\Bbbk)$ such that $A\lambda = \lambda'$. We consider $g \in \mathrm{GL}(A_r)$ given by $g^{-1}(\gamma_i) = \sum_{j=1}^{r} a_{ij}\gamma_j$ for all $i \in \{1, \dots, r\}$. We have

$$\psi_{g.B(\lambda)}(\gamma_i \otimes 1_\Bbbk) = \psi_{B(\lambda)}(g^{-1}(\gamma_i) \otimes 1_\Bbbk) = \sum_{j=1}^{r} \psi_{B(\lambda)}(a_{ij}\gamma_j \otimes 1_\Bbbk) = \sum_{j=1}^{r} a_{ij}\lambda_j = \lambda'_i$$

and conclude $g.B(\lambda) = B(\lambda')$.

(3) We write $B = B(\lambda)$ for $\lambda \in \Bbbk^r \setminus \{0\}$. We have

$$\psi_{D_{K_r}(B(\lambda))} \colon A_r \otimes_\Bbbk \Bbbk^* \longrightarrow \Bbbk^*; a \otimes f \mapsto (f \circ \psi_{B(\lambda)})(a \otimes -) = f \circ a_{B(\lambda)}$$

and denote by $\mathrm{ev}_1 \colon \Bbbk^* \longrightarrow \Bbbk \ ; \ f \mapsto f(1)$ the evaluation isomorphism at 1. Let $i \in \{1, \dots, r\}$ and $f \in \Bbbk^*$. We have

$$(\mathrm{ev}_1 \circ \psi_{D_{K_r}(B(\lambda))})(\gamma_i \otimes f) = (f \circ \psi_{B(\lambda)})(\gamma_i \otimes 1) = f(\lambda_i) = (\psi_{B(\lambda)} \circ (\mathrm{id}_{A_r} \otimes \mathrm{ev}_1))(\gamma_i \otimes f).$$

Hence $(\mathrm{ev}_1, \mathrm{ev}_1) \colon D_{K_r}(B(\lambda)) \longrightarrow B(\lambda)$ defines an isomorphism.

(4) Recall from the previous section that $\overline{\alpha} \colon (0, A_1) \longrightarrow (\Bbbk, A_r) \cong P_1(r)$ is the morphism of representations given by $\overline{\alpha}_2 = \alpha$. The representation $\operatorname{coker} \overline{\alpha}$ is regular with dimension vector $(1, r-1)$. Hence $\underline{\dim}\, \sigma_{K_r}(\operatorname{coker} \overline{\alpha}) = (1, 1)$ and $\sigma_{K_r}(\operatorname{coker} \overline{\alpha})$ is a bristle. The structure maps of $\operatorname{coker} \overline{\alpha}$ and $\sigma_{K_r}(\operatorname{coker} \overline{\alpha})$ are given by

$$\psi_{\operatorname{coker} \overline{\alpha}} \colon A_r \otimes_\Bbbk \Bbbk \longrightarrow A_r / \operatorname{im} \alpha \ ; \ a \otimes 1 \mapsto a + \operatorname{im} \alpha$$

and

$$\psi_{\sigma_{K_r}(\operatorname{coker} \overline{\alpha})} \colon A_r \otimes_\Bbbk \ker \psi_{\operatorname{coker} \overline{\alpha}} \longrightarrow \Bbbk \ ; \ \gamma_i \otimes \sum_{j=1}^{r} \gamma_j \otimes \mu_j \mapsto \mu_i.$$

We have $\ker \psi_{\operatorname{coker} \overline{\alpha}} = \operatorname{im} \alpha \otimes_\Bbbk \Bbbk = \Bbbk(\alpha(\gamma_1) \otimes 1_\Bbbk)$ and

$$\gamma_i.(\alpha(\gamma_1) \otimes 1_\Bbbk) = \gamma_i.(\sum_{j=1}^{r} \gamma_j \otimes \lambda_j) = \lambda_i$$

for all $i \in \{1, \dots, r\}$. Hence $B(\lambda) \cong \sigma_{K_r}(\operatorname{coker} \overline{\alpha})$ and (2) implies $B(\lambda) \cong D_{K_r}(B(\lambda)) \cong D_{K_r}(\sigma_{K_r}(\operatorname{coker} \overline{\alpha}))$. By Corollary 3.3.6 we have $P_1^-(\operatorname{im} \alpha) \cong D_{K_r}(\tau_{K_r}(\operatorname{coker} \overline{\alpha}))$ and conclude

$$\sigma_{K_r}(P_1^-(\operatorname{im} \alpha)) \cong \sigma_{K_r}(D_{K_r}(\tau_{K_r}(\operatorname{coker} \overline{\alpha}))) \cong D_{K_r}(\sigma_{K_r}(\operatorname{coker} \overline{\alpha})) \cong B(\lambda).$$

□

Lemma 3.4.4. *Let $\lambda \in \Bbbk^r \setminus \{0\}$. We have $\dim_\Bbbk \operatorname{Hom}_{K_r}(B(\lambda), C_n(r)) = 1$ and there is exactly one subrepresentation $U_n(\lambda) \subseteq C_n(r)$ such that $B(\lambda) \cong U_n(\lambda)$.*

Proof. We consider at first the case that $\lambda = e_1$ is the first canonical basis vector. We have $Y_1^n \in C_n(r)_1$ and note that $\gamma_1.Y_1^n = 0$ for $i \neq 1$ and $\gamma_i.Y_1^n = Y_1^{n-1}$. Hence the subrepresentation $U_n(e_1) := \langle Y_1^n \rangle \subseteq C_n(r)$ is a bristle isomorphic to $B(e_1)$. Now let $V \subseteq C_n(r)$ be a subrepresentation isomorphic to $B(e_1)$. Since $\underline{\dim}\, V = (1, 1)$, we find $v \in V_1$ such that $\langle v \rangle = V$, $\gamma_i.v = 0$ for all

$i \in \{2, \dots, r\}$ and $\gamma_1.v \neq 0$. We have $v \in C_n(r)_1 = \Bbbk[Y_1, \dots, Y_r]_n$ and write $v = \sum_{|\mu|=n} \beta_\mu Y^\mu$. We note that by definition $\ker((\gamma_j)_{C_n(r)}) = \bigoplus_{|\mu|=n, \mu_j=0} \Bbbk Y^\mu$ for each $j \in \{1, \dots, r\}$. Hence the restriction of each $(\gamma_j)_{C_n(r)}$ to $\bigoplus_{|\mu|=n, \mu_j \neq 0} \Bbbk Y^\mu$ is injective. Let $i \in \{2, \dots, r\}$. Since

$$0 = \gamma_i.v = \gamma_i. \sum_{|\mu|=n, \mu_i=0} \beta_\mu Y^\mu + \gamma_i. \sum_{|\mu|=n, \mu_i \neq 0} \beta_\mu Y^\mu = \gamma_i. \sum_{|\mu|=n, \mu_i \neq 0} \beta_\mu Y^\mu,$$

we conclude that $\beta_\mu = 0$ for all $\mu \in \mathbb{N}_0^r$ with $|\mu| = n$ and $\mu_i \neq 0$. In conclusion, $\beta_\mu \neq 0$ if and only if $\mu = ne_1$ and therefore $v \in \Bbbk Y_1^n$. Clearly $\langle v \rangle = U_n(e_1)$. Let us summarize: We have shown that $C_n(r)$ has exactly one subrepresentation $U_n(e_1)$ isomorphic to $B(e_1)$. Let $f \colon B(e_1) \longrightarrow C_n(r)$ be a non-zero homomorphism. Since the projective representation $P_0(r)$ is not a factor of $B(e_1)$, each factor of $B(e_1)$ is isomorphic to $B(e_1)$ or $I_0(r)$. Since $C_n(r)$ is indecomposable and not isomorphic to $I_0(r)$, we conclude that f is an isomorphism onto its image. Hence $\operatorname{im} f = U_n(e_1)$. Since $B(e_1)$ is a brick, we get $\dim_\Bbbk \operatorname{Hom}_{K_r}(B(e_1), C_n(r)) = \dim_\Bbbk \operatorname{Hom}_{K_r}(B(e_1), U_n(e_1)) = 1$.
Now let λ be arbitrary. Application of Lemma 3.4.3 gives us $g \in \mathrm{GL}(A_r)$ such that $B(\lambda) \cong g.B(e_1)$. Since $C_n(r)$ is homogeneous by Theorem 3.4.1 and $\mathrm{GL}(A_r)$ acts via representation automorphisms, we conclude

$$1 = \dim_\Bbbk \operatorname{Hom}_{K_r}(B(e_1), C_n(r)) = \dim_\Bbbk \operatorname{Hom}_{K_r}(g.B(e_1), g.C_n(r)) = \dim_\Bbbk \operatorname{Hom}_{K_r}(B(\lambda), C_n(r)).$$

As before every non-zero morphism $g \colon B(\lambda) \longrightarrow C_n(r)$ is injective. Hence there is exactly one subrepresentation $U_n(\lambda)$ such that $U_n(\lambda) \cong B(\lambda)$. □

Definition. Let $p \in \Bbbk[Y_1, \dots, Y_r]_n = C_n(r)_1$. We write $p = \sum_{|\mu|=n} \lambda_\mu Y^\mu$ and define a $\Bbbk$-linear map $\varphi(p)_1 \colon \Bbbk[Y_1, \dots, Y_r]_n \longrightarrow \Bbbk$ by setting $\varphi(p)_1(Y^\mu) \coloneqq \lambda_\mu$ for all $\mu \in \mathbb{N}_0^r$ with $|\mu| = n$. We obtain a map

$$\varphi \colon \Bbbk[Y_1, \dots, Y_r]_n = C_n(r)_1 \longrightarrow \operatorname{Hom}_{K_r}(C_n(r), I_0(r)) \; ; \; p \mapsto \varphi(p) \coloneqq (\varphi(p)_1, 0).$$

Given $\lambda \in \Bbbk^r$, we define $\lambda^\mu \coloneqq \lambda_1^{\mu_1} \cdots \lambda_r^{\mu_r}$ for each $\mu \in \mathbb{N}_0^r$. We define a function

$$\Xi_n \colon \Bbbk^r \setminus \{0\} \longrightarrow \Bbbk[Y_1, \dots, Y_r]_n \; ; \; \lambda \mapsto \sum_{\mu \in \mathbb{N}_0^r, |\mu|=n} \lambda^\mu Y^\mu.$$

Lemma 3.4.5. *Let $n \in \mathbb{N}$ and $\lambda \in \Bbbk^r \setminus \{0\}$. Then $\langle \Xi_n(\lambda) \rangle = U_n(\lambda) \subseteq C_n(r)$.*

Proof. By definition we have

$$\gamma_i.\Xi_n(\lambda) = \sum_{\mu \in \mathbb{N}_0^r, |\mu|=n, \mu_i \neq 0} \lambda^\mu Y_1^{\mu_1} \cdots Y_{i-1}^{\mu_{i-1}} Y_i^{\mu_i - 1} Y_{i+1}^{\mu_{i+1}} \cdots Y_r^{\mu_r} = \sum_{\mu \in \mathbb{N}_0^r, |\mu|=n, \mu_i \neq 0} \lambda^\mu Y^{\mu - e_i}.$$

Let us compare the coefficients $\gamma_i.\Xi_n(\lambda)$ and $\gamma_j.\Xi_n(\lambda) \in C_n(r)_2 = \Bbbk[Y_1, \dots, Y_r]_{n-1}$ for $i \neq j$. We denote them by i_η and j_η, respectively for each $\eta \in \mathbb{N}_0^r$ with $|\eta| = n - 1$. We have $i_\eta = \lambda^{\eta + e_i}$ and $j_\eta = \lambda^{\eta + e_j}$. Hence $\lambda_j i_\eta = \lambda_i j_\eta$ and therefore

$$\lambda_j \gamma_i.\Xi_n(\lambda) = \lambda_i \gamma_j.\Xi_n(\lambda)$$

for all $i, j \in \{1, \dots, r\}$. We fix $j \in \{1, \dots, r\}$ such that $\lambda_j \neq 0$. Then $\gamma_j.\Xi_n(\lambda) \neq 0$, $\gamma_i.\Xi_n(\lambda) = \lambda_j^{-1}\lambda_i(\gamma_j.\Xi_n(\lambda))$ for all $i \in \{1 \dots, r\}$ and the $\Bbbk$-linear maps

$$f_1 \colon \Bbbk\Xi_n(\lambda) \longrightarrow \Bbbk \; ; \; \Xi_n(\lambda) \mapsto 1,$$

$$f_2 \colon \Bbbk\gamma_j.\Xi_n(\lambda) \longrightarrow \Bbbk \; ; \; \gamma_j.\Xi_n(\lambda) \mapsto \lambda_j$$

define an isomorphism $f \colon \langle \Xi_n(\lambda) \rangle \longrightarrow B(\lambda)$. Application of Lemma 3.4.4 yields $\langle \Xi_n(\lambda) \rangle = U_n(\lambda)$. □

Lemma 3.4.6. *Let $0 \neq p \in \Bbbk[Y_1, \dots, Y_r]_n$ and $\lambda \in \Bbbk^r \setminus \{0\}$. The following statements are equivalent.*

(1) $\operatorname{Hom}_{K_r}(B(\lambda), \ker \varphi(p)) \neq 0$.

(2) $U_n(\lambda) \subseteq \ker \varphi(p)$.

(3) $\Xi_n(\lambda) \in \ker \varphi(p)_1$.

(4) $\lambda \in \mathcal{V}(p)$[5].

Proof. We write $p = \sum_{\mu \in \mathbb{N}_0^r, |\mu| = n} \delta_\mu Y^\mu$ not all $\delta_\mu = 0$ and recall that $\varphi(p)_1(\sum_{\mu \in \mathbb{N}_0^r, |\mu| = n} \beta_\mu Y^\mu) = \sum_{\mu \in \mathbb{N}_0^r, |\mu| = n} \beta_\mu \delta_\mu$ for all $\sum_{\mu \in \mathbb{N}_0^r, |\mu| = n} \beta_\mu Y^\mu \in \Bbbk[Y_1, \dots, Y_r]_n$.

(1) ⇒ (2). Let $f \colon B(\lambda) \longrightarrow \ker \varphi(p) \subseteq C_n(r)$ be a non-zero homomorphism. Since every proper factor of $B(\lambda)$ is simple injective and $C_n(r) \ncong I_0(r)$ is indecomposable, f is injective and we conclude with Lemma 3.4.4 that $\operatorname{im} f \subseteq \ker \varphi(p) \subseteq C_n(r)$ equals $U_n(\lambda)$.

(2) ⇒ (3). Since $U_n(\lambda) \subseteq \ker \varphi(p)$, we have $U_n(\lambda)_1 \subseteq \ker \varphi(p)_1$. Lemma 3.4.5 tells us that $U_n(\lambda)$ is generated by $\Xi_n(\lambda)$. In particular, we have $\Xi_n(\lambda) \in U_n(\lambda)_1 \subseteq \ker \varphi(p)_1$.

(3) ⇒ (4). Since $\Xi_n(\lambda) \in \ker \varphi(p)_1$, we have

$$0 = \varphi(p)_1\Big(\sum_{\mu \in \mathbb{N}_0^r, |\mu| = n} \lambda^\mu Y^\mu\Big) = \sum_{\mu \in \mathbb{N}_0^r, |\mu| = n} \lambda^\mu \delta_\mu = \sum_{\mu \in \mathbb{N}_0^r, |\mu| = n} \delta_\mu \lambda^\mu = p(\lambda)$$

and conclude that $\lambda \in \mathcal{V}(p)$.

(4) ⇒ (1). Since $\lambda \in \mathcal{V}(p)$, we have $0 = p(\lambda) = \sum_{\mu \in \mathbb{N}_0^r, |\mu| = n} \delta_\mu \lambda^\mu = \varphi(p)_1(\Xi_n(\lambda))$. We conclude with Lemma 3.4.5 that $U_n(\lambda) = \langle \Xi_n(\lambda) \rangle \subseteq \ker \varphi(p)$ and (1) follows from Lemma 3.4.4. □

Definition.

(1) For $\zeta \in \Bbbk[Y_1, \dots, Y_r]_n \setminus \{0\}$ we define $N_\zeta \coloneqq \ker \varphi(\zeta) \subseteq C_n(r)$.

(2) We denote by $\overline{\Xi}_n$ the by $\Xi_n \colon \Bbbk^r \setminus \{0\} \longrightarrow \Bbbk[Y_1, \dots, Y_r]_n$ induced map

$$\overline{\Xi}_n \colon \mathbb{P}^{r-1} \longrightarrow \mathbb{P}(\Bbbk[Y_1, \dots, Y_r]_n) = \mathbb{P}(C_n(r)_1) \; ; \; \Bbbk\lambda \mapsto \Bbbk\Xi_n(\lambda).$$

[5] Recall from Section 3.1 that $\mathcal{V}(\mathcal{F}) \subseteq \Bbbk^r$ denotes the set of common zeros for every subset $\mathcal{F} \in \Bbbk[X_1, \dots, X_r]$.

Let $M \in \operatorname{rep}(K_r)$. Following [Rin17] we call an element $v \in M_1$ *eigenvector* of M, provided the subspace of M_2 generated by all $\gamma_i.v$ is 1-dimensional. We note that v has to satisfy $v \neq 0$ and that v is an eigenvector if and only if all elements in $\Bbbk v \setminus \{0\}$ are eigenvectors. Therefore, $\epsilon(M) \coloneqq \{\Bbbk v \mid v \in M_1, v \text{ eigenvector}\} \subseteq \mathbb{P}(M_1)$ is well-defined. We call $\epsilon(M)$ the *eigenvector variety* of M which is a Zariski closed subset, hence a projective algebraic set. Note that $\Bbbk v \in \epsilon(M)$ if and only if $\langle v \rangle \subseteq M$ is a bristle.

Theorem 3.4.7. *Let $\zeta_1, \dots, \zeta_\ell \in \Bbbk[Y_1, \dots, Y_r]_n \setminus \{0\}$, $M \coloneqq \bigcap_{i=1}^{\ell} N_{\zeta_i} \subseteq C_n(r)$ and $\lambda \in \Bbbk^r \setminus \{0\}$.*

(1) We have $\operatorname{Hom}_{K_r}(B(\lambda), M) \neq 0$ if and only if $\lambda \in \mathcal{V}(\zeta_1, \dots, \zeta_\ell)$.

(2) M has no simple injective direct summands, i.e. M is a reduced Matrix pencil in the sense of [Rin17].

(3) We have $\epsilon(M) = \overline{\Xi}_n(\mathcal{V}^+(\zeta_1, \dots, \zeta_\ell))$[6].

Proof. (1) Let $f\colon B(\lambda) \longrightarrow M$ be non-zero. Then Lemma 3.4.6 implies that $\lambda \in \mathcal{V}(\zeta_i)$ for all $i \in \{1, \dots, \ell\}$. Hence $\lambda \in \bigcap_{i=1}^{\ell} \mathcal{V}(\zeta_i) = \mathcal{V}(\zeta_1, \dots, \zeta_\ell)$. Now assume that $\lambda \in \mathcal{V}(\zeta_1, \dots, \zeta_\ell)$. Then we have $\lambda \in \mathcal{V}(\zeta_i)$ for each $i \in \{1, \dots, \ell\}$ and conclude with Lemma 3.4.6 that $U_n(\lambda) \subseteq N_{\zeta_i}$. Hence $U_n(\lambda) \subseteq \bigcap_{i=1}^{\ell} N_{\zeta_i} = M$. Since $B(\lambda) \cong U_n(\lambda)$, we get $\operatorname{Hom}_{K_r}(B(\lambda), M) \neq 0$.

(2) Let $U \subseteq M$ be an indecomposable direct summand, then $U \subseteq C_n(r)$. By Theorem 3.4.1 the representation $C_n(r)$ is indecomposable and not isomorphic to $I_0(r)$. Hence $U \ncong I_0(r)$.

(3) Let $\Bbbk m \in \epsilon(M)$, then $\langle m \rangle \subseteq M \subseteq C_n(r)$ is a bristle and Lemma 3.4.4 gives us $\lambda \in \Bbbk^r \setminus \{0\}$ such that $U_n(\lambda) = \langle m \rangle$. We conclude with Lemma 3.4.5 that $\Bbbk m = \Bbbk \Xi_n(\lambda) = \overline{\Xi}_n(\Bbbk\lambda)$. Moreover, $\Xi_n(\lambda) \in \Bbbk m \subseteq M_1 = \bigcap_{i=1}^{\ell} \ker(\varphi(\zeta_i)_1)$ implies with Lemma 3.4.6 that $\lambda \in \mathcal{V}(\zeta_1, \dots, \zeta_\ell)$ and therefore $\Bbbk\lambda \in \mathcal{V}^+(\zeta_1, \dots, \zeta_\ell)$. This shows $\epsilon(M) \subseteq \overline{\Xi}_n(\mathcal{V}^+(\zeta_1, \dots, \zeta_\ell))$.

Now we assume that $\Bbbk c \in \overline{\Xi}(\mathcal{V}^+(\zeta_1, \dots, \zeta_\ell))$ for $0 \neq c \in C_n(r)_1$, so there is $\Bbbk\lambda \in \mathcal{V}^+(\zeta_1, \dots, \zeta_\ell)$ with $0 \neq \lambda \in \Bbbk^r$ such that $\Bbbk \Xi_n(\lambda) = \overline{\Xi}_n(\Bbbk\lambda) = \Bbbk c$. Since $\Bbbk\lambda \in \mathcal{V}^+(\zeta_1, \dots, \zeta_\ell)$, we have $\lambda \in \mathcal{V}(\zeta_1, \dots, \zeta_\ell)$ and Lemma 3.4.6 implies $\Xi_n(\lambda) \in \bigcap_{i=1}^{\ell} \ker \varphi(\zeta_i)_1 = M_1$. Hence $0 \neq c \in \Bbbk \Xi_n(\lambda) \subseteq M_1$. By Lemma 3.4.5 the representation $\langle c \rangle = \langle \Xi_n(\lambda) \rangle$ is a bristle. Hence $\Bbbk c \in \epsilon(M)$. □

Let $M \in \operatorname{rep}(K_r)$ and $\mathsf{d} \in \mathbb{N}^2$. The *quiver Grassmannian* $\mathbb{G}_{\mathsf{d}}(M)$ is by definition the set of subrepresentations of M with dimension vector d and carries the structure of a projective variety. Moreover, we call $\beta(M) \coloneqq \mathbb{G}_{(1,1)}(M)$ the *bristle variety* of M.

As mentioned in [Rin17] the bristle variety of a representation M without $I_0(r)$ as a direct summand can be identified with the eigenvector variety $\epsilon(M) \subseteq \mathbb{P}(M_1)$ of M by sending $B \in \beta(M)$ to $B_1 \in \epsilon(M)$. With this observation we get a slightly improved version of [Rin17, Theorem 1'], which was already established by other methods in [Hil15, (1.2)].

Corollary 3.4.8. *Any projective variety $V \subseteq \mathbb{P}^{r-1}$ arises as the bristle variety $\beta(M)$ of a reduced Kronecker representation $M \in \operatorname{rep}(K_r)$, i.e. a representation without $I_0(r)$ as a direct summand.*

[6] Recall from Section 3.1 that $\mathcal{V}^+(\mathcal{F}) \subseteq \mathbb{P}^{r-1}$ denotes the set of common zeros for every subset $\mathcal{F} \in \Bbbk[X_1, \dots, X_r]$.

Proof. Let $n \in \mathbb{N}$. We take the ordering of the monomials $Y^\mu \in \Bbbk[Y_1, \dots, Y_r]_n$ to be the lexicographic order and obtain a bijection $\binom{n+r-1}{n} \longrightarrow \{\mu \in \mathbb{N}_0^r \mid |\mu| = n\}$ and an induced $\Bbbk$-linear isomorphism $\psi\colon \Bbbk^{\binom{n+r-1}{n}} \longrightarrow \Bbbk[Y_1, \dots, Y_r]_n$ that induces an isomorphism of varieties $\overline{\psi}\colon \mathbb{P}^{\binom{n+r-1}{n}-1} \longrightarrow \mathbb{P}(\Bbbk[Y_1, \dots, Y_r]_n)$. For example the isomorphism for $(n, r) = (3, 2)$ is given by

$$\psi\colon \Bbbk^4 \longrightarrow \Bbbk[Y_1, Y_2]_3; \lambda \mapsto \lambda_1 Y_1^3 + \lambda_2 Y_1^2 Y_2 + \lambda_3 Y_1 Y_2^2 + \lambda_4 Y_2^3.$$

We denote by $\nu_n\colon \mathbb{P}^{r-1} \longrightarrow \mathbb{P}^{\binom{n+r-1}{n}-1}$ the *Veronese map* that induces an isomorphism $\mathbb{P}^{r-1} \cong \nu_n(\mathbb{P}^{r-1})$ (see [Per08, (III.11.6(c))]). Hence $\overline{\psi} \circ \nu_n$ induces an isomorphism $\mathbb{P}^{r-1} \cong (\overline{\psi} \circ \nu_n)(\mathbb{P}^{r-1})$ and therefore an isomorphism $V \cong (\overline{\psi} \circ \nu_n)(V)$. By definition we have

$$\overline{\psi} \circ \nu_n = \overline{\Xi}_n$$

and conclude $\overline{\Xi}_n(V) \cong V$. Finally, we apply Theorem 3.4.7 and find a reduced Matrix pencil $M \in \operatorname{rep}(K_r)$ such that

$$\beta(M) \cong \epsilon(M) = \overline{\Xi}_n(V) \cong V.$$

□

Recall from Lemma 3.4.3 that $B(\lambda) \cong B(\lambda')$ if and only $\Bbbk\lambda = \Bbbk\lambda'$. Hence we obtain for each $M \in \operatorname{rep}(K_r)$ a well-defined set

$$\mathbb{B}_M \coloneqq \{\Bbbk\lambda \mid \operatorname{Hom}_{K_r}(B(\lambda), M) \neq 0\} \subseteq \mathbb{P}^{r-1}.$$

Corollary 3.4.9. *Let $V \subseteq \mathbb{P}^{r-1}$ be a non-empty projective algebraic set.*

(1) There exists a representation M without $I_0(r)$ as a direct summand such that $\mathbb{B}_M = V$.

(2) For $r \geq 3$ we can choose M as a regular representation.

Proof. Since V is non-empty, we find $\ell \in \mathbb{N}$ and $f_1, \dots, f_\ell \in \Bbbk[Y_1, \dots, Y_r]$ homogeneous such that $V = \mathcal{V}^+(f_1, \dots, f_\ell)$. Since taking powers of the f_i does not change the corresponding set of zeros, we may assume that $f_1, \dots, f_\ell \in \Bbbk[Y_1, \dots, Y_r]_n$ are homogeneous of the same degree $n \geq 2$. Theorem 3.4.7 yields a subrepresentation $M_V \subseteq C_n(r)$ without simple injective summands such that $\operatorname{Hom}_{K_r}(B(\lambda), M_V) \neq 0$ if and only if $\lambda \in \mathcal{V}(f_1, \dots, f_\ell)$. Let $0 \neq \lambda \in \Bbbk^r$. Then $\lambda \in \mathcal{V}(f_1, \dots, f_\ell)$ if and only if $\Bbbk\lambda \in \mathcal{V}^+(f_1, \dots, f_\ell) = V$.

Now let $r \geq 3$. Since $n \geq 2$, the representation $C_n(r)$ is regular by Theorem 3.4.1. Recall from [ASS06, (VIII.2.13)] that there are no non-zero morphisms from a preinjective representation to a regular representation. Hence $M_V \subseteq C_n(r)$ has no non-zero preinjective direct summands. We write $P \oplus R = M_V$ with P preprojective and R regular. Another application of [ASS06, (VIII.2.13)] gives us

$$\operatorname{Hom}_{K_r}(B(\lambda), M) \cong \operatorname{Hom}_{K_r}(B(\lambda), P) \oplus \operatorname{Hom}_{K_r}(B(\lambda), R) = \operatorname{Hom}_{K_r}(B(\lambda), R)$$

since $B(\lambda)$ is regular. We set $M \coloneqq R$. □

In the remainder of this section, we denote by χ the isomorphism of varieties

$$\chi\colon \mathbb{P}^{r-1} \longrightarrow \mathbb{P}(A_r)\ ;\ \Bbbk\lambda \mapsto \Bbbk(\sum_{i=1}^{r} \lambda_i\gamma_i).$$

Theorem 3.4.10. *(1) Let $M \in \operatorname{rep}_1(K_r)$ be a representation, then $\chi(\mathbb{B}_M) = \mathcal{V}(K_r,1)_{\sigma_{K_r}^{-1}(M)}$.*

(2) Let $V \subseteq \mathbb{P}(A_r)$ be a non-empty closed subset. There exists a representation $N \in \operatorname{rep}(K_r)$ such $V = \mathcal{V}(K_r,1)_N$.

Proof. (1) Let $\mathfrak{v} \in \mathbb{P}(A_r)$ and $\mathfrak{v} = \Bbbk(\sum_{i=1}^{r} \lambda_i\gamma_i)$. By Lemma 3.4.3 we have

$$\sigma_{K_r}(P_1^-(\mathfrak{v})) \cong B(\lambda).$$

Since $M \in \operatorname{rep}_1(K_r)$, we have $\sigma_{K_r}(\sigma_{K_r}^{-1}(M)) \cong M$ and $\sigma_{K_r}\colon \operatorname{rep}_2(K_r) \longrightarrow \operatorname{rep}_1(K_r)$ being an equivalence gives us for every $\mathfrak{v} \in \mathbb{P}(A_r)$ the following equivalences:

$$\operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), \sigma_{K_r}^{-1}(M)) \neq 0 \Leftrightarrow \operatorname{Hom}_{K_r}(\sigma_{K_r}(P_1^-(\mathfrak{v})), M) \neq 0 \Leftrightarrow \operatorname{Hom}_{K_r}(B(\lambda), M) \neq 0$$
$$\Leftrightarrow \chi^{-1}(\mathfrak{v}) = \Bbbk\lambda \in \mathbb{B}_M.$$

Using Theorem 3.3.2, we conclude $\chi(\mathbb{B}_M) = \mathcal{V}(K_r,1)_{\sigma_{K_r}^{-1}(M)}$.

(2) By Corollary 3.4.9 there is $M \in \operatorname{rep}_1(K_r)$ such that $\mathbb{B}_M = \chi^{-1}(V)$. Now we apply (1).

□

Remark 3.4.11. Let $\zeta \in \Bbbk[Y_1,\dots,Y_r]_n \setminus \{0\}$ be homogeneous. We apply Theorem 3.4.10 to the representation $N_\zeta \subseteq C_n(r)$ and obtain $\mathcal{V}(K_r,1)_{\sigma_{K_r}^{-1}(N_\zeta)} = \chi(\mathcal{V}^+(\zeta))$. Therefore, the representations can be seen as an analog to Carlson's L_ζ-modules (cf. [Ben91, §5.9]) introduced for group algebras. Hence we define

$$L_\zeta \coloneqq \sigma_{K_r}^{-1}(N_\zeta).$$

However, in contrast to the group algebra case, the tensor product (see [Ben91, (5.9.2)]) is not available in the category of Kronecker representations.

4. Properties of relative projective representations

In this chapter we introduce the equal socle property for Kronecker representations and show that representations with this property are "almost" relative projective. Here "almost" means that a representation with the equal socle property becomes relative projective after application of $\sigma_{K_r}^{-1}$. Using this fact, we show that relative projective representations form a torsion-free class in $\operatorname{rep}(K_r)$ and study the arrangement of this class in regular Auslander-Reiten components. Parts of this chapter are based on [Bis24].

4.1. The equal socle property

Constant rank type modules have been defined and studied in [CFP12] in the context of elementary abelian p-groups over fields of characteristic p as a generalization of constant Jordan type modules. Inspired by these considerations, representations with the equal socle property have been introduced in [Bis20] for Kronecker representations over fields of arbitrary characteristic.

Let $M \in \operatorname{rep}(K_r)$ be a representation, $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$ and $a \in A_r$. Recall from Section 1.5 that we denote by $a_M \colon M_1 \longrightarrow M_2$ the $\Bbbk$-linear map

$$a_M \colon M_1 \longrightarrow M_2 \,; m \mapsto \psi_M(a \otimes m).$$

Definition. (cf. [Bis20, §2.3]). A representation $M \in \operatorname{rep}(K_r)$ has the *equal d-socle property*, provided $\{0\} = \bigcap_{a \in \mathfrak{v}} \ker a_M$ for all $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$. We denote the corresponding full subcategory of $\operatorname{rep}(K_r)$ by $\operatorname{rep}_{\mathrm{esp}}(K_r, d)$.

Remark 4.1.1. Let $M, N \in \operatorname{rep}(K_r)$, $\mathfrak{v} \in \operatorname{Gr}_d(A_r)$ and $a \in A_r$. Since $\alpha^*(M \oplus N) \cong \alpha^*(M) \oplus \alpha^*(N)$ for all $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$, $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ is closed under direct sums and summands. Moreover, $\ker a_{M \oplus N} = \ker a_M \oplus \ker a_N$ implies that $\operatorname{rep}_{\mathrm{esp}}(K_r, d)$ is closed under direct sums and summands. It follows from Lemma 3.3.1 and the definition of $\operatorname{rep}_{\mathrm{esp}}(K_r, d)$ that $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ and $\operatorname{rep}_{\mathrm{esp}}(K_r, d)$ are closed under subrepresentations.

Relative projective representations and representations with the equal socle property are closely related:

Proposition 4.1.2. *Let $N \in \operatorname{rep}(K_r)$ and $1 \leq d < r$. The following statements are equivalent.*

(1) $N \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$.

(2) $\sigma_{K_r}(N) \in \mathrm{rep}_{\mathrm{esp}}(K_r, r-d)$.

Proof. We define $M \coloneqq \sigma_{K_r}(N)$.

(1) $\Rightarrow$ (2). We assume that $N \notin \mathrm{rep}_{\mathrm{proj}}(K_r, d)$. By Theorem 3.3.2 we find $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ such that

$$\psi_N|_{\mathfrak{v}\otimes_{\Bbbk} N_1} : \mathfrak{v}\otimes_{\Bbbk} N_1 \longrightarrow N_2 \ ; \ a\otimes n \mapsto \psi_N(a\otimes n)$$

is not injective. Let $(a_1, \ldots, a_d)$ be a basis of $\mathfrak{v}$ and $x = \sum_{j=1}^{d} a_j \otimes n_j$ be a non-zero element in $\ker \psi_N|_{\mathfrak{v}\otimes_{\Bbbk} N_1}$. We write $a_j = \sum_{i=1}^{r} \beta_{ij}\gamma_i$ for $1 \leq j \leq d$ and set $n_i' \coloneqq \sum_{j=1}^{d} \beta_{ij} n_j$ for $1 \leq i \leq r$. By definition we have

$$x = \sum_{j=1}^{d}\sum_{i=1}^{r} \beta_{ij}\gamma_i \otimes n_j = \sum_{i=1}^{r} \gamma_i \otimes \sum_{j=1}^{d} \beta_{ij} n_j = \sum_{i=1}^{r} \gamma_i \otimes n_i'.$$

Recall that

$$M_2 = N_1, M_1 = \ker \psi_N \text{ and } \psi_M(\gamma_j \otimes \sum_{i=1}^{r} \gamma_i \otimes x_i) = x_j \text{ for all } j \in \{1, \ldots, r\}.$$

Hence $\sum_{i=1}^{r} \gamma_i \otimes n_i' = x \in M_1 \setminus \{0\}$. Let $A \coloneqq \{\delta \in \Bbbk^r \mid \sum_{i=1}^{r} \delta_i n_i' = 0\}$. Since $\sum_{i=1}^{r} \Bbbk n_i' \subseteq \sum_{i=1}^{d} \Bbbk n_i$, we have $\dim_{\Bbbk} A \geq r-d$. We fix a subspace $B \subseteq A$ of dimension $r-d$. Let $\mathfrak{u} \coloneqq \{\sum_{i=1}^{r} \delta_i\gamma_i \mid \delta \in B\} \in \mathrm{Gr}_{r-d}(A_r)$. Let $a = \sum_{i=1}^{r} \delta_i\gamma_i \in \mathfrak{u}$, then

$$\psi_M(a\otimes x) = \sum_{i=1}^{r} \delta_i \psi_M(\gamma_i \otimes x) = \sum_{i=1}^{r} \delta_i n_i' = 0.$$

Hence $0 \neq x \in \bigcap_{a\in\mathfrak{u}} \ker a_M$ and $M \notin \mathrm{rep}_{\mathrm{esp}}(K_r, r-d)$.

(2) $\Rightarrow$ (1). Assume that $M \notin \mathrm{rep}_{\mathrm{esp}}(K_r, r-d)$. We find $\mathfrak{u} \in \mathrm{Gr}_{r-d}(A_r)$ and $0 \neq m \in \bigcap_{a\in\mathfrak{u}} \ker a_M$. By definition of M_1, we have $m = \sum_{i=1}^{r} \gamma_i \otimes n_i \in A_r \otimes_{\Bbbk} N_1$ and $0 = \psi_N(m)$. Let $a = \sum_{i=1}^{r} \lambda_i\gamma_i \in \mathfrak{u}$, then $0 = a_M(m) = \sum_{i=1}^{r} \lambda_i \psi_M(\gamma_i \otimes m) = \sum_{i=1}^{r} \lambda_i n_i$. Hence $\{\delta \in \Bbbk^r \mid \sum_{i=1}^{r} \delta_i n_i = 0\}$ is a vector space of dimension at least $r-d$ and $\sum_{i=1}^{r} \Bbbk n_i$ a vector space of dimension at most d. Let $(x_1, \ldots, x_m)$ be a basis of $\sum_{i=1}^{r} \Bbbk n_i$. We write $n_i = \sum_{j=1}^{m} \lambda_{ij} x_j$ for $1 \leq i \leq r$ and let $b_l \coloneqq \sum_{j=1}^{r} \lambda_{jl}\gamma_j$ for $1 \leq l \leq m \leq d$. We fix $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ such that $\sum_{l=1}^{m} \Bbbk b_l \subseteq \mathfrak{v}$. We have

$$\mathfrak{v}\otimes_{\Bbbk} N_1 \ni \sum_{l=1}^{m} b_l \otimes x_l = \sum_{l=1}^{m}(\sum_{j=1}^{r} \lambda_{jl}\gamma_j) \otimes x_l = \sum_{j=1}^{r} \gamma_j \otimes (\sum_{l=1}^{m} \lambda_{jl} x_l) = \sum_{j=1}^{r} \gamma_j \otimes n_j = m.$$

and $\psi_N|_{\mathfrak{v}\otimes_{\Bbbk} N_1}(m) = 0$. Hence $N \notin \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ by Theorem 3.3.2. □

Remark 4.1.3.

(1) Let $1 \leq d < r$. Note that $\sigma_{K_r}^{-1}(I_0(r)) = 0 \in \mathrm{rep}_{\mathrm{proj}}(K_r, r-d)$ but $I_0(r) \notin \mathrm{rep}_{\mathrm{esp}}(K_r, d)$.

(2) Let $M \in \mathrm{rep}_1(K_r)$, then $\sigma_{K_r}(\sigma_{K_r}^{-1}(M)) \cong M$ and Proposition 4.1.2 implies

$$M \in \mathrm{rep}_{\mathrm{esp}}(K_r, d) \Leftrightarrow \sigma_{K_r}(\sigma_{K_r}^{-1}(M)) \in \mathrm{rep}_{\mathrm{esp}}(K_r, d) \Leftrightarrow \sigma_{K_r}^{-1}(M) \in \mathrm{rep}_{\mathrm{proj}}(K_r, r-d).$$

Since $\mathrm{rep}_{\mathrm{esp}}(K_r, d) \subseteq \mathrm{rep}_1(K_r)$, we conclude

$$M \in \mathrm{rep}_{\mathrm{esp}}(K_r, d) \Rightarrow \sigma_{K_r}^{-1}(M) \in \mathrm{rep}_{\mathrm{proj}}(K_r, r-d).$$

Corollary 4.1.4. *The following statements hold.*

(1) We have

$$\mathrm{rep}_{\mathrm{proj}}(K_r, r-1) \subseteq \cdots \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, 1) = \mathrm{rep}_{\mathrm{esp}}(K_r, 1) \subseteq \cdots \subseteq \mathrm{rep}_{\mathrm{esp}}(K_r, r-1).$$

(2) We have $\sigma_{K_r}^{-1}(\mathrm{rep}_{\mathrm{proj}}(K_r, d)) \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, r-1)$ *for all* $1 \leq d \leq r-1$.

(3) We have $\sigma_{K_r}^{-1}(\mathrm{rep}_{\mathrm{esp}}(K_r, d)) \subseteq \mathrm{rep}_{\mathrm{esp}}(K_r, 1)$ *for all* $1 \leq d \leq r-1$.

Proof. (1) By definition and Theorem 3.3.2 we have $\mathrm{rep}_{\mathrm{proj}}(K_r, 1) = \mathrm{rep}_{\mathrm{esp}}(K_r, 1)$. The statement follows since we have $\mathrm{rep}_{\mathrm{proj}}(K_r, d) \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, d')$ and $\mathrm{rep}_{\mathrm{esp}}(K_r, d') \subseteq \mathrm{rep}_{\mathrm{esp}}(K_r, d)$ for all $1 \leq d' \leq d < r$.

(2) By (1) and Remark 4.1.3 we have

$$\sigma_{K_r}^{-1}(\mathrm{rep}_{\mathrm{proj}}(K_r, d)) \subseteq \sigma_{K_r}^{-1}(\mathrm{rep}_{\mathrm{esp}}(K_r, 1)) \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, r-1).$$

(3) By (1) and Remark 4.1.3 we have

$$\sigma_{K_r}^{-1}(\mathrm{rep}_{\mathrm{esp}}(K_r, d)) \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, r-d) \subseteq \mathrm{rep}_{\mathrm{esp}}(K_r, 1).$$

□

4.2. Arranging relative projective representations in AR components

Definition. Let $r \geq 3$ and $\mathcal{C}$ be a regular component of the Auslander-Reiten of $\mathrm{rep}(K_r)$. Given a quasi-simple representation $M \in \mathcal{C}$, we denote by

$$(M \rightarrow) := \{\tau_{K_r}^{-n}(M)_{[i]} \mid n \in \mathbb{N}_0, i \in \mathbb{N}\}$$

the *cone of successors* of M in $\mathcal{C}$.

Theorem 4.2.2 is one of the main results of this thesis and will allow us later on to apply Auslander-Reiten theory to Steiner bundles. A proof using the description established in Theorem 3.3.2 may be found in [BF24]. We give a new proof using the following result, which was proved in [Bis23, (4.10)].

Theorem 4.2.1. *Let* $M \in \mathrm{rep}(K_r)$ *be indecomposable such that* $q_r(\underline{\dim}\, M) + \Delta_M \geq 1$. *Then* $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$.

Theorem 4.2.2. *Let* $r \geq 2$ *and* $d \in \{1, \ldots, r-1\}$.

(1) The category $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ *is a torsion-free class closed under direct summands,* $\sigma_{K_r}^{-1}$ *and* $\tau_{K_r}^{-1}$.

(2) *The category* $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ *contains all preprojective representations and no non-zero preinjective representation.*

(3) *Let* $r \geq 3$ *and* $\mathcal{C}$ *be a regular component of the Auslander-Reiten component. There exists a quasi-simple representations* $M_{\mathcal{C},d} \in \mathcal{C}$ *such that*

$$\mathcal{C} \cap \operatorname{rep}_{\mathrm{proj}}(K_r, d) = (M_{\mathcal{C},d} \to).$$

(4) *Let* $r \geq 3$ *and* $\mathcal{C}$ *be a regular component of the Auslander-Reiten component. There are only the following two possibilities:*

(i) Either $M_{\mathcal{C},1} = M_{\mathcal{C},2} = \cdots = M_{\mathcal{C},r-1}$*, or*

(ii) $M_{\mathcal{C},1} = \tau_{K_r}(M_{\mathcal{C},r-1})$ *and in this case there exists a unique number* $1 < b \leq r-1$ *such that*

$$M_{\mathcal{C},1} = M_{\mathcal{C},2} = \cdots = M_{\mathcal{C},b-1}, \text{and}$$

$$M_{\mathcal{C},b} = M_{\mathcal{C},b+1} = \cdots = M_{\mathcal{C},r-1}.$$

Proof. (1) By Remark 4.1.1 the category $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ is closed under subrepresentations, direct products in $\operatorname{rep}(K_r)$ (i.e. direct sums) and direct summands. Let

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

be a short exact sequence with $A, B \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ and $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$. The representations $\alpha^*(A)$ and $\alpha^*(B)$ are projective. Since $\alpha^*(B)$ is projective, we have $\alpha^*(E) \cong \alpha^*(A) \oplus \alpha^*(B)$ and conclude that $\alpha^*(E)$ is projective. Hence $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ is closed under extensions. Let $C \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$, then Corollary 4.1.4 implies $\sigma_{K_r}^{-1}(C) \in \operatorname{rep}_{\mathrm{proj}}(K_r, r-1) \subseteq \operatorname{rep}_{\mathrm{proj}}(K_r, d)$. The statement follows since $\tau_{K_r}^{-1} \cong \sigma_{K_r}^{-1} \circ \sigma_{K_r}^{-1}$.

(2) The statements follow from (1) since $P_0(r) \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ and $I_0(r) \notin \operatorname{rep}_{\mathrm{proj}}(K_r, d)$.

(3) Let $X \in \mathcal{C}$ be a quasi-simple representation and $Y \coloneqq \sigma_{K_r}^{-1}(X)$. Then

$$(*) \quad \Delta_Y - \Delta_X = (\dim_{\Bbbk} Y_2 - \dim_{\Bbbk} Y_1) - (\dim_{\Bbbk} X_2 - \dim_{\Bbbk} X_1) = (r-2)\dim_{\Bbbk} X_2 > 0.$$

Since $q_r(\underline{\dim}\, X) = q_r(\underline{\dim}\, \sigma_{K_r}^{\ell}(X))$ for all $l \in \mathbb{Z}$ and $\sigma_{K_r} \circ \sigma_{K_r} \cong \tau_{K_r}$, we find $k \in \mathbb{N}_0$ such that

$$q_r(\underline{\dim}\, \tau_{K_r}^{-k}(X)) + \Delta_{\tau_{K_r}^{-k}(X)} \geq 1.$$

Now Theorem 4.2.1 implies $\tau_{K_r}^{-k}(X) \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1) \cap \mathcal{C}$. By the same token, $(*)$ implies that we find $\ell \in \mathbb{N}_0$ such that $\Delta_{\tau_{K_r}^{\ell}(X)} < 0$. In particular, $\tau_{K_r}^{\ell}(X) \notin \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$. We conclude with (1) that the statements hold for $d = 1$. The general case now follows from the case $d = 1$ using (1) and Corollary 4.1.4.

(4) This follows immediately from Corollary 4.1.4.

□

Remark 4.2.3. Note that Theorem 4.2.1 allows us to easily compute an upper bound for the distance between any representation $X \in \mathcal{C}$ and the cone $(M_{\mathcal{C},d} \to)$.

Figure 4.1 and Figure 4.2 illustrate both cases that can occur for a regular component $\mathcal{C}$ of the Auslander-Reiten of K_r for $r \geq 3$.

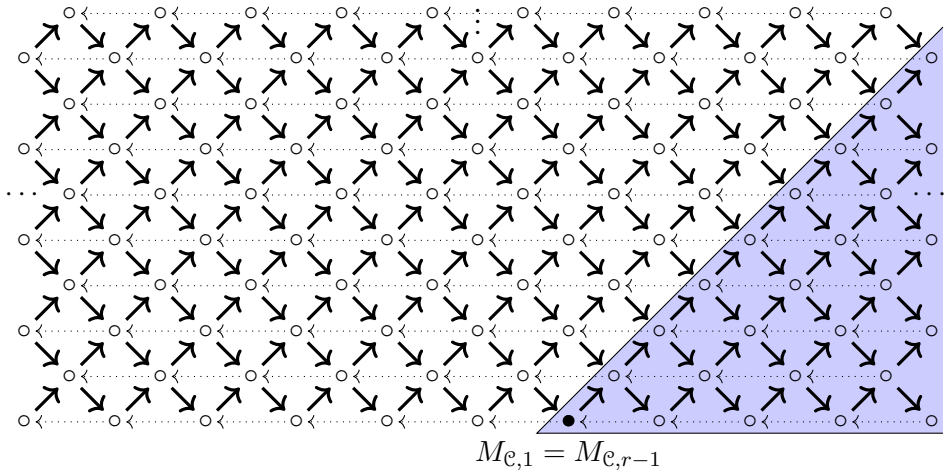


Figure 4.1.: Regular component with $M_{\mathcal{C},1} = M_{\mathcal{C},r-1}$.

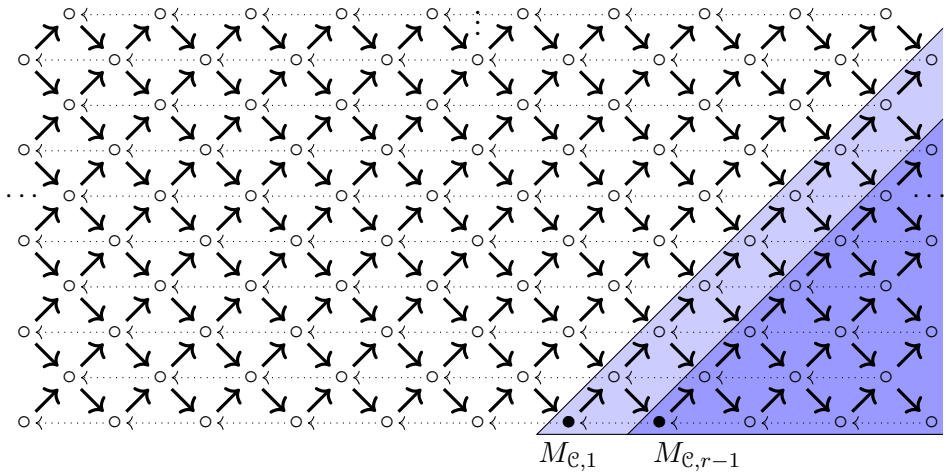


Figure 4.2.: Regular component with $M_{\mathcal{C},1} \neq M_{\mathcal{C},r-1}$.

5. The variety of representations

We fix a pair of vector spaces (V_1, V_2) and consider the irreducible space of representations

$$\operatorname{rep}(K_r; V_1, V_2) \coloneqq \operatorname{Hom}_{\Bbbk}(A_r \otimes_{\Bbbk} V_1, V_2).$$

The algebraic group $G(V_1, V_2) \coloneqq \operatorname{GL}(V_2) \times \operatorname{GL}(V_1)$ acts on the variety $\operatorname{rep}(K_r; V_1, V_2)$ via

$$(g_2, g_1).\psi \coloneqq g_2 \circ \psi \circ (g_1^{-1} \otimes \operatorname{id}_{V_1})$$

and the orbits of this action are in bijection with the isomorphism classes of Kronecker representations defined on the pair (V_1, V_2) via the map

$$G(V_1, V_2).\psi \mapsto [(V_1, V_2, \psi)]^1.$$

Given $\mathcal{S} \subseteq \operatorname{rep}(K_r)$ and $\mathcal{T} \subseteq \operatorname{rep}(K_r; V_1, V_2)$ we define

$$\mathcal{S} \cap \mathcal{T} \coloneqq \mathcal{T} \cap \mathcal{S} \coloneqq \{g \in \mathcal{T} \mid (V_1, V_2, g) \in \mathcal{S}\} \subseteq \operatorname{rep}(K_r; V_1, V_2).$$

We devote this chapter to showing that the equal socle property and being relative d-projective are open properties in $\operatorname{rep}(K_r; V_1, V_2)$. Furthermore, we investigate which constraints on the pair

$$\underline{\dim}(V_1, V_2) \coloneqq (\dim_{\Bbbk} V_1, \dim_{\Bbbk} V_2)$$

guarantee that the corresponding open sets are not empty. Parts of this chapter are based on [Bis24].

5.1. Bricks

Recall that a representation $M \in \operatorname{rep}(K_r)$ is called *brick*, provided $\operatorname{End}_{K_r}(M) = \Bbbk \operatorname{id}_M$. We define

$$\mathcal{B}(V_1, V_2) \coloneqq \{\psi \in \operatorname{rep}(K_r; V_1, V_2) \mid (V_1, V_2, \psi) \text{ is a brick}\}$$

and say that $\underline{\dim}(V_1, V_2)$ is a *Schur root*, if $\mathcal{B}(V_1, V_2)$ is non-empty. It is well-known that $\mathcal{B}(V_1, V_2)$ is a dense open subset of $\operatorname{rep}(K_r; V_1, V_2)$ if and only if $q_r(\underline{\dim}(V_1, V_2)) \leq 1$. To the best of our knowledge, the result was first proven by Kac. For the convenience of the reader, we provide a proof.

Proposition 5.1.1. *Let (V_1, V_2) be a pair of $\Bbbk$-vector spaces such that $V_1 \oplus V_2 \neq \{0\}$. The following statements hold.*

[1] We write (V_1, V_2, ψ) to make clear that we mean an element in $\operatorname{rep}(K_r)$.

(1) *The map* $\varphi\colon \operatorname{rep}(K_r; V_1, V_2) \longrightarrow \mathbb{N}_0\ ;\ M \mapsto \dim_{\Bbbk} \operatorname{End}_{K_r}(M)$ *is upper semicontinuous.*

(2) *The set* $\mathcal{B}(V_1, V_2)$ *is open in* $\operatorname{rep}(K_r; V_1, V_2)$.

(3) *The set* $\mathcal{B}(V_1, V_2)$ *is a dense subset of* $\operatorname{rep}(K_r; V_1, V_2)$ *if and only if* $q_r(\underline{\dim}(V_1, V_2)) \leq 1$.

Proof. (1) The proof is based on [Cra20, §3.5]. We define $V \coloneqq \operatorname{End}_{\Bbbk}(V_1) \times \operatorname{End}_{\Bbbk}(V_2)$. As a composition of a morphism and a lower semicontinuous map (see Example 3.2.1), the map

$$\kappa\colon \operatorname{rep}(K_r; V_1, V_2) \times V \longrightarrow \operatorname{Hom}_{\Bbbk}(A_r \otimes_{\Bbbk} V_1, V_2) \longrightarrow \mathbb{N}_0$$

$$(\psi, (f_1, f_2)) \mapsto f_2 \circ \psi - \psi \circ (f_1 \otimes \mathrm{id}_{V_1}) \mapsto \operatorname{rk}(f_2 \circ \psi - \psi \circ (f_1 \otimes \mathrm{id}_{V_1}))$$

is lower semicontinuous. Hence

$$Z \coloneqq \kappa^{-1}(\mathbb{N}_{\leq 0}) = \{(\psi, (f_1, f_2)) \mid f_2 \circ \psi - \psi \circ (f_1 \otimes \mathrm{id}_{V_1}) = 0\} \subseteq \operatorname{rep}(K_r; V_1, V_2) \times V$$

is closed in $\operatorname{rep}(K_r; V_1, V_2) \times V$ and a variety. We consider the projection morphism

$$\operatorname{pr}\colon Z \longrightarrow \operatorname{rep}(K_r; V_1, V_2)\ ;\ (\psi, (f_1, f_2)) \mapsto \psi.$$

Using Example 3.2.1(2) we get an upper semicontinuous map

$$\varepsilon_{\operatorname{pr}}\colon Z \longrightarrow \mathbb{N}_0\ ;\ z \mapsto \dim_z \operatorname{pr}^{-1}(\operatorname{pr}(z)).$$

Precomposing $\varepsilon_{\operatorname{pr}}$ with the morphism $\operatorname{rep}(K_r; V_1, V_2) \longrightarrow Z\ ;\ \psi \mapsto (\psi, (0, 0))$ gives the upper semicontinuous map

$$\varphi\colon \operatorname{rep}(K_r; V_1, V_2) \longrightarrow Z \longrightarrow \mathbb{N}_0$$

$$\psi \mapsto (\psi, (0, 0)) \mapsto \dim_{(\psi,(0,0))} \operatorname{pr}^{-1}(\operatorname{pr}(\psi, (0, 0))).$$

We note that

$$\operatorname{pr}^{-1}(\operatorname{pr}(\psi, (0, 0))) = \{(\psi, (f_1, f_2)) \mid \psi \circ (f_1 \otimes \mathrm{id}_{V_1}) = f_2 \circ \psi\} = \{\psi\} \times \operatorname{End}_{K_r}((V_1, V_2, \psi))$$

and conclude

$$\varphi(\psi) = \dim_{(\psi,(0,0))}\{\psi\} \times \operatorname{End}_{K_r}((V_1, V_2, \psi)) = \dim_{\Bbbk} \operatorname{End}_{K_r}((V_1, V_2, \psi)).$$

(2) We have $\mathcal{B}(V_1, V_2) = \operatorname{rep}(K_r; V_1, V_2) \setminus \varphi^{-1}(\mathbb{N}_{>1})$. By (1) the set $\varphi^{-1}(\mathbb{N}_{>1})$ is closed.

(3) Since a brick is an indecomposable representation, it follows from Kac's Theorem that $q_r(\underline{\dim}(V_1, V_2)) \leq 1$ is a necessary condition.

Since $\operatorname{rep}(K_r; V_1, V_2)$ is irreducible and $\mathcal{B}(V_1, V_2)$ is open, it suffices to show that $\mathcal{B}(V_1, V_2)$ is not empty whenever $q_r(\underline{\dim}(V_1, V_2)) \leq 1$. If $q_r(\underline{\dim}(V_1, V_2)) = 1$, we find $i \in \mathbb{N}_0$ such that $\underline{\dim}(V_1, V_2) = \underline{\dim}\, P_i(r)$ or $\underline{\dim}(V_1, V_2) = \underline{\dim}\, I_i(r)$. Since preinjective and preprojective indecomposable representations are bricks (see [ASS06, (VIII.2.7)]), we assume $q_r(\underline{\dim}(V_1, V_2)) \leq 0$. In this case [Che13, (3.6)] shows that $\mathcal{B}(V_1, V_2)$ is not empty.

□

5.2. General representations

Let $\mathsf{d} := (\dim_{\Bbbk} V_1, \dim_{\Bbbk} V_2) \neq (0,0)$. For $\mathsf{e} \leq \mathsf{d} \in \mathbb{N}_0^2$ (componentwise) we let $\operatorname{rep}(K_r; V_1, V_2)_{\mathsf{e}}$ be the Zariski-closed subset (cf. [Sch92, (3.1)]) of $\operatorname{rep}(K_r; V_1, V_2)$ consisting of all representations admitting a subrepresentation of dimension vector e. We write $\mathsf{e} \hookrightarrow \mathsf{d}$ if $\operatorname{rep}(K_r; V_1, V_2)_{\mathsf{e}} = \operatorname{rep}(K_r; V_1, V_2)$. Otherwise we write $\mathsf{e} \not\hookrightarrow \mathsf{d}$. Schofield gave in [Sch92] a criterion (in a more general setting) in characteristic zero to decide whether $\mathsf{e} \hookrightarrow \mathsf{d}$ holds. Crawley-Boevey extended this criterion in [Cra96] to positive characteristic. The statement in the Kronecker setting reads as follows:

Theorem 5.2.1 (Crawley-Boevey, Schofield)**.** *We have* $\mathsf{e} \hookrightarrow \mathsf{d}$ *if and only if* $\langle \mathsf{f}, \mathsf{d} - \mathsf{e} \rangle_r \geq 0$ *for all* $\mathsf{f} \hookrightarrow \mathsf{e}$.

In the case of imaginary roots, the statement can be simplified as follows:

Proposition 5.2.2. *(see [Rei24, (3.4)]) Assume that* $q_r(\mathsf{d}) \leq 0$*. The following statements are equivalent for* $\mathsf{e} \in \mathbb{N}_0^2$ *with* $\mathsf{e} \leq \mathsf{d}$*.*

(1) $\mathsf{e} \hookrightarrow \mathsf{d}$.

(2) $\langle \mathsf{e}, \mathsf{d} - \mathsf{e} \rangle_r \geq 0$.

We adapt the proof of Reineke by using a theorem of Schofield on Schur roots to show:

Proposition 5.2.3. *The following statements are equivalent for* $\mathsf{e} \in \mathbb{N}_0^2$ *with* $\mathsf{e} \leq \mathsf{d}$ *and* $q_r(\mathsf{e}) \leq 1$*.*

(1) $\mathsf{e} \hookrightarrow \mathsf{d}$.

(2) $\langle \mathsf{e}, \mathsf{d} - \mathsf{e} \rangle_r \geq 0$.

Proof. (1) $\Rightarrow$ (2). Apply Theorem 5.2.1 for $\mathsf{f} = \mathsf{e}$.

(2) $\Rightarrow$ (1). Let $\mathsf{f} \hookrightarrow \mathsf{e}$. In view of Theorem 5.2.1, it suffices to show that $\langle \mathsf{f}, \mathsf{d} - \mathsf{e} \rangle_r \geq 0$. Since $q_r(\mathsf{e}) \leq 1$ holds, e is a Schur root (see Proposition 5.1.1). Hence [Sch92, (6.1)] implies $0 < \langle \mathsf{f}, \mathsf{e} \rangle_r - \langle \mathsf{e}, \mathsf{f} \rangle_r = r(e_1 f_2 - e_2 f_1)$. In particular, $e_1 \neq 0$ and $f_2 > \frac{e_2 f_1}{e_1}$. We conclude with $d_2 - e_2 \geq 0$

$$\begin{aligned} \langle \mathsf{f}, \mathsf{d} - \mathsf{e} \rangle_r &= f_1(d_1 - e_1 - r(d_2 - e_2)) + f_2(d_2 - e_2) \geq f_1(d_1 - e_1 - r(d_2 - e_2)) + \frac{f_1 e_2}{e_1}(d_2 - e_2) \\ &= \frac{f_1}{e_1} \langle \mathsf{e}, \mathsf{d} - \mathsf{e} \rangle_r \geq 0. \end{aligned}$$

□

5.3. The equal socle property

The goal of this section is to characterize dimension vectors admitting representations with the equal socle property. This description plays a crucial role in Chapter 6, where we determine the dimension vectors admitting an elementary representation. To be able to use Proposition 5.2.3, we prove a characterization of $\mathrm{rep}_{\mathrm{esp}}(K_r, d)$ and $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$ in terms of the absence of subrepresentations:

Proposition 5.3.1. *Let $M \in \mathrm{rep}(K_r)$.*

(1) The following statements are equivalent.

(i) $M \notin \mathrm{rep}_{\mathrm{esp}}(K_r, d)$.

(ii) There exists $a \in \{0, \dots, r-d\}$ and a subrepresentation $X \subseteq M$ with dimension vector $(1, a)$.

(2) The following statements are equivalent.

(i) $M \notin \mathrm{rep}_{\mathrm{proj}}(K_r, d)$.

(ii) There exist $a \in \{1, \dots, d\}$, $a' \in \{0, \dots, ar-1\}$ and a subrepresentation $X \subseteq M$ with dimension vector (a, a').

(3) A subrepresentation X as in (1)(ii) and (2)(ii) is not preprojective.

Proof. (1) (i) $\Rightarrow$ (ii). Let $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ and $0 \neq x \in \bigcap_{a \in \mathfrak{v}} \ker a_M$. We denote by X the representation generated by x. Let $\mathfrak{u} \in \mathrm{Gr}_{r-d}(A_r)$ such that $\mathfrak{u} \oplus \mathfrak{v} = A_r$. Then $X_2 = \mathrm{im}\, \psi_M(\mathfrak{v} \otimes_{\Bbbk} X_1) + \mathrm{im}\, \psi_M(\mathfrak{u} \otimes_{\Bbbk} X_1) = \mathrm{im}\, \psi_M(\mathfrak{u} \otimes_{\Bbbk} X_1)$. Since $\dim_{\Bbbk} X_1 = 1$, we obtain $\dim_{\Bbbk} \psi_M(\mathfrak{u} \otimes_{\Bbbk} X_1) \leq \dim_{\Bbbk} \mathfrak{u} = r - d$.

(ii) $\Rightarrow$ (i). Let $x \in X_1 \setminus \{0\}$. Then x generates an indecomposable representation $\langle x \rangle \subseteq X$ with $\underline{\dim} \langle x \rangle = (1, u)$ for some $0 \leq u \leq r - d$ and

$$\psi_M|_{A_r \otimes_{\Bbbk} \Bbbk x} \colon A_r \otimes_{\Bbbk} \Bbbk x \longrightarrow \langle x \rangle_2 \; ; \; a \otimes m \mapsto \psi_M(a \otimes m)$$

is surjective. We have $\dim_{\Bbbk} \ker \psi_M|_{A_r \otimes_{\Bbbk} \Bbbk x} = r - u \geq r - (r-d) = d$. Hence we find $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ such that $\psi_M(\{a \otimes x \mid a \in \mathfrak{v}\}) = \{0\}$, hence $0 \neq x \in \bigcap_{a \in \mathfrak{v}} a_M$ and $M \notin \mathrm{rep}_{\mathrm{esp}}(K_r, d)$.

(2) (i) $\Rightarrow$ (ii). We find $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ (see Theorem 3.3.2) such that $\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1} \colon \mathfrak{v} \otimes_{\Bbbk} M_1 \longrightarrow M_2$ is not injective. We fix a basis $(v_1, \dots, v_d)$ of $\mathfrak{v}$ and an element $0 \neq x = \sum_{i=1}^{d} v_i \otimes m_i$ in the kernel of $\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}$. We consider the representation $X \subseteq M$ generated by $\{m_1, \dots, m_d\}$, then $1 \leq \dim_{\Bbbk} X_1 \leq d$. Let $\mathfrak{u} \in \mathrm{Gr}_{r-d}(A_r)$ be a direct complement of $\mathfrak{v}$ in A_r. We have

$$\begin{aligned} \dim_{\Bbbk} X_2 &\leq \dim_{\Bbbk} \psi_M(\mathfrak{v} \otimes_{\Bbbk} X_1) + \dim_{\Bbbk} \psi_M(\mathfrak{u} \otimes_{\Bbbk} X_1) \\ &\leq d \dim_{\Bbbk} X_1 - 1 + (r-d) \dim_{\Bbbk} X_1 \\ &= r \dim_{\Bbbk} X_1 - 1. \end{aligned}$$

(ii) $\Rightarrow$ (i). We write $X = Y \oplus \ell P_0(r)$ such that $P_0(r)$ is not a direct summand of Y. Then

$$\psi_M|_{A_r \otimes_{\Bbbk} Y_1} \colon A_r \otimes_{\Bbbk} Y_1 \longrightarrow Y_2$$

is surjective. We have $\dim_{\Bbbk} Y_1 = a$, $\dim_{\Bbbk} Y_2 \leq ar - 1$ and obtain $\dim_{\Bbbk} \ker(\psi_M|_{A_r \otimes_{\Bbbk} Y_1}) \geq ra - (ar - 1) = 1$. Let $(v_1, \dots, v_a)$ be a basis of Y_1. We find $0 \neq x = \sum_{i=1}^{a} y_i \otimes v_i \in \ker(\psi_M|_{A_r \otimes_{\Bbbk} Y_1})$ and $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ containing $y_1, \dots, y_a$. Hence we have $0 \neq x \in \ker \psi|_{\mathfrak{v} \otimes_{\Bbbk} M_1}$.

(3) Let X be as in (1)(ii) or (2)(ii). Since $X \subseteq X$, the representation is not in $\mathrm{rep}_{\mathrm{esp}}(K_r, d)$ or $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$, respectively. By Corollary 4.1.4 we have $\mathrm{rep}_{\mathrm{proj}}(K_r, d) \subseteq \mathrm{rep}_{\mathrm{esp}}(K_r, d)$ and conclude $X \notin \mathrm{rep}_{\mathrm{proj}}(K_r, d)$. Now the statement follows from Theorem 4.2.2(2).

□

Theorem 5.3.2. *Let V_1, V_2 be vector spaces such that $V_1 \oplus V_2 \neq 0$. The following statements hold.*

(1) The set $\mathrm{rep}_{\mathrm{esp}}(K_r, d) \cap \mathrm{rep}(K_r; V_1, V_2)$ is open in $\mathrm{rep}(K_r; V_1, V_2)$.

(2) The following statements are equivalent.

(i) $\mathrm{rep}_{\mathrm{esp}}(K_r, d) \cap \mathrm{rep}(K_r; V_1, V_2) \neq \varnothing$.

(ii) $V_1 = 0$ or $\nabla_{(V_1,V_2)}(d) \geq d(r-d)$[2].

Proof. (1) By Proposition 5.3.1 we have

$$\mathrm{rep}_{\mathrm{esp}}(K_r, d) \cap \mathrm{rep}(K_r; V_1, V_2) = \mathrm{rep}(K_r; V_1, V_2) \setminus \bigcup_{i=0}^{r-d} \mathrm{rep}(K_r; V_1, V_2)_{(1,i)}.$$

(2) (i) $\Rightarrow$ (ii) Assume that $\mathrm{rep}_{\mathrm{esp}}(K_r, d) \cap \mathrm{rep}(K_r; V_1, V_2) \neq \varnothing$. We assume that $V_1 \neq 0$. Then $\dim_{\Bbbk} V_2 > r - d$ by Proposition 5.3.1. Another application of Proposition 5.3.1 implies $(1, r-d) \not\hookrightarrow (\dim_{\Bbbk} V_1, \dim_{\Bbbk} V_2)$. We have $q_r(1, r-d) = 1 - d(r-d) \leq 1$ and conclude with Proposition 5.2.3

$$\begin{aligned} 0 &> \langle (1, r-d), (\dim_{\Bbbk} V_1, \dim_{\Bbbk} V_2) - (1, r-d) \rangle_r \\ &= \dim_{\Bbbk} V_1 - d \dim_{\Bbbk} V_2 - (1 - d(r-d)) \\ &= -\nabla_{(V_1,V_2)}(d) + d(r-d) - 1. \end{aligned}$$

(ii) $\Rightarrow$ (i). If $V_1 = 0$, the statement is clear. Hence we assume $\dim_{\Bbbk} V_1 \neq 0$ and $\nabla_{(V_1,V_2)}(d) \geq d(r-d)$. We have $d \dim_{\Bbbk} V_2 \geq \nabla_{(V_1,V_2)}(d) \geq d(r-d)$ and conclude $\dim_{\Bbbk} V_2 \geq r - d$. Hence $(1, r-d) \leq \underline{\dim}(V_1, V_2)$ with

$$\langle (1, r-d), \underline{\dim}(V_1, V_2) - (1, r-d) \rangle_r = -\nabla_{(V_1,V_2)}(d) + d(r-d) - 1 \leq -1.$$

Since $q_r(1, r-d) \leq 1$, we conclude with Proposition 5.2.3 that the set $\mathrm{rep}(K_r; V_1, V_2) \setminus \mathrm{rep}(K_r; V_1, V_2)_{(1,r-d)}$ is non-empty. Note that $\dim_{\Bbbk} V_2 \geq r - d$ and $V_1 \neq 0$ imply

$$\mathrm{rep}(K_r; V_1, V_2)_{(1,r-d)} = \bigcup_{i=0}^{r-d} \mathrm{rep}(K_r; V_1, V_2)_{(1,i)}.$$

□

Given $M \in \mathrm{rep}(K_r)$ indecomposable, Kac's Theorem implies $q_r(\underline{\dim}\, M) \leq 1$. Therefore, the following result describes all dimension vectors that can be realized by indecomposable elements in $\mathrm{rep}_{\mathrm{esp}}(K_r, d)$.

[2] Recall the definition of ∇ from Section 3.3.

Corollary 5.3.3. *Let V_1, V_2 be a pair of vector spaces such that $V_1 \oplus V_2 \neq 0$ and $q_r(\underline{\dim}(V_1, V_2)) \leq 1$. The following statements are equivalent.*

(i) $\operatorname{rep}_{\text{esp}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2) \neq \varnothing$.

(ii) $\operatorname{rep}_{\text{esp}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$ *is a dense open subset* $\operatorname{rep}(K_r; V_1, V_2)$.

(iii) There exists a brick $N \in \operatorname{rep}_{\text{esp}}(K_r, d)$ *with dimension vector* $\underline{\dim}\, N = \underline{\dim}(V_1, V_2)$.

(iv) $\nabla_{(V_1,V_2)}(d) \geq d(r-d)$ *or* $\underline{\dim}(V_1, V_2) = (0, 1)$.

Proof. (i) $\Rightarrow$ (ii). This is clear since $\operatorname{rep}_{\text{esp}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$ is open in $\operatorname{rep}(K_r; V_1, V_2)$ by Theorem 5.3.2 and $\operatorname{rep}(K_r; V_1, V_2)$ is irreducible.

(ii) $\Rightarrow$ (iii). Since $q_r(\underline{\dim}(V_1, V_2)) \leq 1$ and $V_1 \oplus V_2 \neq 0$, we know by Proposition 5.1.1 that the open set

$$\mathcal{B}(V_1, V_2) = \{\psi \in \operatorname{rep}(K_r; V_1, V_2) \mid (V_1, V_2, \psi) \text{ is a brick}\}$$

is dense in $\operatorname{rep}(K_r; V_1, V_2)$. Hence $\mathcal{B}(V_1, V_2) \cap \operatorname{rep}_{\text{esp}}(K_r, d)$ lies also dense $\operatorname{rep}(K_r; V_1, V_2)$ and is in particular non-empty.

(iii) $\Rightarrow$ (iv). Follows from Theorem 5.3.2 and the fact that a representation with dimension vector $(0, \dim_{\Bbbk} V_2)$ is isomorphic to $(\dim_{\Bbbk} V_2)P_0(r)$.

(iv) $\Rightarrow$ (i) Follows from Theorem 5.3.2. □

5.4. Relative projective representations

We will prove in Section 8.2 that relative projective representations are closely related to Steiner bundles on Grassmann varieties. Hence a better understanding of relative projective representations is of major importance. The next result is one of the main results of [BF24] (cf. [BF24, (2.3.2), (3.3.2)]). We give a new proof using the theory of general representations.

Theorem 5.4.1. *Let V_1, V_2 be vector spaces such that $V_1 \oplus V_2 \neq 0$. The following statements hold.*

(1) The set $\operatorname{rep}_{\text{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$ *is open in* $\operatorname{rep}(K_r; V_1, V_2)$.

(2) The following statements are equivalent.

(i) $\operatorname{rep}_{\text{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2) \neq \varnothing$.

(ii) $\Delta_{(V_1,V_2)}(d) \geq \min\{d(r-d), \dim_{\Bbbk} V_1(r-d)\}$[3].

Proof. (1) In view of Proposition 5.3.1, the set

$$\operatorname{rep}_{\text{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2) = \operatorname{rep}(K_r; V_1, V_2) \setminus \bigcup_{\mathsf{e} \in \mathfrak{M}} \operatorname{rep}(K_r; V_1, V_2)_{\mathsf{e}}$$

for $\mathfrak{M} := \{(a, a') \mid a \in \{1, \dots, d\}, a' \in \{0, \dots, ar-1\}\}$ is open.

[3] Recall the definition of Δ from Section 3.3.

(2) (i) $\Rightarrow$ (ii). Let $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$. In particular, $M_1 = V_1$ and $M_2 = V_2$. We write $M = aP_0(r) \oplus bP_1(r) \oplus N$ such that $P_0(r), P_1(r) \nmid N$. If $N \neq 0$, we have $\sigma_{K_r}(M) \cong \sigma_{K_r}(N) \oplus bP_0(r)$ and Proposition 4.1.2 implies $\sigma_{K_r}(N) \in \operatorname{rep}_{\mathrm{esp}}(K_r, r-d)$. We have

$$\begin{aligned}\underline{\dim}\,\sigma_{K_r}(N) &= \sigma_r(\dim_{\Bbbk} M_1 - b, \dim_{\Bbbk} M_2 - a - rb)\\ &= (r(\dim_{\Bbbk} M_1 - b) - \dim_{\Bbbk} M_2 + a + rb, \dim_{\Bbbk} M_1 - b)\\ &= (r\dim_{\Bbbk} M_1 - \dim_{\Bbbk} M_2 + a, \dim_{\Bbbk} M_1 - b).\end{aligned}$$

Since N is not projective, we have $\sigma_{K_r}(N)_1 \neq 0$ and conclude with Theorem 5.3.2

$$\begin{aligned}d(r-d) &\leq \nabla_{\sigma_{K_r}(N)}(r-d)\\ &= (r-d)(\dim_{\Bbbk} M_1 - b) - (r\dim_{\Bbbk} M_1 - \dim_{\Bbbk} M_2 + a)\\ &= \dim_{\Bbbk} M_2 - d\dim_{\Bbbk} M_1 - b(r-d) - a = \Delta_M(d) - b(r-d) - a.\end{aligned}$$

Hence

$$d(r-d) \leq d(r-d) + b(r-d) + a \leq \Delta_M(d).$$

Now assume that $N = 0$, i.e. M is projective. Then $\Delta_M(d) = b(r-d) + a \geq b(r-d) = \dim_{\Bbbk} M_1(r-d)$.

(ii) $\Rightarrow$ (i). At first we consider the case $\Delta_{(V_1,V_2)}(d) \geq \dim_{\Bbbk} V_1(r-d)$. Then we have $\dim_{\Bbbk} V_2 \geq r\dim_{\Bbbk} V_1$, i.e. $\Delta_{(V_1,V_2)}(r) \geq 0$. Since $(\dim_{\Bbbk} V_1)P_1(r) \oplus \Delta_{(V_1,V_2)}(r)P_0(r) \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ has dimension vector $\underline{\dim}(V_1, V_2)$, we conclude $\operatorname{rep}_{\mathrm{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2) \neq \varnothing$.

Now we consider the case $\Delta_{(V_1,V_2)}(d) \geq d(r-d)$. By the first case, we may assume that $r\dim_{\Bbbk} V_1 - \dim_{\Bbbk} V_2 > 0$ holds. We consider $(r\dim_{\Bbbk} V_1 - \dim_{\Bbbk} V_2, \dim_{\Bbbk} V_1) \in \mathbb{N}^2$. Then $\nabla_{(r\dim_{\Bbbk} V_1 - \dim_{\Bbbk} V_2, \dim_{\Bbbk} V_1)}(r-d) = (r-d)\dim_{\Bbbk} V_1 - (r\dim_{\Bbbk} V_1 - \dim_{\Bbbk} V_2) = \Delta_{(V_1,V_2)}(d) \geq d(r-d)$. We apply Theorem 5.3.2 and find $M \in \operatorname{rep}_{\mathrm{esp}}(K_r, r-d)$ with dimension vector $(r\dim_{\Bbbk} V_1 - \dim_{\Bbbk} V_2, \dim_{\Bbbk} V_1)$. Since $\operatorname{rep}_{\mathrm{esp}}(K_r, r-d)$ does not contain $I_0(r) = S(1)$, we conclude $\underline{\dim}\,\sigma_{K_r}^{-1}(M) = \underline{\dim}(V_1, V_2)$ and Remark 4.1.3 implies $\sigma_{K_r}^{-1}(M) \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$. □

This previous result motivates the following definition.

Definition. Let $d \in \{1, \ldots, r-1\}$. We say that $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ has *minimal type*, provided $\Delta_M(d) = d(r-d)$.

We shall see later that the following result reflects the scarcity of indecomposable Steiner bundles $\mathcal{F}$ on $\mathrm{Gr}_d(A_r)$ of rank $\mathrm{rk}(\mathcal{F}) < d(r-d)$.

Corollary 5.4.2. *Let $M, N \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ be representations.*

(1) Suppose that $\Delta_M(d) < d(r-d)$.

(a) If M is indecomposable, then $M \cong P_0(r)$, or $d \neq 1$ and $M \cong P_1(r)$.

(b) $M \cong \Delta_M(r)P_0(r) \oplus (\dim_{\Bbbk} M_1)P_1(r)$ is projective.

(2) Suppose that $\Delta_M(d) = d(r-d)$.

(a) If M is not projective, then every $f \in \operatorname{Hom}_{K_r}(M,N) \setminus \{0\}$ is injective.

(b) M is a brick or projective.

(c) If M is not projective, then $\dim_{\Bbbk} M_1 \geq d+1$.

(d) If $\dim_{\Bbbk} M_1 \geq d+1$, then M is not projective.

Proof. (1a) Suppose first that $M \ncong P_0(r)$ is indecomposable. Then $\psi_M \colon A_r \otimes_{\Bbbk} M_1 \longrightarrow M_2$ is surjective. Hence $\Delta_M(r) \leq 0$ and

$$(*) \quad 0 \longrightarrow -\Delta_M(r)P_0(r) \longrightarrow (\dim_{\Bbbk} M_1)P_1(r) \longrightarrow M \longrightarrow 0$$

is a projective resolution of M (see [ASS06, (I.5.8), (III.2.2)]. Direct computation shows

$$(\dim_{\Bbbk} M_1)(r-d) = \Delta_M(d) - \Delta_M(r).$$

Our current assumption in conjunction with Theorem 5.4.1 implies $d > \dim_{\Bbbk} M_1 \geq 1$ and $\Delta_M(d) \geq (\dim_{\Bbbk} M_1)(r-d)$. We conclude $\Delta_M(d) \geq (\dim_{\Bbbk} M_1)(r-d) = \Delta_M(d) - \Delta_M(r)$, whence $\Delta_M(r) \geq 0$. Consequently, $\Delta_M(r) = 0$, so that $(*)$ and the indecomposability of M show that $M \cong P_1(r)$.

(1b) Let $M = \bigoplus_{i=1}^n M_i$ with $M_1, \ldots, M_n$ indecomposable. Since $\operatorname{rep}_{\operatorname{proj}}(K_r, d)$ is closed under taking subrepresentations, the statement follows from (a) using $0 \leq \Delta_{M_i}(d)$ (see Theorem 5.4.1) and $\sum_{i=1}^n \Delta_{M_i}(d) = \Delta_M(d) < d(r-d)$.

(2a) Let $f \in \operatorname{Hom}_{K_r}(M,N) \setminus \{0\}$. Then the terms of the short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} \operatorname{im} f \longrightarrow 0$$

belong to $\operatorname{rep}_{\operatorname{proj}}(K_r, d)$ and

$$(\dagger) \quad \Delta_M(d) = \Delta_{\ker f}(d) + \Delta_{\operatorname{im} f}(d).$$

Let $X \in \operatorname{rep}_{\operatorname{proj}}(K_r, d)$ such that $\Delta_X(d) = 0$, then Theorem 5.4.1 implies

$$\min\{d(r-d), \dim_{\Bbbk} X_1(r-d)\} \leq \Delta_X(d) = 0$$

and therefore $\dim_{\Bbbk} X_1 = 0$. Moreover, $0 = \Delta_X(d) = \dim_{\Bbbk} X_2 - d \dim_{\Bbbk} X_1 = \dim_{\Bbbk} X_2$. This shows that $X = 0$.

We have $\operatorname{im} f \neq 0$ and assume $\ker f \neq 0$. Using $(\dagger)$ and the above considerations, we conclude $d(r-d) > \Delta_{\ker f}(d), \Delta_{\operatorname{im} f}(d) > 0$. By (1a) $\ker f$ and $\operatorname{im} f$ are projective and therefore also $M \cong \ker f \oplus \operatorname{im} f$, a contradiction.

(2b) Suppose that M is not projective and let $f \in \operatorname{End}_{K_r}(M) \setminus \{0\}$. In view of (2a), f is bijective. Hence there is a non-zero eigenvalue $\alpha \in \Bbbk$ and another application of (2a) yields $f = \alpha \operatorname{id}_M$.

(2c) By (2b) M is a non-projective brick and in particular indecomposable. As in (1a) we obtain the exact sequence $(*)$ and $\Delta_M(r) < 0$. Moreover, we have

$$(\dim_{\Bbbk} M_1)(r-d) = \Delta_{(\dim_{\Bbbk} M_1)P_1(r)}(d) = -\Delta_M(r) + \Delta_M(d) > \Delta_M(d) = d(r-d),$$

so that $\dim_{\Bbbk} M_1 \geq d+1$.

(2d) Suppose that M is projective. Then $M \cong \Delta_M(r)P_0(r) \oplus (\dim_{\Bbbk} M_1)P_1(r)$, and we have

$$0 \leq \Delta_M(r) = \Delta_M(d) - (\dim_{\Bbbk} M_1)(r-d) = d(r-d) - (\dim_{\Bbbk} M_1)(r-d) \leq -(r-d),$$

a contradiction. Hence M is not projective.

□

Remark 5.4.3.

(1) It follows that every regular representation $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ of minimal type is a brick.

(2) Let (V_1, V_2) be a pair of $\Bbbk$-vector spaces. If $\Delta_{(V_1,V_2)}(d) \geq d(r-d)$, then Theorem 5.4.1 ensures that $\operatorname{rep}_{\mathrm{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$ is not empty. Alternatively, this set is either empty (namely, if and only if $\Delta_M(r) < 0$), or it consists of one $(\mathrm{GL}(V_2) \times \mathrm{GL}(V_1))$-orbit of projective representations.

Corollary 5.4.4. *Suppose that $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ is non-projective and of minimal type.*

(1) For $1 \leq d < r-1$ we have $M \notin \operatorname{rep}_{\mathrm{proj}}(K_r, d+1)$.

(2) Either M is a regular brick and $M = M_{\mathcal{C},d}$, where $\mathcal{C}$ is the regular component containing M, or $d = r-1$ and $M \cong P_2(r)$.

Proof. (1) Since M is non-projective and of minimal type, Corollary 5.4.2(2c) provides $x \geq d+1$ such that $\underline{\dim}\, M = (x, d(r-d+x))$. We thus have

$$(*) \quad \Delta_M(d+1) = d(r-d) - x.$$

Suppose that $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d+1)$. Since $x \geq d+1$, Theorem 5.4.1 in conjunction with $(*)$ gives $d(r-d) - x \geq (d+1)(r-d-1)$, whence $d+1 \leq x \leq 2d+1-r$. As this contradicts $d \leq r-1$, we conclude that $M \notin \operatorname{rep}_{\mathrm{proj}}(K_r, d+1)$.

(2) By Corollary 5.4.2(2b), M is a brick and therefore indecomposable. Since $\operatorname{rep}_{\mathrm{proj}}(K_r, d)$ does not contain non-zero preinjective representations (see Theorem 4.2.2(2)), we conclude that M is regular or preprojective. We proceed in steps.

(i) M is regular for $1 \leq d < r-1$: This follows from (1) in conjunction with Theorem 4.2.2(2).

(ii) For $d = r-1$ the representation $P_2(r)$ is of minimal type: We have $\Delta_{P_2(r)}(r-1) = (r^2-1) - (r-1)r = r-1 = (r-1)(r-(r-1))$.

(iii) For $d = r-1$, we have $M \cong P_2(d)$ or M is regular: We consider $N := \sigma_{K_r}(M) \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ (see Proposition 4.1.2) with $\Delta_N(1) = \dim_{\Bbbk} M_1 - (r \dim_{\Bbbk} M_1 - \dim_{\Bbbk} M_2) = \Delta_M(r-1) = r-1 = 1(r-1)$. Hence N is of minimal type for $d = 1 < r-1$. If $M \ncong P_2(r)$, then $N \ncong P_1(r)$ is not projective and (i) implies that N is regular. We conclude that M is regular.

(iv) If M is regular, then $M = M_{\mathcal{C},d}$: Regular bricks for the Kronecker quiver are quasi-simple (see [Ker94, (9.2), (9.4)]). If $d < r-1$, then (1) implies that $M \notin \operatorname{rep}_{\text{proj}}(K_r, d+1)$. Twofold application of Theorem 4.2.2 gives $\tau_{K_r}^{-1}(M) = M_{\mathcal{C},d+1}$ and $M = M_{\mathcal{C},d}$. For $d = r-1$, we conclude as in (iii) that $N := \sigma_{K_r}(M) \in \operatorname{rep}_{\text{proj}}(K_r, 1)$ is regular and has minimal type for $d = 1$. Again, (1) implies that $N \notin \operatorname{rep}_{\text{proj}}(K_r, r-1)$. Since $\operatorname{rep}_{\text{proj}}(K_r, r-1)$ is closed under $\sigma_{K_r}^{-1}$ by Theorem 4.2.2, we conclude $\tau_{K_r}(M) \cong \sigma_{K_r}^2(M) \notin \operatorname{rep}_{\text{proj}}(K_r, r-1)$ and $M = M_{\mathcal{C},r-1}$.

□

We record two further direct consequences of Theorem 5.4.1, which we will use in the next chapter in which we examine elementary representations.

Definition. Let $1 \leq d < r$. We define $\operatorname{rep}_{\text{inj}}(K_r, d) := D_{K_r}(\operatorname{rep}_{\text{proj}}(K_r, d))$

Corollary 5.4.5. *Let V_1, V_2 be vector spaces such that $V_1 \oplus V_2 \neq 0$. The following statements hold.*

(1) The set $\operatorname{rep}_{\text{inj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$ is open in $\operatorname{rep}(K_r; V_1, V_2)$.

(2) The following statements are equivalent.

(i) $\operatorname{rep}_{\text{inj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2) \neq \varnothing$.

(ii) $-\nabla_{(V_1,V_2)}(d) \geq \min\{d(r-d), \dim_{\Bbbk} V_2(r-d)\}$.

Proof. (1) Note that $\operatorname{rep}_{\text{inj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$ is open in $\operatorname{rep}(K_r; V_1, V_2)$, since the duality $D_{K_r} \colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_r)$ induces an isomorphism of varieties

$$\operatorname{rep}(K_r; V_2, V_1) \longrightarrow \operatorname{rep}(K_r; V_1^*, V_2^*) \cong \operatorname{rep}(K_r; V_1, V_2)$$

that takes $\operatorname{rep}_{\text{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_2, V_1)$ to $\operatorname{rep}_{\text{inj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$.

(2) We have $-\nabla_{(V_1,V_2)}(d) = -(d \dim_{\Bbbk} V_2 - \dim_{\Bbbk} V_1) = \Delta_{(V_2,V_1)}(d)$. Since the set $\operatorname{rep}_{\text{inj}}(K_r, d) \cap \operatorname{rep}(K_r; V_1, V_2)$ is non-empty if and only if $\operatorname{rep}_{\text{proj}}(K_r, d) \cap \operatorname{rep}(K_r; V_2, V_1)$ is non-empty, the statement follows from Theorem 5.4.1.

□

Corollary 5.4.6. *The following statements hold.*

(1) Let $M \in \operatorname{rep}_{\text{proj}}(K_r, d)$ with $\dim_{\Bbbk} M_1 \leq d$, then M is projective.

(2) Let $M \in \operatorname{rep}_{\text{inj}}(K_r, d)$ with $\dim_{\Bbbk} M_2 \leq d$, then M is injective.

Proof. (1) Let N be an indecomposable direct summand of M not isomorphic to $P_0(r)$. Then $N_1 \neq 0$, $N \in \operatorname{rep}_{\text{proj}}(K_r, d)$ and $\dim_{\Bbbk} N_1 \leq d$. By Theorem 5.4.1 we have

$$\dim_{\Bbbk} N_2 - d \dim_{\Bbbk} N_1 = \Delta_N(d) \geq \min\{d(r-d), \dim_{\Bbbk} N_1(r-d)\} = \dim_{\Bbbk} N_1(r-d)$$

and conclude $\dim_{\Bbbk} N_2 \geq r \dim_{\Bbbk} N_1$. Since $\dim_{\Bbbk} N_1 \neq 0$, we also have a projective resolution (cf. the proof of Corollary 5.4.2)

$$0 \longrightarrow (r \dim_{\Bbbk} N_1 - \dim_{\Bbbk} N_2) P_0(r) \longrightarrow (\dim_{\Bbbk} N_1) P_1(r) \longrightarrow N \longrightarrow 0$$

and conclude $r \dim_{\Bbbk} N_1 - \dim_{\Bbbk} N_2 = 0$ as well as $N \cong (\dim_{\Bbbk} N_1) P_1(r)$.

(2) This follows from duality.

□

5.5. Restriction to tame subquivers

Let $1 < d < r$. We are interested in representations $M \in \operatorname{rep}(K_r)$ such that $\alpha^*(M) \in \operatorname{rep}(K_d)$ is preprojective for all $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$. Assume that M has this property, and let $\beta \in \operatorname{Inj}_{\Bbbk}(A_{d-1}, A_r)$. We consider the canonical embedding $\iota\colon A_{d-1} \longrightarrow A_d$ and find $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ such that $\alpha \circ \iota = \beta$. Since $\alpha^*(M) \in \operatorname{rep}(K_d)$ is preprojective, Theorem 4.2.2 implies $\alpha^*(M) \in \operatorname{rep}_{\operatorname{proj}}(K_d, d-1)$. Hence $\beta^*(M) = \iota^*(\alpha^*(M))$ is projective. This shows $M \in \operatorname{rep}_{\operatorname{proj}}(K_r, d-1)$.

Throughout this section we will always assume that $d = 2$. The above considerations tell us that a representation $M \in \operatorname{rep}(K_r)$ with $M|_{\mathfrak{v}}$ preprojective for every $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$ must be an element in $\operatorname{rep}_{\operatorname{proj}}(K_r, 1)$. This is the reason why, in the following, we intersect the considered open subsets of the representation variety with $\operatorname{rep}_{\operatorname{proj}}(K_r, 1)$.

As before we fix a pair (V_1, V_2) of $\Bbbk$-vector spaces and define for $\mathsf{e} \in \mathbb{N}_0^2$ the sets

$$\mathcal{O}(\mathsf{e}, \not\hookrightarrow) \coloneqq \operatorname{rep}(K_r; V_1, V_2) \setminus \operatorname{rep}(K_r; V_1, V_2)_{\mathsf{e}}$$

and

$$\mathcal{O}(\mathsf{e}, \not\twoheadrightarrow) \coloneqq \operatorname{rep}(K_r; V_1, V_2) \setminus \operatorname{rep}(K_r; V_1, V_2)_{\underline{\dim}(V_1,V_2)-\mathsf{e}}.$$

By definition $\mathcal{O}(\mathsf{e}, \not\hookrightarrow) \subseteq \operatorname{rep}(K_r; V_1, V_2)$ is an open subset consisting of all representations in $\operatorname{rep}(K_r; V_1, V_2)$ that do not have a subrepresentation with dimension vector e. By the same token, $\mathcal{O}(\mathsf{e}, \not\twoheadrightarrow) \subseteq \operatorname{rep}(K_r; V_1, V_2)$ is the open set consisting of all representations that do not have a factor with dimension vector e. Since the duality $D_{K_r}\colon \operatorname{rep}(K_r) \longrightarrow \operatorname{rep}(K_r)$ takes quotients to subrepresentations and vice versa, we note that

$$\begin{aligned}\mathcal{O}(\mathsf{e}, \not\twoheadrightarrow) &= \operatorname{rep}(K_r; V_1, V_2) \setminus \operatorname{rep}(K_r; V_1, V_2)_{\underline{\dim}(V_1,V_2)-\mathsf{e}} \\ &\cong \operatorname{rep}(K_r; V_2, V_1) \setminus \operatorname{rep}(K_r; V_2, V_1)_{\delta(\mathsf{e})},\end{aligned}$$

where $\delta\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ denotes the twist.

For $n \in \mathbb{N}$, we let

$$\mathsf{f}_{n,r}^- \coloneqq (n+1, (r-1)(n+1)+1) \text{ and } \mathsf{f}_{n,r}^+ \coloneqq (n(r-1)-1, n)$$

and note $\underline{\dim}\, \mathsf{f}_{n,r}^{\pm} = \underline{\dim}\, P_n^{\pm}(\mathfrak{v})$ for all $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$ (cf. Section 2.7).

Proposition 5.5.1. *Let $n \in \mathbb{N}$ and (V_1, V_2) be a pair of $\Bbbk$-vector spaces.*

(1) If $n \dim_{\Bbbk} V_2 - (n+1) \dim_{\Bbbk} V_1 \geq n(n+1)(r-2)$, then $\mathcal{O}(\mathsf{f}_{n,r}^-, \not\hookrightarrow) \neq \varnothing$.

(2) Assume that $\dim_{\Bbbk} V_2 \geq (n+1)(r-1)+1$ and let $M \in \mathcal{O}(\mathsf{f}_{n,r}^-, \not\hookrightarrow) \cap \operatorname{rep}_{\operatorname{proj}}(K_r, 1)$. We have $0 = \operatorname{Hom}_{K_r}(P_n^-(\mathfrak{v}), M)$ for all $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$.

Proof. (1) We can assume that $\mathsf{f}^-_{n,r} \leq \underline{\dim}(V_1, V_2)$, otherwise $\mathcal{O}(\mathsf{f}^-_{n,r}, \not\twoheadrightarrow) = \operatorname{rep}(K_r; V_1, V_2)$ and we are done. We have

$$\begin{aligned}
&\langle \mathsf{f}^-_{n,r}, \underline{\dim}(V_1, V_2) - \mathsf{f}^-_{n,r} \rangle_r = \langle (n+1, (r-1)(n+1)+1), \underline{\dim}(V_1, V_2) \rangle_r - q_r(\mathsf{f}^-_{n,r}) \\
=&(n+1)\dim_{\Bbbk} V_1 - n \dim_{\Bbbk} V_2 - q_r(\mathsf{f}^-_{n,r}) \leq -(n(n+1)(r-2)) - q_r(\sigma_r(\mathsf{f}^-_{n,r})) \\
=&-(n(n+1)(r-2)) - (q_2(P_n(2)) - (r-2)n(n+1)) = -1 < 0.
\end{aligned}$$

Since $q_r(\underline{\dim}\, \mathsf{f}^-_{n,r}) = q_r(n, n+1) \leq 1$, we can apply Proposition 5.2.3.

(2) Let $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$. We assume that there exists $f\colon P^-_n(\mathfrak{v}) \longrightarrow M$ non-zero. Since $\operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ is closed under subrepresentations, we have $0 \neq \operatorname{im} f \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$. For $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ such that $\operatorname{im}\alpha = \mathfrak{v}$, we have $\alpha^*(\operatorname{im} f) \in \operatorname{rep}_{\mathrm{proj}}(K_2, 1)$. Hence $\operatorname{im} f|_{\mathfrak{v}}$ is preprojective by [Wor13b, (4.3)] we write $\operatorname{im} f|_{\mathfrak{v}} = \bigoplus_{i \geq 0} b_i(\mathfrak{v}) P_i(2)$. Now Proposition 2.7.4 implies

$$0 \neq \dim_{\Bbbk} \operatorname{Hom}_{K_r}(P^-_n(\mathfrak{v}), \operatorname{im} f) = \sum_{i \geq n+1} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}_{K_2}(P_{n+1}(2), P_i(2)).$$

Hence $b_i(\mathfrak{v}) > 0$ for some $i \geq n+1$. In particular, $n+1 = \dim_{\Bbbk}(P^-_n(\mathfrak{v}))_1 \geq \dim_{\Bbbk}(\operatorname{im} f)_1 \geq \dim_{\Bbbk}(P_i(2))_1 = i \geq n+1$. We conclude $i = n+1 = \dim_{\Bbbk}(\operatorname{im} f)_1$. By assumption we have

$$\ell := \dim_{\Bbbk}(\operatorname{im} f)_2 \leq \dim_{\Bbbk}(P^-_n(\mathfrak{v}))_2 = (n+1)(r-1)+1 \leq \dim_{\Bbbk} M_2.$$

Hence can extend $\operatorname{im} f$, by adding a direct summand of the form $[(n+1)(r-1)+1-\ell]P_0(r)$, to a subrepresentation of M with dimension vector $(n+1, (n+1)(r-1)+1) = \mathsf{f}^-_{n,r}$. This is a contradiction to the assumption.

□

Proposition 5.5.2. *Let $n \in \mathbb{N}$ and (V_1, V_2) be a pair of $\Bbbk$-vector spaces.*

(1) If $(n+1)\dim_{\Bbbk} V_1 - n \dim_{\Bbbk} V_2 \geq n(n+1)(r-2)$ and $\Delta_{(V_1,V_2)} \geq 0$, then $\bigcap_{l=1}^n \mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow) \neq \varnothing$.

(2) Assume that $\dim_{\Bbbk} V_1 \geq n(r-1)-1$.

(i) Let $M \in \operatorname{rep}(K_r; V_1, V_2)$ and assume that there is $l \in \{1, \ldots, n\}$, a representation $U \in \operatorname{rep}(K_r)$ with $\dim_{\Bbbk} U_1 \leq l(r-1)-1$, $\dim_{\Bbbk} U_2 = l$ and an epimorphism $M \twoheadrightarrow U$. Then $M \notin \bigcap_{i=1}^n \mathcal{O}(\mathsf{f}^+_{i,r}, \not\twoheadrightarrow)$.

(ii) We have $0 = \operatorname{Hom}_{K_r}(M, P^+_n(\mathfrak{v}))$ for all $M \in \bigcap_{l=1}^n \mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow)$ and all $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$.

(3) If $(n+1)\dim_{\Bbbk} V_1 - n\dim_{\Bbbk} V_2 \geq n(n+1)(r-2)$ and $\Delta_{(V_1,V_2)} \geq r-1$, then $\dim_{\Bbbk} V_1 \geq n(r-1)-1$ and $\bigcap_{l=1}^n \mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow) \neq \varnothing$. Moreover, we have $0 = \dim_{\Bbbk} \operatorname{Hom}_{K_r}(M, P^+_n(\mathfrak{v}))$ for all $M \in \bigcap_{l=1}^n \mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow)$ and all $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$.

Proof. (1) Let $1 \leq l \leq n$. We have

$$\begin{aligned}
(l+1)\dim_{\Bbbk} V_1 - l \dim_{\Bbbk} V_2 &= (n+1)\dim_{\Bbbk} V_1 - n \dim_{\Bbbk} V_2 + (n-l)\Delta_{(V_1,V_2)} \\
&\geq n(n+1)(r-2) \geq l(l+1)(r-2).
\end{aligned}$$

and

$$\begin{aligned} q_r(\delta(\mathsf{f}^+_{l,r})) = q_r(\mathsf{f}^+_{l,r}) = q_r(\sigma_r^{-1}(\mathsf{f}^+_{l,r})) = q_r(\underline{\dim}\, P_l(2)) = q_2(\underline{\dim}\, P_l(2)) - (r-2)l(l+1) \\ = 1 - (r-2)l(l+1), \end{aligned}$$

where $\delta\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2; (x,y) \mapsto (y,x)$. We conclude

$$\begin{aligned} \langle \delta(\mathsf{f}^+_{l,r}), \underline{\dim}(V_2,V_1) - \delta(\mathsf{f}^+_{l,r})\rangle_r &= l\dim_{\Bbbk} V_2 - (l+1)\dim_{\Bbbk} V_1 - (1-(r-2)l(l+1)) \\ &\leq -l(l+1)(r-2) - 1 + (r-2)l(l+1) < 0. \end{aligned}$$

Moreover, we have $q_r(\delta(\mathsf{f}^+_{l,r})) \leq 1$. As in the proof of Proposition 5.5.1 we may assume without loss of generality $\mathsf{f}^+_{l,r} \leq \underline{\dim}(V_1,V_2)$. Now Proposition 5.2.3 and taking duals imply $\mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow) \neq \varnothing$. The statement follows since $\operatorname{rep}(K_r; V_1, V_2)$ is irreducible and we take an intersection of finitely many non-empty open sets.

(2) (i) Denote by $K = (K_1, K_2, \psi_M|_{A_r \otimes_{\Bbbk} K_1})$ the kernel of the epimorphism $M \twoheadrightarrow U$ under consideration. We have

$$\begin{aligned} \dim_{\Bbbk} K_1 &= \dim_{\Bbbk} V_1 - \dim_{\Bbbk} U_1 \geq \dim_{\Bbbk} V_1 - (l(r-1)-1) \\ &\geq \dim_{\Bbbk} V_1 - (n(r-1)-1) \geq 0. \end{aligned}$$

In particular, we find a subspace $W_1 \subseteq K_1$ of dimension $\dim_{\Bbbk} V_1 - (l(r-1)-1)$. We consider the subrepresentation W of K corresponding to (W_1, K_2). We have $\underline{\dim}\, W = (\dim_{\Bbbk} V_1 - (l(r-1)-1), \dim_{\Bbbk} V_2 - l)$ and therefore $\underline{\dim}(M/W) = (l(r-1)-1, l) = \mathsf{f}^+_{l,r}$. Hence $M \notin \mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow) \supseteq \bigcap_{i=1}^n \mathcal{O}(\mathsf{f}^+_{i,r}, \not\twoheadrightarrow)$.

(ii) Let $M \in \bigcap_{l=1}^n \mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow)$ and assume that $0 \neq \operatorname{Hom}_{K_r}(M, P_n^+(\mathfrak{v}))$ for some $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$. We find $l \in \{1,\dots,n\}$ minimal such that $0 \neq \operatorname{Hom}_{K_r}(M, P_l^+(\mathfrak{v}))$, and a non-zero morphism $f\colon M \longrightarrow P_l^+(\mathfrak{v})$. We set $U := \operatorname{im} f \subseteq P_l^+(\mathfrak{v})$ and note that $\dim_{\Bbbk} U_2 \neq 0$, since $I_0(r)$ is not a direct summand of $P_l^+(\mathfrak{v})$ by Proposition 2.7.5.

We write $U|_{\mathfrak{v},\mathrm{pproj}} = \bigoplus_{i\geq 0} b_i(\mathfrak{v}) P_i(2)$ and conclude with Proposition 2.7.4(3)

$$(*) \quad 0 \neq \dim_{\Bbbk} \operatorname{Hom}_{K_r}(U, P_l^+(\mathfrak{v})) = \sum_{i=0}^{l-1} b_i(\mathfrak{v}) \dim_{\Bbbk} \operatorname{Hom}_{K_2}(P_i(2), P_{l-1}(2)).$$

If $l = 1$, we have $\underline{\dim}\, P_l^+(\mathfrak{v}) = \mathsf{f}^+_{1,r} = (r-2,1)$ and therefore $0 < \dim_{\Bbbk} U_2 \leq 1 = l$, i.e. $\dim_{\Bbbk} U_2 = l$ and $\underline{\dim}\, U = (k,l)$ with $k \leq r-2 = l(r-1)-1$. This is a contradiction to (i). Hence $l > 1$. Minimality of l implies $\operatorname{Hom}_{K_r}(M, P_{l-1}^+(\mathfrak{v})) = 0$ and left exactness of $\operatorname{Hom}_{K_r}(-, P_{l-1}^+(\mathfrak{v}))$ in conjunction with Proposition 2.7.4(3) implies

$$0 = \dim_{\Bbbk} \operatorname{Hom}_{K_r}(U, P_{l-1}^+(\mathfrak{v})) = \sum_{i=0}^{l-2} b_i(\mathfrak{v}) \underbrace{\dim_{\Bbbk} \operatorname{Hom}_{K_2}(P_i(2), P_{l-2}(2))}_{\neq 0}.$$

Hence $b_i(\mathfrak{v}) = 0$ for all $0 \leq i \leq l-2$ and $(*)$ simplifies to

$$0 \neq b_{l-1}(\mathfrak{v}) \operatorname{Hom}_{K_2}(P_{l-1}(2), P_{l-1}(2)) = b_{l-1}(\mathfrak{v}).$$

In particular, $\dim_{\Bbbk} U_2 \geq \dim_{\Bbbk}(P_{l-1}(2))_2 = l$. Since

$$\underline{\dim}\, U \leq \underline{\dim}\, P_l^+(\mathfrak{v}) = (l(r-1)-1, l),$$

we get $\dim_{\Bbbk} U_1 \leq l(r-1)-1$ and $\dim_{\Bbbk} U_2 = l$. This is a contradiction to (i).

(3) By (1) we have $\bigcap_{l=1}^{n} \mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow) \neq \varnothing$ and

$$\dim_{\Bbbk} V_1 \geq n(n+1)(r-2) + n\Delta_{(V_1,V_2)} \geq n(n+1)(r-2) + n(r-1) \geq n(r-1) - 1.$$

Hence the conditions of (2) are satisfied and the statement follows from (2)(ii).

□

Now we can prove the main result of this section.

Theorem 5.5.3. *Let $n \in \mathbb{N}$, $r \geq 3$ and (V_1, V_2) be a pair of $\Bbbk$-vector spaces such that*

$$(n+1)\dim_{\Bbbk} V_1 - n\dim_{\Bbbk} V_2 \geq n(n+1)(r-2) \quad \text{(I)}$$

and

$$(n+1)\dim_{\Bbbk} V_2 - (n+2)\dim_{\Bbbk} V_1 \geq (n+1)(n+2)(r-2) \quad \text{(II)}.$$

Then the following statements hold.

(1) The open subset $\mathcal{O} \coloneqq B(V_1,V_2) \cap \operatorname{rep}_{\mathrm{proj}}(K_r,1) \cap \bigcap_{l=1}^{n} \mathcal{O}(\mathsf{f}^+_{l,r}, \not\twoheadrightarrow) \cap \mathcal{O}(\mathsf{f}^-_{n+1,r}, \not\hookrightarrow) \subseteq \operatorname{rep}(K_r; V_1, V_2)$ is non-empty.

(2) For $M \in \mathcal{O}$ the representation $M|_{\mathfrak{v}}$ is preprojective for all $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$ and we have

$$M|_{\mathfrak{v}} \cong [(n+1)\dim_{\Bbbk} V_2 - (n+2)\dim_{\Bbbk} V_1]P_n(2) \oplus [(n+1)\dim_{\Bbbk} V_1 - n\dim_{\Bbbk} V_2]P_{n+1}(2).$$

Proof. Note that (I) + (II) gives us

$$(*) \quad \Delta_{(V_1,V_2)} \geq (2n+2)(n+1)(r-2) \geq 8(r-2) \geq r-1.$$

This shows that the assumptions of Theorem 5.4.1, Proposition 5.5.2(1) and Proposition 5.5.1(1) are satisfied. In particular,

$$X \coloneqq \operatorname{rep}_{\mathrm{proj}}(K_r,1) \cap \bigcap_{i=1}^{n} \mathcal{O}(\mathsf{f}^+_{i,r}, \not\twoheadrightarrow) \cap \mathcal{O}(\mathsf{f}^-_{n+1,r}, \not\hookrightarrow)$$

is a non-empty and open subset of $\operatorname{rep}(K_r; V_1, V_2)$. Let $M \in X$ and $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$, then $M \in \operatorname{rep}_{\mathrm{proj}}(K_r,1)$, $M|_{\mathfrak{v}}$ is preprojective (see [Wor13b, (4.3)]), and Proposition 5.5.2(3) gives us $0 = \operatorname{Hom}_{K_r}(M, P_n^+(\mathfrak{v}))$. We conclude with Proposition 2.7.4(4) and $n \geq 1$ that $P_0(2)$ is not a direct summand of $M|_{\mathfrak{v}}$. Hence $P_0(r)$ is not a direct summand of M.

Let N be a direct summand of M, then $N_1 \neq 0$. The assumption $N \in \operatorname{rep}_{\mathrm{proj}}(K_r,2)$ yields with Theorem 5.4.1 that $\Delta_N(2) \geq \min\{2(r-2), \dim_{\Bbbk} N_1(r-2)\} \geq r-2 \neq 0$ and $N|_{\mathfrak{v}} = \Delta_N(2)P_0(2) \oplus (\dim_{\Bbbk} N_1)P_1(2)$ is a direct summand of $M|_{\mathfrak{v}}$, a contradiction. We conclude with Theorem 4.2.2(2) that M can not have a preprojective representation as a direct summand. Since $\operatorname{rep}_{\mathrm{proj}}(K_r,1)$ does not contain non-zero preinjective representations, every indecomposable direct summand of M is therefore regular. Hence Corollary 1.5.2 implies that $\frac{1}{L_1}\dim_{\Bbbk} V_2 < \dim_{\Bbbk} V_1 < L_r \dim_{\Bbbk} V_2$ and $q_r(\underline{\dim}(V_1,V_2)) \leq 0$. Now Proposition 5.1.1 implies that $\mathcal{B}(V_1,V_2) \cap X = \mathcal{O}$ is also non-empty.

We note that (I) implies $\dim_{\Bbbk} V_1 \geq n(n+1)(r-2) + n\Delta_{(V_1,V_2)} \geq n(n+1)(r-2) + n(r-1)$ and $\Delta_{(V_1,V_2)} \geq (2n+2)(n+1)(r-2)$ established in $(*)$, gives

$$\begin{aligned}\dim_{\Bbbk} V_2 &\geq \dim_{\Bbbk} V_1 + (2n+2)(n+1)(r-2)\\ &\geq n(n+1)(r-2) + n(r-1) + (2n+2)(n+1)(r-2)\\ &= (3n+2)(n+1)(r-2) + n(r-1)\\ &\geq 10(r-2) + n(r-1) \geq 2(r-1) + 1 + n(r-1)\\ &= (n+2)(r-1) + 1.\end{aligned}$$

Hence we may apply Proposition 5.5.1(2) and Proposition 5.5.2(3) to conclude that for every element in $\mathcal{O}$ the condition of Theorem 2.7.6(3) is satisfied. Finally, we recall that $a_n(2) = n$ for all $n \in \mathbb{N}_0$. □

The following result shows a possible application of Theorem 5.5.3.

Corollary 5.5.4. *Let $n \in \mathbb{N}$ and $r \geq 3$.*

(a) There exists a brick $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ such that $M|_{\mathfrak{v}}$ is independent of $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$ and given by

$$M|_{\mathfrak{v}} \cong [(n+1)\dim_{\Bbbk} M_2 - (n+2)\dim_{\Bbbk} M_1]P_n(2) \oplus [(n+1)\dim_{\Bbbk} M_1 - n\dim_{\Bbbk} M_2]P_{n+1}(2).$$

(b) Moreover, we can find M as in (1) with the property that

$$M|_{\mathfrak{v}} \cong [(n+2)(n+1)(r-2)]P_n(2) \oplus [n(n+1)(r-2)]P_{n+1}(2).$$

In particular, $\dim_{\Bbbk} M_1 = n(n+1)(2n+3)(r-2)$ and $\Delta_M = (n+1)(2n+2)(r-2)$.

Proof. (a) It suffices to show that we can find a pair of vector spaces (V_1, V_2) that satisfies the assumptions of Theorem 5.5.3. Let $x \geq (2n^2+3n)(r-2)$ be a natural number. We define $a_x := (n+1)x$, $b_x := (n+2)(x+r-2)$ and fix a pair of vector spaces (V_1, V_2) such that $\underline{\dim}(V_1, V_2) = (a_x, b_x)$. We have

$$\begin{aligned}(n+1)\dim_{\Bbbk} V_1 - n\dim_{\Bbbk} V_2 &= (n+1)(n+1)x - n(n+2)(x+r-2)\\ &= x - n(n+2)(r-2)\\ &\geq (2n^2+3n)(r-2) - n(n+2)(r-2) = n(n+1)(r-2).\end{aligned}$$

By definition we have

$$\begin{aligned}(n+1)\dim_{\Bbbk} V_2 - (n+2)\dim_{\Bbbk} V_1 &= (n+1)(n+2)(x+r-2) - (n+2)(n+1)x\\ &= (n+2)(n+1)(r-2).\end{aligned}$$

(b) Consider the case $x = (2n^2+3n)(r-2)$ in (a).

□

We also record the following Lemma.

Lemma 5.5.5. *Let $n \in \mathbb{N}$, $r \geq 3$ and (V_1, V_2) be a pair of $\Bbbk$-vector spaces.*

(1) Assume that $n \dim_\Bbbk V_2 - (n+1) \dim_\Bbbk V_1 \geq n(n+1)(r-2)$ and $\dim_\Bbbk V_2 \geq (n+1)(r-1)+1$. A general representation M[4] of $\operatorname{rep}(K_r; V_1, V_2)$ is in $\operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ and for each $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$ we have

$$M|_{\mathfrak{v}} = M|_{\mathfrak{v},\mathrm{pproj}} \in \operatorname{add}(\{P_i(2) \mid 0 \leq i \leq n\}).$$

(2) Assume that $(n+1) \dim_\Bbbk V_1 - n \dim_\Bbbk V_2 \geq n(n+1)(r-2)$ and $\Delta_{(V_1,V_2)} \geq r-1$. A general representation M of $\operatorname{rep}(K_r; V_1, V_2)$ is in $\operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ and for each $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$ we have

$$M|_{\mathfrak{v}} = M|_{\mathfrak{v},\mathrm{pproj}} \in \operatorname{add}(\{P_i(2) \mid i \geq n\}).$$

Proof. (1) By assumption we have $\Delta_{(V_1,V_2)} = \dim_\Bbbk V_2 - \dim_\Bbbk V_1 \geq (n+1)(r-2) + \frac{\dim_\Bbbk V_1}{n} \geq (n+1)(r-2) \geq 2(r-2) \geq r-1$. We combine Proposition 5.5.1, Theorem 5.4.1 and Proposition 2.7.4 and obtain a non-empty open subset $\mathcal{O} \subseteq \operatorname{rep}(K_r; V_1, V_2) \cap \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ such that for all $M \in \mathcal{O}$ we have $M|_{\mathfrak{v}} = M|_{\mathfrak{v},\mathrm{pproj}} \in \operatorname{add}(\{P_i(2) \mid 0 \leq i \leq n\})$.

(2) This follows from Proposition 5.5.2(3), Theorem 5.4.1 and Proposition 2.7.4.

□

[4] By "general representation" we mean that there exists a non-empty open subset O of $\operatorname{rep}(K_r; V_1, V_2)$ such that every representation in O has the described property.

6. Elementary representations

In the previous chapters we have used the adjoint pairs $(\sigma_{K_r}^{-1} \circ \inf, \sigma_{K_d} \circ \mathrm{res})$ and $(\sigma_{K_d}^{-1} \circ \mathrm{res}, \sigma_{K_r} \circ \inf)$ to construct families of test representations $(X^{\pm}(\mathfrak{v}))_{\mathfrak{v} \in \mathrm{Gr}_d(A_r)}$ from a homogeneous representation $X \in \mathrm{rep}(K_d)$. We were able to prove strong results for the special case $X = P_1(d)$ (cf. Theorem 3.3.2 and Theorem 4.2.2) and have shown that $P_1^-(\mathfrak{v})$ and $P_1^+(\mathfrak{v})$ are elementary representations (see Proposition 3.3.5).

Before we transfer the results we have proven so far into the category of Steiner bundles on Grassmann varieties, this chapter is devoted exclusively to the study of elementary representations for the Kronecker quiver. We prove constraints on the corresponding dimension vectors and compare our finding with the results proven for $r = 3$ in [Rin16].

This chapter is based on [Bis24].

6.1. General results

Let Q be a wild quiver. Since regular representations are closed under extensions, there is a uniquely determined smallest class of regular representation $\mathcal{E} \subseteq \mathrm{reg}(Q)$ closed under isomorphisms, whose extension-closure is $\mathrm{reg}(Q)$. In particular, every representation M possesses a finite filtration (in general not uniquely determined)

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{l-1} \subset M_l = M$$

with $M_i/M_{i-1} \in \mathcal{E}$ for all $i \in \{1, \ldots, l\}$. The representations in $\mathcal{E}$ are called *elementary* and are the simple objects in the category of regular representations. The definition of elementary representations is due to Crawley-Boevey and is a natural generalization of quasi-simple representations living in regular tubes of tame hereditary algebras.

Among other things, elementary representations are of interest because they are closely related (see [KL96, (3.1)]) to the graph of domination (see [Ker94, (15.2), (15.3)] for a precise definition), whose sinks are given by Kronecker algebras corresponding to the quivers K_r for $r \geq 3$. We begin by recalling the definition of elementary representations (see Section 3.3).

Definition. A non-zero regular representation $E \in \mathrm{rep}(Q)$ is called *elementary*, provided there is no short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

with $A, B \in \mathrm{rep}(Q)$ regular and non-zero.

By definition, the elementary representations are the simple objects in the full subcategory of regular representations. Elementary representations for wild quivers were first systematically studied in [KL96] and [Luk92]. There the authors showed that, parallel to the tame situation,

there exist only finitely many Coxeter-orbits of dimension vectors of elementary representations. A very useful characterization of elementary representations, established more recently in [Rin16, Appendix A], is the following. For completeness, we recall the result and the proof.

Proposition 6.1.1. *Let $E \in \operatorname{rep}(Q)$ be a non-zero regular representation. The following statements are equivalent.*

(1) E is elementary.

(2) Given any subrepresentation U of E, U is preprojective or the quotient E/U is preinjective.

Proof. (1) $\Rightarrow$ (2). Let U be a representation that is not preprojective. Since $U \subseteq E$ and there are no non-zero homomorphisms from preinjective representations to regular representations (see [ASS06, (VIII.2.13)]), we may write $U = P \oplus R$ with P preprojective and $R \neq 0$ regular. Since E is elementary, E/R is preinjective by [KL96, (1.3)] and the presence of the projection $E/R \longrightarrow E/U$ in conjunction with [ASS06, (VIII.2.13)] implies that E/U is preinjective.
(2) $\Rightarrow$ (1). This is clear. □

We return to the case $Q = K_r$ for $r \geq 3$. It is well known (see for example [Rin76, (3.4)]) that the region

$$\mathcal{C}_r := \{(x, y) \in \mathbb{N}^2 \mid \frac{1}{r-1}x \leq y < (r-1)x\}$$

is a fundamental domain for the action of the Coxeter transformation

$$\Phi_r = \begin{pmatrix} r^2 - 1 & -r \\ r & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})^1$$

on the set

$$\mathcal{R}_r := \{(x, y) \in \mathbb{N}^2 \mid x^2 + y^2 - rxy < 1\}$$

of dimension vectors of regular representations in $\operatorname{rep}(K_r)$. Ultimately, we are interested in the set

$$\widetilde{\mathcal{E}}_r := \{(x, y) \in \mathcal{C}_r \mid \exists E \in \operatorname{rep}(K_r) \text{ elementary}, \underline{\dim}\, E = (x, y)\}.$$

By [Rin16, Section 2] (see also [Kac80, §2.6]) the set

$$\mathcal{F}_r := \{(x, y) \in \mathbb{N}^2 \mid \frac{2}{r}x \leq y \leq x\} \subseteq \mathcal{C}_r$$

is a fundamental domain for the action of the group $G_r \subseteq \mathrm{GL}_2(\mathbb{Z})$ generated by σ_r and the twist function $\delta \colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2; (x, y) \mapsto (y, x)$ on $\mathcal{R}$. Figure 6.1 shows $\mathcal{C}_3$ and $\mathcal{F}_3$. We define

$$\mathcal{E}_r := \widetilde{\mathcal{E}}_r \cap \mathcal{F}_r = \{(x, y) \in \mathcal{F}_r \mid \exists E \in \operatorname{rep}(K_r) \text{ elementary}, \underline{\dim}\, E = (x, y)\}.$$

Given $M \in \operatorname{rep}(K_r)$ regular indecomposable, we have

$$\underline{\dim}\, \sigma_{K_r}(M) = \sigma_r(\underline{\dim}\, M) \text{ and } \underline{\dim}\, D_{K_r}(M) = \delta(\underline{\dim}\, M).$$

Since M is elementary if and only if its dual (respectively its σ_{K_r}-shift) is elementary, and $\sigma_r \circ \sigma_r = \Phi_r$, the determination of $\mathcal{E}_r$ only necessitates the knowledge of $\widetilde{\mathcal{E}}_r$. The set $\mathcal{E}_3$ has been determined in [Rin16] and is given by

$$\mathcal{E}_3 = \{(1, 1), (2, 2)\}.$$

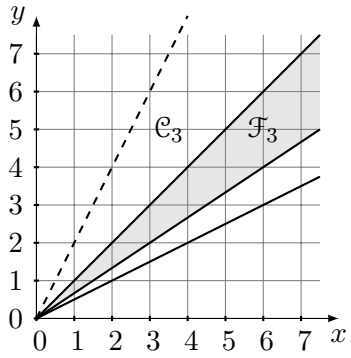


Figure 6.1.: Illustration of $\mathcal{C}_3$ and $\mathcal{F}_3$.

In the following we will determine the set $\mathcal{E}_r$ for any $r \geq 3$. We begin our considerations with the following simple observations that will be needed later. Recall the definitions of ∇ and Δ from Section 3.3, which we need in the following results.

Lemma 6.1.2. *Let $M \in \operatorname{rep}(K_r)$ and $\dim_{\Bbbk} M_2 \leq 2(r-1)$.*

(1) If M is preinjective, then $M \in \operatorname{add}(I_0(r) \oplus I_1(r) \oplus I_2(r))$.

(2) If $\underline{\dim}\, M \in \mathcal{F}_r$ and $U \subseteq M$ is such that M/U is preinjective, then $M/U \in \operatorname{add}(I_0(r) \oplus I_1(r))$ and $M/U \cong -\nabla_{M/U}(r) I_0(r) \oplus \dim_{\Bbbk}(M/U)_1 I_1(r)$.

(3) If $\underline{\dim}\, M \in \mathcal{F}_r$ and $U \subseteq M$ is such that

$$r(\dim_{\Bbbk}(M/U)_2) > \dim_{\Bbbk}(M/U)_1,$$

then M/U is not preinjective.

(4) A representation $N \in \operatorname{rep}(K_r)$ with $\dim_{\Bbbk} N_1 < L_r \dim_{\Bbbk} N_2$ is not preinjective.

(5) A representation $N \in \operatorname{rep}(K_r)$ with $2 \leq \dim_{\Bbbk} N_2$ and $\dim_{\Bbbk} N_1 < 2r$ is not preinjective.

Proof. (1) We have $\dim_{\Bbbk}(I_l(r))_2 \geq \dim_{\Bbbk}(I_3(r))_2 = r^2 - 1$ for all $l \geq 3$. Moreover, we have $r^2 - 1 > 2(r-1) \geq \dim_{\Bbbk} M_2$ since $r \geq 3$.

(2) We apply (1) to M/U and get $M/U \in \operatorname{add}(I_0(r) \oplus I_1(r) \oplus I_2(r))$. Moreover, we have

$$\dim_{\Bbbk}(M/U)_1 \leq \dim_{\Bbbk} M_1 \leq \frac{r}{2} \dim_{\Bbbk} M_2 \leq \frac{r}{2} 2(r-1) = r^2 - r < r^2 - 1 = \dim_{\Bbbk}(I_2(r))_1.$$

Hence $M/U \in \operatorname{add}(I_0(r) \oplus I_1(r))$ and therefore $M/U \cong -\nabla_{M/U}(r) I_0(r) \oplus \dim_{\Bbbk}(M/U)_2 I_1(r)$.

(3) This is a direct consequence of (2) since $\underline{\dim}\, I_1(r) = (r, 1)$ and $\underline{\dim}\, I_0(r) = (1, 0)$.

(4) This follows immediately by applying Corollary 1.5.2 to the direct summands of N.

[1] We identify $\Phi_{K_r} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ and the *Coxeter-matrix* Φ_r with its natural action on $\mathbb{Z}^2$ by left multiplication.

(5) We have $\dim_{\Bbbk}(I_l(r))_1 \geq \dim_{\Bbbk}(I_2(r))_1 = r^2 - 1 > 2r$ for all $l \geq 2$. Assume that N is preinjective. Then $N \in \operatorname{add}(I_0(r) \oplus I_1(r))$. Since $\underline{\dim}\, I_0(r) = (1,0)$ and $\underline{\dim}\, I_1(r) = (r,1)$, we conclude with $\dim_{\Bbbk} N_2 \geq 2$ that $\dim_{\Bbbk} N_1 \geq 2r$, a contradiction.

□

Lemma 6.1.3. *Let $M \in \operatorname{rep}(K_r)$ be a representation with $(1, r-d) \leq \underline{\dim}(M_1, M_2)$. We assume that one of the following conditions holds:*

(i) $\nabla_M(d) < d(r-d)$, *or*

(ii) $M \notin \operatorname{rep}_{\mathrm{esp}}(K_r, d)$.

Then there exists a non-preprojective subpresentation U_{r-d} of M with dimension vector $(1, r-d)$.

Proof. In case (i) we conclude with $\nabla_M(d) < d(r-d)$ and Theorem 5.3.2 that $M \notin \operatorname{rep}_{\mathrm{esp}}(K_r, d)$, since $M_1 \neq \{0\}$. In case (ii) we apply Proposition 5.3.1 and find a subrepresentation $X \subseteq M$ with dimension vector $(1, a)$ for some $a \in \{0, \ldots, r-d\}$. Since $\dim_{\Bbbk} M_2 \geq r-d$, we can extend X to a subrepresentation U_{r-d} with dimension vector $(1, r-d)$ and Proposition 5.3.1(3) shows that U_{r-d} is not preprojective. □

6.2. Restricting y

Proposition 6.2.1. *Let $(x,y) \in \mathcal{F}_r$ with $y \geq r$ and E be a representation with dimension vector $\underline{\dim}\, E = (x,y)$. Then E is not elementary.*

Proof. Since elementary representations are bricks (see [KL96, (1.4)]), we can assume that E is a regular indecomposable representation. From now on we proceed in steps. Since $\nabla_E(1) = \dim_{\Bbbk} E_2 - \dim_{\Bbbk} E_1 \leq 0 < 1(r-1)$ and $\dim_{\Bbbk} E_2 = y \geq r-1$, we can apply Lemma 6.1.3 and find a non-preprojective subrepresentation $U_{r-1} \subseteq E$ with dimension vector $(1, r-1)$.

At first we assume that $y \geq 2(r-1)$. The quotient E/U_{r-1} has dimension vector $(x-1, y-r+1)$. We claim that this dimension vector can not belong to a preinjective representation. Indeed, since $r - 1 < L_r$, we have $r - 2L_r < 0$ and conclude

$$\begin{aligned}(x-1) - (y-r+1)L_r &\overset{x \leq \frac{r}{2}y}{\leq} (\frac{r}{2} - L_r)y + L_r(r-1) - 1 = (\frac{r-2L_r}{2})y + L_r(r-1) - 1 \\ &\overset{y \geq 2(r-1)}{\leq} (\frac{r-2L_r}{2})2(r-1) + L_r(r-1) - 1 \\ &= (r-2L_r)(r-1) + L_r(r-1) - 1 = (r-1)(r-L_r) - 1.\end{aligned}$$

Recall that L_r is a root of the polynomial $f = X^2 - rX + 1 \in \mathbb{R}[X]$. Hence

$$(x-1) - (y-r+1)L_r \leq (r-L_r)(r-1) - 1 = (r-L_r)(r-1) + L_r(L_r - r) = (r-L_r)(r-1-L_r) < 0,$$

since $r - 1 < L_r < r$. Now Lemma 6.1.2(4) implies that E/U_{r-1} is not preinjective. We conclude with Proposition 6.1.1 that E is not elementary.

Therefore we can assume from now on that $r \leq y < 2(r-1)$. Given $d \in \mathbb{N}$ we define

$$\Omega(d) \coloneqq r(r-d).$$

We begin with the case $\Omega(1) \leq ry - x$. We have $\underline{\dim}\, E/U_{r-1} = (x-1, y-(r-1))$ and $y-(r-1) \neq 0$. Therefore

$$r \dim_{\Bbbk}(E/U_{r-1})_2 = r(y-(r-1)) = ry - \Omega(1) \geq x > x-1 = \dim_{\Bbbk}(E/U_{r-1})_1.$$

Since $\dim_{\Bbbk} E_2 = y < 2(r-1)$, we can apply Lemma 6.1.2(3) and conclude that E/U_{r-1} is not preinjective. Now Proposition 6.1.1 implies that E is not elementary.

Now we assume that $ry - x < \Omega(1)$. Since $ry - x \geq r\frac{2}{r}x - x = x \geq y \geq r = \Omega(r-1)$ and

$$\Omega(1) > \Omega(2) > \ldots > \Omega(r-2) > \Omega(r-1),$$

we find a natural number $2 \leq d \leq r-1$ such that

$$\Omega(d) \leq ry - x < \Omega(d-1).$$

We consider two cases:

- $d \in \{2, \ldots, r-2\}$, then $d-(r-1) \leq -1$ and

$$\begin{aligned}\nabla_{(x,y)}(d) - d(r-d) &= ry - x - y(r-d) - d(r-d) < \Omega(d-1) - (y+d)(r-d) \\ &\leq \Omega(d-1) - (r+d)(r-d) = r(r-d+1) - r^2 + d^2 \\ &= -rd + d^2 + r = r + d(d-r) \\ &= (d-1)(d-(r-1)) + 1 \leq (d-1)\cdot(-1) + 1 \leq 2 - d \leq 0,\end{aligned}$$

  By Lemma 6.1.3 there is a subrepresentation $U_{r-d} \subseteq E$ with dimension vector $(1, r-d)$ that is not preprojective. We have $\underline{\dim}\, E/U_{r-d} = (x-1, y-(r-d))$ and the choice of d gives us

$$r \dim_{\Bbbk}(E/U_{r-d})_2 = r(y-(r-d)) = ry - \Omega(d) \geq x > x-1 = \dim_{\Bbbk}(E/U_{r-d})_1.$$

  Therefore E/U_{r-d} is not preinjective by Lemma 6.1.2(3) and Proposition 6.1.1 implies that E is not elementary.

- $d = r-1$, i.e. $\Omega(r-1) \leq ry - x < \Omega(r-2)$. We get

$$x > r(y-2) \geq r(\frac{2}{r}x - 2) = 2x - 2r \Leftrightarrow x < 2r$$

  and conclude with $r(y-2) < x < 2r$ that $y < 4$. Since $3 \leq r \leq y < 4$, we conclude $r = 3$. Hence the statement follows since $\mathcal{E}_r = \{(1,1), (2,2)\}$ by [Rin16].

□

6.3. Existence of elementary representations

For $x, y \in \mathbb{N}_0$, we define

$$\mathcal{E}(x,y) := \{g \in \operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y) \mid (\Bbbk^x, \Bbbk^y, g) \text{ elementary}\},$$

and

$$\mathcal{B}(x,y) := \mathcal{B}(\Bbbk^x, \Bbbk^y) = \{g \in \operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y) \mid (\Bbbk^x, \Bbbk^y, g) \text{ brick}\}.$$

Since elementary representations are bricks (see [KL96, (1.4)]) we have $\mathcal{E}(x,y) \subseteq \mathcal{B}(x,y)$.

From now on we assume that $(x,y) \in \mathcal{F}_r$. We recall from Proposition 5.1.1 that $\mathcal{B}(x,y)$ is a dense open subset of $\operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y)$ since $q_r(x,y) \leq 1$.

In the following we determine under which assumptions on (x,y) the set $\mathcal{E}(x,y)$ is non-empty. Since Proposition 6.2.1 implies that $\mathcal{E}(x,y) \neq \varnothing$ can only happen for $y < r$, we assume from now on that $y < r$. Before we tackle the general case, we consider an example that illustrates the strategy of proof. Recall that

$$\operatorname{rep}_{\text{inj}}(K_r, d) := D_{K_r}(\operatorname{rep}_{\text{proj}}(K_r, d))$$

for all $1 \leq d < r$.

Example 6.3.1. *We have $(6,3) \in \mathcal{F}_4$ with $3 < 4 = r$ and claim that $(6,3) \in \mathcal{E}_4$. We have $\nabla_{(6,3)}(3) = 3 \geq 3 = 3(4-3)$. Moreover, we have $-\nabla_{(6,3)}(1) = 3 \geq 1(4-1)$. Hence Theorem 5.3.2 and Corollary 5.4.5 imply that $\operatorname{rep}_{\text{esp}}(K_r, 3) \cap \operatorname{rep}_{\text{inj}}(K_r, 1) \cap \mathcal{B}(6,3)$ is non-empty. We fix a representation E in this set. Let $0 \neq U \subseteq E$ be a non-preprojective representation. We now show that E/U is preinjective. Since U is not projective, we find $0 \neq u \in U_1$. We consider the subrepresentation $\langle u \rangle \subseteq U \subseteq E$ generated by u. Then $\underline{\dim}\langle u \rangle = (1,z)$ for some $z \in \{0,1,2,3\}$. Since $\operatorname{rep}_{\text{esp}}(K_r, 3)$ is closed under subrepresentations, we have $\langle u \rangle \in \operatorname{rep}_{\text{esp}}(K_r, 3)$ and conclude with Theorem 5.3.2 that $3z - 1 = \nabla_{(1,z)}(3) \geq 3(4-3) = 3$. Hence $z \geq 2$. Therefore we have $\underline{\dim}\, E/\langle u \rangle = (5,b)$ with $b \in \{0,1\}$. Since $E \in \operatorname{rep}_{\text{inj}}(K_r, 1)$ and $\operatorname{rep}_{\text{inj}}(K_r, 1)$ is closed under images ($\operatorname{rep}_{\text{proj}}(K_r, 1)$ is closed under subrepresentations), we have $E/\langle u \rangle \in \operatorname{rep}_{\text{inj}}(K_r, 1)$. Now we apply Corollary 5.4.6 to conclude that $E/\langle u \rangle$ is (pre)injective. Finally, the presence of the canonical epimorphism $E/\langle u \rangle \longrightarrow E/U$ implies that E/U is (pre)injective (see [ASS06, (VIII.2.13)]).*

Now we consider the general case and start with the following simple observation.

Lemma 6.3.2. *Let $y = 1$, then $\mathcal{E}(x,y) = \mathcal{B}(x,y) \neq \varnothing$.*

Proof. Let $M \in \mathcal{B}(x,y)$, then M is indecomposable and regular. Let $U \subseteq M$ be a proper subrepresentation. Then $0 \neq \dim_{\Bbbk} U_2$ and therefore $M/U \in \operatorname{add}(I_0(r))$ is injective. In particular, M is elementary. This shows $\varnothing \neq \mathcal{B}(x,y) = \mathcal{E}(x,y)$. □

We assume from now on that $1 < y < r$ and set $b := \lceil \frac{x}{r} \rceil \in \mathbb{N}$. Note that b is the unique natural number that satisfies

$$(b-1)r < x \leq br.$$

Remark 6.3.3.

(1) We have $1 \leq b < y < r$: Assume that $\lceil \frac{x}{r} \rceil = b \geq y$. Then $\frac{x}{r} > y - 1$ and therefore

$$\frac{r}{2}y \geq x = \frac{x}{r}r > ry - r.$$

Hence $2 > y$, a contradiction since we assume $2 \leq y$.

(2) We extend the definition of $\operatorname{rep}_{\text{inj}}(K_r, d)$ to $d \in \{0, \ldots, r-1\}$ by setting $\operatorname{rep}_{\text{inj}}(K_r, 0) \coloneqq \operatorname{rep}(K_r)$.

Proposition 6.3.4. *Let $(x, y) \in \mathcal{F}_r$ with $1 < y < r$ and $b \coloneqq \lceil \frac{x}{r} \rceil$. The following statements hold.*

(1) If $\mathcal{E}(x, y)$ is non-empty, then

$$(b-1)(y + r - (b-1)) \leq x \leq b(r - y + b)$$

and $\mathcal{E}(x, y) \subseteq \mathcal{B}(x, y) \cap \operatorname{rep}_{\text{esp}}(K_r, r - y + b) \cap \operatorname{rep}_{\text{inj}}(K_r, b - 1)$.

(2) If

$$(b-1)(y + r - (b-1)) \leq x \leq b(r - y + b),$$

then $\mathcal{E}(x, y)$ is a non-empty open set given by

$$\mathcal{E}(x, y) = \mathcal{B}(x, y) \cap \operatorname{rep}_{\text{esp}}(K_r, r - y + b) \cap \operatorname{rep}_{\text{inj}}(K_r, b - 1).$$

Proof. (1) Let E be an elementary representation with dimension vector $\underline{\dim}\, E = (x, y)$. We denote by $F \coloneqq D_{K_r}(E)$ the dual representation with dimension vector (y, x). We proceed in steps.

(i) We have $(b-1)(y + r - (b-1)) \leq x$ and $E \in \operatorname{rep}_{\text{inj}}(K_r, b-1)$: We assume that $x < (b-1)(y + r - (b-1))$ or $E \notin \operatorname{rep}_{\text{inj}}(K_r, b-1)$. In both cases we conclude $b \neq 1$ and therefore $b - 1 \in \{1, \ldots, r-1\}$ by Remark 6.3.3. If $x < (b-1)(y + r - (b-1))$, we have

$$\Delta_F(b-1) = x - (b-1)y < (b-1)(r - (b-1)).$$

Moreover, $x \leq \frac{r}{2}y < ry$ gives us

$$\begin{aligned} \Delta_F(b-1) = x - (b-1)y &< ry - (b-1)y = y(r - (b-1)) \\ &= \dim_{\Bbbk} F_1(r - (b-1)). \end{aligned}$$

Hence Theorem 5.4.1 implies $F \notin \operatorname{rep}_{\text{proj}}(K_r, b-1)$. If $E \notin \operatorname{rep}_{\text{inj}}(K_r, b-1)$, we immediately get $F \notin \operatorname{rep}_{\text{proj}}(K_r, b-1)$ from the definition.

The book-keeping: In both cases we arrive at $b - 1 \neq 0$ and $F \notin \operatorname{rep}_{\text{proj}}(K_r, b-1)$.

In view of Proposition 5.3.1, we find $a \in \{1, \ldots, b-1\}$ and a subrepresentation $X \notin \operatorname{rep}_{\text{proj}}(K_r, b-1)$ of F with $\underline{\dim}\, X = (a, a')$ and $a' \leq ar - 1 \leq (b-1)r - 1$. The inequality $(b-1)r - 1 < x$ ensures that we can extend X with a semisimple projective direct summand to a subrepresentation Y of F with dimension vector $(a, (b-1)r - 1)$. Since

X is not in $\operatorname{rep}_{\text{proj}}(K_r, b-1)$, the same is true for Y. In particular, Y is not preprojective by Theorem 5.4.1.

Since F is elementary, we can apply Proposition 6.1.1 to conclude that $(y-a, x-(b-1)r+1) = \underline{\dim}\, F/X$ belongs to a preinjective representation. But this is impossible since $x-(b-1)r+1 \geq 1$, $y-a < r$ and $\dim_{\Bbbk}(I_l(r))_1 \geq r$ for all $l \geq 1$ and $\underline{\dim}\, I_0(r) = (1,0)$. Hence $(b-1)(y+r-(b-1)) \leq x$ and $E \in \operatorname{rep}_{\text{inj}}(K_r, b-1)$.

(ii) We have $x \leq b(r-y+b)$ and $E \in \operatorname{rep}_{\text{esp}}(K_r, r-y+b)$: We assume that $x > b(r-y+b)$. We set $d := r-(y-b)$ and note that $d \in \{1, \ldots, r-1\}$ by Remark 6.3.3. We get

$$\begin{aligned}\nabla_E(d) - d(r-d) &= d(y-r+d) - x \\ &= b(r-y+b) - x < 0.\end{aligned}$$

Hence $E \notin \operatorname{rep}_{\text{esp}}(K_r, d)$ by Theorem 5.3.2. Since $r-d = y-b \leq y$, we conclude with Lemma 6.1.3 that there exists a non-preprojective subrepresentation $U_{r-d} \subseteq E$ with dimension vector $(1, r-d)$. Once again we apply Proposition 6.1.1 and conclude that E/U_{r-d} with dimension vector $(x-1, y-(r-d)) = (x-1, b)$ is preinjective. We apply Lemma 6.1.2(3) and conclude $br \leq x-1$. But this is a contradiction to the definition of b since $x \leq br$.
We note that this also shows $E \in \operatorname{rep}_{\text{esp}}(K_r, d) = \operatorname{rep}_{\text{esp}}(K_r, r-y+b)$.

(2) We set $d := r-(y-b) \in \{1, \ldots, r-1\}$ and have

$$\nabla_E(d) - d(r-d) = d(y-r+d) - x = b(r-y+b) - x \geq 0,$$

and

$$-\nabla_E(b-1) = x-(b-1)y \geq (b-1)(r-(b-1)).$$

We can apply Theorem 5.3.2 and Corollary 5.4.5 (for $b \neq 1$) to conclude that $\mathcal{B}(x,y) \cap \operatorname{rep}_{\text{esp}}(K_r, d) \cap \operatorname{rep}_{\text{inj}}(K_r, b-1)$ is non-empty (for $b=1$ we have $\operatorname{rep}_{\text{inj}}(K_r, b-1) = \operatorname{rep}(K_r)$). We fix a representation E in this space and show now that E is elementary.

Let $U \subseteq E$ be a non-preprojective representation, then we find $0 \neq u \in U_1$. Recall from Remark 4.1.1 that $\operatorname{rep}_{\text{esp}}(K_r, d)$ is closed under subrepresentations. Therefore the subrepresentation $\langle u \rangle$ generated by u is in $\operatorname{rep}_{\text{esp}}(K_r, d)$ and $\underline{\dim}\langle u \rangle = (1, z)$ for some $z \in \{1, \ldots, y\}$. We conclude with Theorem 5.3.2 that

$$dz - 1 = \nabla_{\langle u \rangle}(d) \geq d(r-d) \Leftrightarrow d(z-(r-d)) \geq 1.$$

In particular, $z \geq r-d+1 = y-(b-1)$. In other words, $E/\langle u \rangle$ satisfies $\underline{\dim}\, E/\langle u \rangle = (x-1, a)$ with $0 \leq a \leq b-1$. If $b=1$, we conclude that $a=0$ and therefore $\underline{\dim}\, E/\langle u \rangle$ is injective and the presence of the canonical epimorphism $E/\langle u \rangle \longrightarrow E/U$ implies that E/U is injective. If $b \neq 1$ we have $E \in \operatorname{rep}_{\text{inj}}(K_r, b-1)$ with $b-1 \neq 0$. Since $\operatorname{rep}_{\text{inj}}(K_r, b-1)$ is closed under images ($\operatorname{rep}_{\text{proj}}(K_r, b-1)$ is closed under subrepresentation), we have $\underline{\dim}\, E/\langle u \rangle \in \operatorname{rep}_{\text{inj}}(K_r, b-1)$ and can apply Corollary 5.4.6 to conclude that $E/\langle u \rangle$ is injective and presence of the canonical epimorphism $E/\langle u \rangle \to E/U$ implies that E/U is preinjective (see [ASS06, (VIII.2.13)]). Hence E is elementary by Proposition 6.1.1.

The book-keeping: We have shown that

$$\varnothing \neq \mathcal{B}(x,y) \cap \operatorname{rep}_{\text{esp}}(K_r, r-y+b) \cap \operatorname{rep}_{\text{inj}}(K_r, b-1) \subseteq \mathcal{E}(x,y).$$

Now we apply (1) to finish the proof.

□

Remark 6.3.5. We extend the definition of $\mathrm{rep}_{\mathrm{esp}}(K_r, d)$ to $\{1, \dots, r\}$ by setting $\mathrm{rep}_{\mathrm{esp}}(K_r, r) \coloneqq \mathrm{rep}(K_r)$.

Theorem 6.3.6. *Let $(x, y) \in \mathcal{F}_r$.*

(1) $\mathcal{E}(x, y) \neq \varnothing$ implies $y < r$.

(2) For $y < r$ the following statements are equivalent.

(i) $\mathcal{E}(x, y) \neq \varnothing$.

(ii) $(\lceil \frac{x}{r} \rceil - 1)(y + r - (\lceil \frac{x}{r} \rceil - 1)) \leq x \leq \lceil \frac{x}{r} \rceil (r - y + \lceil \frac{x}{r} \rceil)$.

If one the equivalent statements holds, we have

$$\mathcal{E}(x, y) = \mathcal{B}(x, y) \cap \mathrm{rep}_{\mathrm{esp}}(K_r, r - y + \lceil \frac{x}{r} \rceil) \cap \mathrm{rep}_{\mathrm{inj}}(K_r, \lceil \frac{x}{r} \rceil - 1).$$

Proof. (1) This is the statement of Proposition 6.2.1.

(2) At first we assume that $y = 1$. Then $x \leq \frac{r}{2} y < r$ and $\lceil \frac{x}{r} \rceil = 1$. So in this case the inequalities in (ii) are always satisfied and by Lemma 6.3.2 we have $\mathcal{E}(x, y) = \mathcal{B}(x, y) \neq \varnothing$ as well as

$$\begin{aligned} \mathcal{E}(x, y) = \mathcal{B}(x, y) &= \mathcal{B}(x, y) \cap \mathrm{rep}_{\mathrm{esp}}(K_r, r) \cap \mathrm{rep}_{\mathrm{inj}}(K_r, 0) \\ &= \mathcal{B}(x, y) \cap \mathrm{rep}_{\mathrm{esp}}(K_r, r - y + \lceil \frac{x}{r} \rceil) \cap \mathrm{rep}_{\mathrm{inj}}(K_r, \lceil \frac{x}{r} \rceil - 1). \end{aligned}$$

Now we assume that $1 < y < r$. Then the equivalence of (i) and (ii) is precisely the statement of Proposition 6.3.4.

□

Corollary 6.3.7. *Let $(x, y) \in \mathcal{F}_r$ be such that $y \leq x < r$. The following statements are equivalent:*

(1) $\mathcal{E}(x, y) \neq \varnothing$.

(2) $x + y \leq r + 1$.

In this case we have

$$\mathcal{E}(x, y) = \mathcal{B}(x, y) \cap \mathrm{rep}_{\mathrm{esp}}(K_r, r - y + 1).$$

Proof. We have $\lceil \frac{x}{r} \rceil = 1$. Hence $(x, y) \in \mathcal{F}_r$ satisfies the inequality of Theorem 6.3.6 if and only if $x \leq r - y + 1$. Moreover, in this case we have

$$\mathcal{E}(x, y) = \mathcal{B}(x, y) \cap \mathrm{rep}_{\mathrm{esp}}(K_r, r - y + 1) \cap \mathrm{rep}_{\mathrm{inj}}(K_r, 0) = \mathcal{B}(x, y) \cap \mathrm{rep}_{\mathrm{esp}}(K_r, r - y + 1).$$

□

Corollary 6.3.8. *Let $(x, y) \in \mathcal{F}_r$. The following statements are equivalent.*

(1) $\mathcal{E}(x, y) \neq \varnothing$.

(2) $\lfloor \frac{x}{r} \rfloor (y + r - \lfloor \frac{x}{r} \rfloor) \leq x \leq \lceil \frac{x}{r} \rceil (r - y + \lceil \frac{x}{r} \rceil)$ *and* $y < r$.

(3) $y \leq \min\{\lfloor \frac{x}{r} \rfloor + \frac{x}{\lfloor \frac{x}{r} \rfloor} - r, \lceil \frac{x}{r} \rceil - \frac{x}{\lceil \frac{x}{r} \rceil} + r, r-1\}$, *where we interpret* $\lfloor \frac{x}{r} \rfloor + \frac{x}{\lfloor \frac{x}{r} \rfloor} - r$ *as* ∞ *for* $1 \leq x < r$.

If one of the equivalent statements holds, we have

$$\mathcal{E}(x,y) = \mathcal{B}(x,y) \cap \mathrm{rep}_{\mathrm{esp}}(K_r, r - y + \lceil \frac{x}{r} \rceil) \cap \mathrm{rep}_{\mathrm{inj}}(K_r, \lfloor \frac{x}{r} \rfloor).$$

Proof. Assume that $\frac{x}{r} \in \mathbb{N}$. In this case we have $\frac{x}{r} = \lceil \frac{x}{r} \rceil$ and

$$x < ry \Leftrightarrow x > x - \frac{x}{r}y + (\frac{x}{r})^2 \Leftrightarrow x > \lceil \frac{x}{r} \rceil (r - y + \lceil \frac{x}{r} \rceil).$$

Now Theorem 6.3.6 implies $\mathcal{E}(x,y) = \varnothing$. Hence we can assume $\frac{x}{r} \notin \mathbb{N}$. Then $\lceil \frac{x}{r} \rceil - 1 = \lfloor \frac{x}{r} \rfloor$ and Theorem 6.3.6 implies the equivalence of (1) and (2).

The equivalence of (2) and (3) follows from direct computation and Corollary 6.3.7. □

Proposition 6.3.9. *We have*

$$\mathcal{E}_r = \{(x,y) \in \mathbb{N}_{\leq \frac{r(r-1)}{2}} \times \mathbb{N}_{\leq r-1} \mid \frac{2x}{r} \leq y \leq \min\{\lfloor \frac{x}{r} \rfloor + \frac{x}{\lfloor \frac{x}{r} \rfloor} - r, \lceil \frac{x}{r} \rceil - \frac{x}{\lceil \frac{x}{r} \rceil} + r, x\}\}.$$

Proof. Recall that $(x,y) \in \mathcal{F}_r$ with $\mathcal{E}(x,y) \neq \varnothing$ implies $\frac{2x}{r} \leq y \leq x$ and $y \leq r-1$. In particular, $x \leq \frac{r(r-1)}{2}$. □

Example 6.3.10. *In the following we discuss the cases* $r = 3, 4$ *in detail to illustrate how to apply our formulas.*

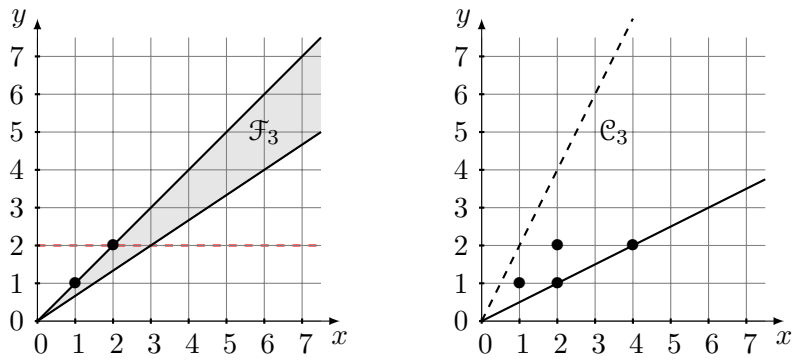
(1) *The case* $r = 3$. *We have* $1 \leq x \leq \frac{r(r-1)}{2} = 3$ *and* $1 \leq y \leq r - 1 = 2$. *We consider the inequalties*

$$\underline{x = 1} : \frac{2}{3} \leq y \leq \min\{\infty, 3, x = 1, r - 1 = 2\} = 1, \quad \underline{x = 2} : \frac{4}{3} \leq y \leq \min\{\infty, 2, 2, 2\} = 2,$$
$$\underline{x = 3} : 2 \leq y \leq \min\{1, 1, 3, 2\} = 1.$$

This shows $\mathcal{E}_3 = \{(1,1), (2,2)\}$. *Moreover, we have*

$$\mathcal{E}(1,1) = \mathcal{B}(1,1) \cap \mathrm{rep}_{\mathrm{esp}}(K_3, 3) \cap \mathrm{rep}_{\mathrm{inj}}(K_3, 0) = \mathcal{B}(1,1) \text{ and}$$
$$\mathcal{E}(2,2) = \mathcal{B}(2,2) \cap \mathrm{rep}_{\mathrm{esp}}(K_3, 2) \cap \mathrm{rep}_{\mathrm{inj}}(K_3, 0) = \mathcal{B}(1,1) \cap \mathrm{rep}_{\mathrm{esp}}(K_3, 2).$$

The following figure on the left-hand side shows the elementary dimensions vector in $\mathcal{E}_3$ *and the figure on the right hand side shows* $\widetilde{\mathcal{E}}_3$. *The dashed red line is the restriction* $y \leq r - 1 = 2$.



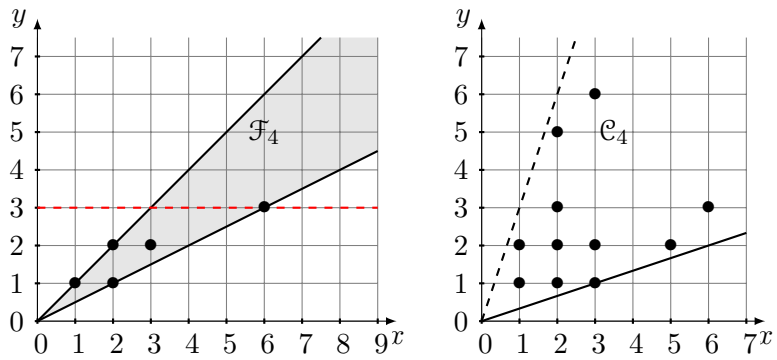
(2) The case $r = 4$. We have $1 \leq x \leq \frac{r(r-1)}{2} = 6$ and $y \leq r - 1 = 3$. We consider the inequalties

$$\underline{x = 1} : \frac{1}{2} \leq y \leq \min\{\infty, 4, 1, 3\} = 1, \qquad \underline{x = 2} : 1 \leq y \leq \min\{\infty, 3, 2, 3\} = 2$$
$$\underline{x = 3} : \frac{3}{2} \leq y \leq \min\{\infty, 2, 3, 3\} = 2, \qquad \underline{x = 4} : 2 \leq y \leq \min\{1, 1, 4, 3\} = 1$$
$$\underline{x = 5} : \frac{5}{2} \leq y \leq \min\{2, \frac{7}{2}, 5, 3\} = 2, \qquad \underline{x = 6} : 3 \leq y \leq \min\{3, 3, 6, 3\} = 3.$$

Hence $\mathcal{E}_4 = \{(1,1), (2,1), (2,2), (3,2), (6,3)\}$. Moreover, we have

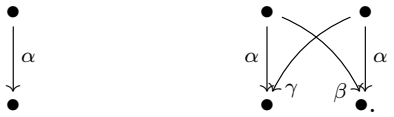
$$\begin{aligned}
\mathcal{E}(1,1) &= \mathcal{B}(1,1) \cap \mathrm{rep}_{\mathrm{esp}}(K_4, 4) \cap \mathrm{rep}_{\mathrm{inj}}(K_4, 0) = \mathcal{B}(1,1),\\
\mathcal{E}(2,1) &= \mathcal{B}(2,1) \cap \mathrm{rep}_{\mathrm{esp}}(K_4, 4) \cap \mathrm{rep}_{\mathrm{inj}}(K_4, 0) = \mathcal{B}(2,1),\\
\mathcal{E}(2,2) &= \mathcal{B}(2,2) \cap \mathrm{rep}_{\mathrm{esp}}(K_4, 3) \cap \mathrm{rep}_{\mathrm{inj}}(K_4, 0) = \mathcal{B}(2,2) \cap \mathrm{rep}_{\mathrm{esp}}(K_4, 3),\\
\mathcal{E}(3,2) &= \mathcal{B}(3,2) \cap \mathrm{rep}_{\mathrm{esp}}(K_4, 3) \cap \mathrm{rep}_{\mathrm{inj}}(K_4, 0) = \mathcal{B}(3,2) \cap \mathrm{rep}_{\mathrm{esp}}(K_4, 3),\\
\mathcal{E}(6,3) &= \mathcal{B}(6,3) \cap \mathrm{rep}_{\mathrm{esp}}(K_4, 3) \cap \mathrm{rep}_{\mathrm{inj}}(K_4, 1).
\end{aligned}$$

The figure on the left-hand side shows the elementary dimensions vector in $\mathcal{E}_4$ and the figure on the right hand side shows $\widetilde{\mathcal{E}}_4$.



6.4. Orbits of elementary representations

It has been shown in [Rin16] that elementary representations E with dimension vector in $\mathcal{E}_3 = \{(1,1), (2,2)\}$ can be described combinatorially in terms of their coefficient quiver. More precisely: There exists a basis α, β, γ of the arrow space A_3 such that the coefficient quiver of E has one of the following two forms:



In the following we rephrase this result in terms of an algebraic group that acts on the variety of representations. Let M_1, M_2 be vector spaces and $\varnothing \neq \mathcal{O} \subseteq \mathrm{rep}(K_r; M_1, M_2)$ be a non-empty set. We consider the action of $G(M_1, M_2) = \mathrm{GL}(M_2) \times \mathrm{GL}(M_1)$ on $\mathrm{rep}(K_r; M_1, M_2)$[2] and assume that

[2]See the beginning the Chapter 5.

$\mathcal{O}$ is invariant under the action of $G(M_1, M_2)$. The orbits in $\mathcal{O}/G(M_1, M_2)$ are, via the map

$$G(M_1, M_2).\psi \mapsto [(M_1, M_2, \psi)],$$

in bijection with the isomorphism classes of representations in $\operatorname{rep}(K_r)$ defined on the pair (M_1, M_2) such that the corresponding structure map is contained in $\mathcal{O}$. We denote by $[\mathcal{O}]$ the set of the corresponding isomorphism classes. The algebraic group

$$G_{(A_r,M_1,M_2)} \coloneqq \operatorname{GL}(A_r) \times \operatorname{GL}(M_2) \times \operatorname{GL}(M_1)$$

also acts on $\operatorname{rep}(K_r; M_1, M_2)$ via

$$\begin{aligned} G_{(A_r,M_1,M_2)} \times \operatorname{rep}(K_r; M_1, M_2) &\longrightarrow \operatorname{rep}(K_r; M_1, M_2) \\ ((g, h_2, h_1), \psi) &\mapsto h_2 \circ \psi \circ (g^{-1} \otimes h_1^{-1}). \end{aligned}$$

If $\mathcal{O}$ is $G_{(A_r,M_1,M_2)}$-invariant, then $\mathcal{O}$ is $G(M_1, M_2)$- and $\operatorname{GL}(A_r)$-invariant under the action

$$g.\psi \coloneqq \psi \circ (g^{-1} \otimes \mathrm{id}_{M_1}).$$

We obtain an induced action of $\operatorname{GL}(A_r)$ on $[\mathcal{O}]$. By definition $[\mathcal{O}]/\operatorname{GL}(A_r)$ is the set of orbits under the action of $\operatorname{GL}(A_r)$ on the set of isomorphism classes with representatives in $\mathcal{O}$. Moreover, we have a one-to-one correspondence between $\mathcal{O}/G_{(A_r,M_1,M_2)}$ and $[\mathcal{O}]/\operatorname{GL}(A_r)$ given by

$$G_{(A_r,M_1,M_2)}.\psi \mapsto \operatorname{GL}(A_r).[(M_1, M_2, \psi)].$$

For $(x, y) \in \mathbb{N}^2$ we let

$$G_{(r,x,y)} \coloneqq \operatorname{GL}(A_r) \times \operatorname{GL}(\Bbbk^y) \times \operatorname{GL}(\Bbbk^x).$$

Since regular representations are $\operatorname{GL}(A_r)$-invariant (see Example 2.6.1), the set

$$\mathcal{E}(x, y) \subseteq \operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y)$$

is $G_{(r,x,y)}$-invariant. Since $\operatorname{GL}(A_r)$ acts transitively on bases of A_r, we can rephrase the aforementioned results of Ringel as follows.

Theorem 6.4.1. *(see [Rin16, Theorem]) The following statements hold.*

(1) We have $\mathcal{E}_3 = \{(1, 1), (2, 2)\}$.

(2) The sets $\mathcal{E}(1, 1)$, $\mathcal{E}(2, 2)$ are orbits under the action of $G_{(3,1,1)}$ and $G_{(3,2,2)}$ on $\operatorname{rep}(K_3; \Bbbk, \Bbbk)$ and $\operatorname{rep}(K_3; \Bbbk^2, \Bbbk^2)$, respectively.

(3) Let $M \in \operatorname{rep}(K_3)$ be a representation with dimension vector $(1, 1)$. The representation is elementary if and only if there is $g \in \operatorname{GL}(A_3)$ such that $g.M \cong (\Bbbk, \Bbbk, (\mathrm{id}_\Bbbk, 0, 0))$.

(4) Let $M \in \operatorname{rep}(K_3)$ be a representation with dimension vector $(2, 2)$. The representation is elementary if and only if there is $g \in \operatorname{GL}(A_3)$ such that $g.M \cong (\Bbbk^2, \Bbbk^2, (\mathrm{id}_{\Bbbk^2}, \beta, \gamma))$ with $\beta(a, b) = (0, a)$ and $\gamma(a, b) = (b, 0)$ for all $(a, b) \in \Bbbk^2$.

In the following we show that we can not hope for such a nice classification if $r \geq 4$.

Lemma 6.4.2. *Let $\varnothing \neq \mathcal{O} \subseteq \operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y)$ be a non-empty open and $G_{(r,x,y)}$-invariant subset of $\operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y)$ such that $q_r(x,y) < -r^2$. Then $\mathcal{O}/G_{(r,x,y)}$ is not finite.*

Proof. We set $G := G_{(r,x,y)}$. We assume that $\mathcal{O}/G$ is finite and fix $T_1, \dots, T_n \in \mathcal{O}$ such that $\mathcal{O} = \bigcup_{i=1}^n G.T_i$. Hence

$$\operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y) = \overline{\mathcal{O}} = \bigcup_{i=1}^n \overline{G.T_i}.$$

Because $\operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y)$ is irreducible, we find $i \in \{1, \dots, n\}$ such that $\operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y) = \overline{G.T_i}$. Since orbits are open in their closure ([Hum12, (8.3)]), we conclude with [Har77, (1.10)] that

$$\dim G \geq \dim G.T_i = \dim \overline{G.T_i} = \dim \operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y) = rxy.$$

In particular, we have

$$0 \leq \dim G - rxy = r^2 + x^2 + y^2 - rxy = r^2 + q_r(x,y),$$

in contradiction to the assumption. □

Corollary 6.4.3. *Let $(x,y) \in \mathcal{E}_r$ be such that $q_r(x,y) < -r^2$. Then $\mathcal{E}(x,y)/G_{(r,x,y)}$ is not finite.*

Proof. Since $(x,y) \in \mathcal{E}_r$, we can apply Theorem 6.3.6 and conclude that

$$\mathcal{E}(x,y) = \mathcal{B}(x,y) \cap \operatorname{rep}_{\text{esp}}(K_r, r - y + \lceil \frac{x}{r} \rceil) \cap \operatorname{rep}_{\text{inj}}(K_r, \lceil \frac{x}{r} \rceil - 1)$$

is a non-empty open subset of $\operatorname{rep}(K_r; \Bbbk^x, \Bbbk^y)$. Moreover, $\mathcal{E}(x,y)$ is $G_{(r,x,y)}$-invariant, since regular representations are $\operatorname{GL}(A_r)$-invariant. Now we apply Lemma 6.4.2. □

Theorem 6.4.4. *Let $r \geq 4$. Then there are infinitely many pairwise non-isomorphic elementary representations with dimension vector $(r+2, 3)$ that all are in different $\operatorname{GL}(A_r)$-orbits.*

Proof. We set $x = r + 2$ and $y = 3 < r$. Then $(x,y) \in \mathcal{F}_r$ and

$$\lfloor \frac{x}{r} \rfloor (y + r - \lfloor \frac{x}{r} \rfloor) = r + 2 \leq x \leq 2(r-1) = \lceil \frac{x}{r} \rceil (r - y + \lceil \frac{x}{r} \rceil).$$

Now Corollary 6.3.8 implies that $(r+2, 3) \in \mathcal{E}_r$. Moreover, we have $q_{K_r}(r+2, 3) = -2r^2 - 2r + 13 < -r^2$ and can apply Corollary 6.4.3 □

Corollary 6.4.5. *Let $(x,y) \in \mathbb{N}^2$ be such that $q_r(x,y) < -r^2$. The number of different $\operatorname{GL}(A_r)$-orbits of isomorphism classes of elementary representations with dimension vector (x,y) is either 0 or infinite.*

Proof. We can assume that there is $E \in \operatorname{rep}(K_r)$ elementary with dimension vector (x,y). By applying D_{K_r} and powers of σ_{K_r} to E we find an elementary representation F with dimension vector $\underline{\dim} F \in \mathcal{E}_r$. Since σ_{K_r} and D_{K_r} do not change the quadratic form, we have

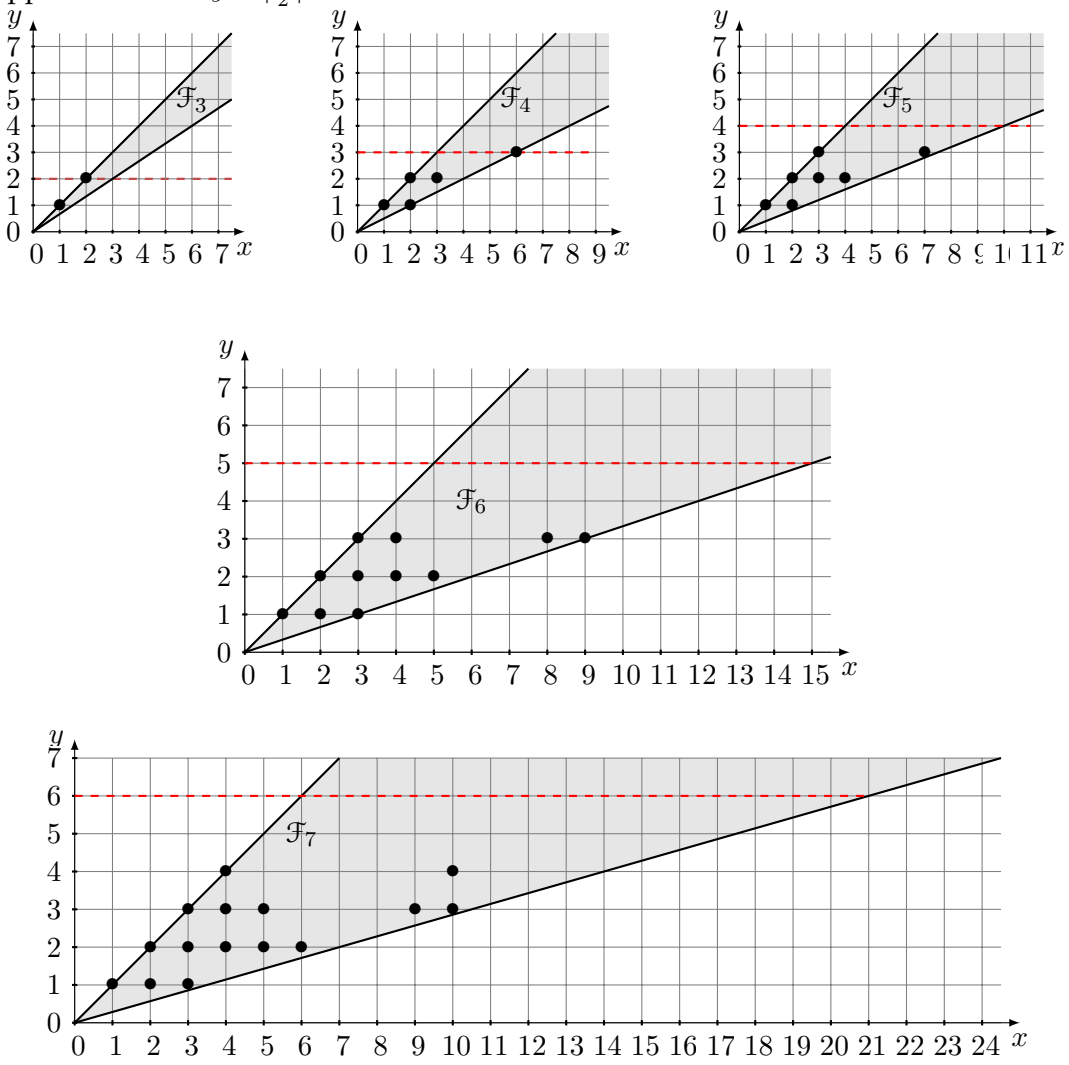
$$q_r(\underline{\dim} F) = q_r(x,y) < -r^2.$$

Now Corollary 6.4.3 implies that we get infinitely many orbits. Since D_{K_r} and σ_r respect $\operatorname{GL}(A_r)$-orbits (see Lemma 2.6.2), the statement follows. □

Remark 6.4.6. Let $E \in \operatorname{rep}(K_3)$ be elementary. Then [Rin16] implies that $q_3(\underline{\dim} E) \in \{-1, -4\}$. Hence $q_3(\underline{\dim} E) \geq -9 = -r^2$.

6.5. Examples

The following figures illustrate our findings for $r \in \{3, 4, 5, 6, 7\}$. The dashed red line is the restriction $y \leq r - 1$. We would like to remark that simulations for $5 \leq r \leq 500$ indicate that a sharp upper bound for y is $\lceil \frac{r}{2} \rceil$.



7. Vector bundles on varieties

In this chapter we recall well-known results from the theory of vector bundles (understood as locally free $\mathcal{O}_X$-modules of finite rank) over a variety X, which will be needed in the following chapter in the special case of the Grassmann variety $\mathrm{Gr}_d(A_r)$ of d-dimensional subspaces of the arrow space A_r. A discussion of the connection between vector bundles and locally trivial linear fibrations and the theory of Chern classes for the Grassmann variety $\mathrm{Gr}_d(A_r)$, also needed in Chapter 8, may be found in Appendix A and Appendix B, respectively.

7.1. Vector bundles as locally free sheaves

We assume that the reader is familiar with the basic definitions and properties of sheaves, (locally) ringed spaces, and modules of ringed spaces, and refer to [GW10, §2.5, §7] for unexplained terminology.

Definition. (1) Given a sheaf $\mathcal{F}$ over a topological space X we denote by $\mathcal{F}_x$ the *stalk* of $\mathcal{F}$ at x. For $U \subseteq \mathcal{F}$ open, we denote by $\mathcal{F}(U)$ the set of *sections* of $\mathcal{F}$ on U. Given a section $s \in \mathcal{F}(U)$ and $x \in U \subseteq X$ we denote by $s_x = [(U, s)] \in \mathcal{F}_x$ the *germ of s at x*.

(2) Let $(X, \mathcal{O}_X)$ be a locally ringed space and $x \in X$. By definition the stalk $\mathcal{O}_{X,x} \coloneqq (\mathcal{O}_X)_x$ is a local ring and we denote by $\mathfrak{m}_x \trianglelefteq \mathcal{O}_{X,x}$ the unique maximal ideal of $\mathcal{O}_{X,x}$. Moreover, we write $k(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x$ for the fiber of $\mathcal{O}_X$ at x and note that $k(x)$ is a field.

(3) Let $(X, \mathcal{O}_X)$ be a ringed space, then $\mathrm{Mod}(X) = \mathrm{Mod}\,\mathcal{O}_X$ denotes the category of $\mathcal{O}_X$-modules. Given an $\mathcal{O}_X$-homomorphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ we denote the image, kernel and cokernel sheaf of φ by $\mathsf{Im}\,\varphi$, $\mathsf{Ker}\,\varphi$, and $\mathsf{Coker}\,\varphi$, respectively.

(4) Let $(X, \mathcal{O}_X)$ be a locally ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module. We call the $k(x)$-vector space

$$\mathcal{F}(x) \coloneqq \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

the *fiber* of $\mathcal{F}$ at x. For a section $s \in \mathcal{F}(U)$ with $x \in U \subseteq X$ open, we set $s(x) \coloneqq s_x + \mathfrak{m}_x\mathcal{F}_x$.

Example 7.1.1. *(1) Let $(X, \mathcal{O}_X)$ be a space with functions and $x \in X$. We have a surjective morphism of $\Bbbk$-algebras*

$$\mathrm{ev}_x\colon \mathcal{O}_{X,x} \longrightarrow \Bbbk\ ;\ [(U, r)] \mapsto r(x)$$

with kernel $\ker(\mathrm{ev}_x) = \{[(U, r)] \mid r(x) = 0\}$. Let $[(U, r)] \in \mathcal{O}_{X,x} \setminus \ker(\mathrm{ev}_x)$, then $x \in D(r) \subseteq U$, $\frac{1}{r} \in \mathcal{O}_X(D(r))$ and $[(D(\frac{1}{r}), \frac{1}{r})] \in \mathcal{O}_{X,x}$. We obtain $[(D(r), \frac{1}{r})] \cdot [(U, r)] = [(D(r), \frac{1}{r})] \cdot [(D(r), r)] = [(D(r), 1)] = 1_{\mathcal{O}_{X,x}}$. Hence $[(U, r)]$ is a unit. Therefore, $\mathfrak{m}_x \coloneqq \ker(\mathrm{ev}_x)$ is the unique maximal ideal of $\mathcal{O}_{X,x}$ and $(X, \mathcal{O}_X)$ is a locally ringed space.

(2) *Let V be a finite dimensional $\Bbbk$-vector space and $X \subseteq \mathbb{A}(V)$ be an affine algebraic set. We consider the affine variety $(X, \mathcal{O}_{\mathbb{A}(V)|X})$, let $\mathcal{I}(X) \coloneqq \{f \in S(V^*) \mid \forall x \in X \colon f(x) = 0\} \trianglelefteq S(V^*)$ and define $\Bbbk[X] \coloneqq S(V^*)/\mathcal{I}(X)$. Let $x \in V$. We consider the maximal ideal $\mathfrak{M}_x \coloneqq \{f \in \Bbbk[X] \mid f(x) = 0\} \trianglelefteq \Bbbk[X]$ and define a map*

$$\varphi \colon \Bbbk[X]_{\mathfrak{M}_x} \longrightarrow \mathcal{O}_{X,x} \; ; \; \frac{f}{g} \mapsto [(D(g), \frac{f}{g})].$$

According to [Kun97, (IV.3.2)], φ is an isomorphism of $\Bbbk$-algebras. In particular, $\mathcal{O}_{X,x}$ is a finitely generated $\Bbbk$-algebra.

(3) *Let $(X, \mathcal{O}_X)$ be a variety and $x \in X$. We find $x \in U \subseteq X$ open such that $(U, \mathcal{O}_{X|U})$ is affine. We have $\mathcal{O}_{X,x} \cong \mathcal{O}_{X|U,x}$ and conclude with (2) that $\mathcal{O}_{X,x}$ is a finitely generated $\Bbbk$-algebra. By Hilbert's Nullstellensatz (see [GW10, (1.7)]) $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \colon \Bbbk$ is a finite field extension and $\Bbbk$ being algebraically closed implies $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x = \Bbbk$.*

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be $\mathcal{O}_X$-modules.

(1) Let $f \colon \mathcal{F} \longrightarrow \mathcal{G}, g \colon \mathcal{G} \longrightarrow \mathcal{H}$ be morphisms of $\mathcal{O}_X$-modules. The sequence $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ is *exact* at $\mathcal{G}$ if $\mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x \xrightarrow{g_x} \mathcal{H}_x$ is exact for all $x \in X$. Exactness of sequences of $\mathcal{O}_X$-modules is defined in the obvious way.

(2) Let $U \subseteq X$ be open. We denote by $\mathcal{F}|_U$ the $\mathcal{O}_{X|U}$-module obtained by considering open sets $V \subseteq U$.

(3) We say that $\mathcal{F}$ is of *finite type* if for every $x \in X$ there is an open neighborhood $x \in U_x \subseteq X$, $n(x) \in \mathbb{N}_0$ and a short exact sequence of $\mathcal{O}_X|_{U_x}$-modules

$$\mathcal{O}_X^{n(x)}|_{U_x} \longrightarrow \mathcal{F}|_{U_x} \longrightarrow 0.$$

(4) We call $\mathcal{F}$ *locally free* if for all $x \in X$ there exists an open neighborhood $x \in U_x$ and a set I_x such that $\mathcal{O}_X^{(I_x)}|_{U_x} \cong \mathcal{F}|_{U_x}$.

Remark 7.1.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, $x \in X$ and $x \in U \subseteq X$ be open such that $\mathcal{F}|_U \cong \mathcal{O}_{X|U}^{(I_x)}$ as $\mathcal{O}_{X|U}$-modules for a set I_x. Since taking stalks commutes with direct sums (see [GW10, (7.4.1)]), we have an isomorphism of free $\mathcal{O}_{X,x}$-modules

$$\mathcal{F}_x \cong (\mathcal{F}|_U)_x \cong (\mathcal{O}_{X|U})_x^{(I_x)} \cong \mathcal{O}_{X,x}^{(I_x)}.$$

In particular, $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module and the cardinality of I_x does not depend on U but only on x. We define $\operatorname{rk}_x(\mathcal{F}) \coloneqq |I_x|$ for $x \in X$.

Definition. Let $(X, \mathcal{O}_X)$ be a locally ringed space.

(1) A locally free $\mathcal{O}_X$-module $\mathcal{F}$ such that $\operatorname{rk}_x(\mathcal{F})$ is finite for all $x \in X$ is called *locally free of finite rank*.

(2) Let $\mathcal{F}$ be locally free of finite rank. For $x \in X$ we call $\mathrm{rk}_x(\mathcal{F}) \in \mathbb{N}_0$ the *rank of* $\mathcal{F}$ *at* x.

(3) If $(X, \mathcal{O}_X)$ is a variety and $\mathcal{F}$ is locally free of finite rank, we call $\mathcal{F}$ a *vector bundle on* X.

Remark 7.1.3. Let $(X, \mathcal{O}_X)$ be a ringed space and le $\mathcal{F}$ be locally free of finite rank.

(1) The map $\mathrm{rk}\colon X \longrightarrow \mathbb{N}_0\ ;\ x \mapsto \mathrm{rk}_x(\mathcal{F})$ is well-defined and by definition locally constant, hence constant on connected components of X (cf. Lemma A.1.3).

(2) If $(X, \mathcal{O}_X)$ is locally ringed, we conclude with $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\mathrm{rk}_x(\mathcal{F})}$ that $\dim_{k(x)} \mathcal{F}(x) = \mathrm{rk}_x(\mathcal{F})$ for all $x \in X$. For a variety X we obtain $\mathrm{rk}_x(\mathcal{F}) = \dim_{\Bbbk} \mathcal{F}(x)$.

Definition. Let $(X, \mathcal{O}_X)$ be a connected variety and $\mathcal{F}$ be a vector bundle. Then $\mathrm{rk}\colon X \longrightarrow \mathbb{N}_0\ ;\ x \mapsto \mathrm{rk}_x(\mathcal{F})$ is constant and we call $\mathrm{rk}(\mathcal{F}) := \mathrm{rk}_x(\mathcal{F}) \in \mathbb{N}_0$ the *rank* of $\mathcal{F}$ for any $x \in X$.

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module.

(1) We call $\mathcal{F}$ *coherent* if $\mathcal{F}$ is of finite type and if for every open subset $U \subseteq X$, every integer $n \in \mathbb{N}_0$ and for every homomorphism $\varphi\colon (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U$ the kernel of φ is of finite type (over U).

(2) We denote by $\mathrm{Coh}(X) = \mathrm{Coh}(\mathcal{O}_X)$ the full subcategory of $\mathrm{Mod}\,\mathcal{O}_X$ consisting of coherent modules.

(3) If $(X, \mathcal{O}_X)$ is a variety, we denote by $\mathrm{Vect}(X) = \mathrm{Vect}(\mathcal{O}_X) \subseteq \mathrm{Coh}(X)$ (cf. [GW10, (7.40), (7.45)]) the full subcategory of vector bundles.

Let $(X, \mathcal{O}_X)$ be a variety, then $\mathrm{Coh}(X)$ is an abelian category (cf. [GW10, (7.46)]) and $\mathrm{Vect}(X)$ is a Krull-Schmidt category (cf. [Ati56]). Thus, the indecomposable objects of any full subcategory $\mathcal{C} \subseteq \mathrm{Vect}(X)$ that is closed under taking direct summands are just the indecomposable vector bundles belonging to $\mathcal{C}$.

7.2. Tests for vector bundles

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be $\mathcal{O}_X$-modules.

(1) A morphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ is called *locally split injective*, provided $\varphi\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is a split monomorphism for each $x \in X$, i.e. there is a $\mathcal{O}_{X,x}$-module homomorphism $g_x\colon \mathcal{G}_x \longrightarrow \mathcal{F}_x$ such that $g_x \circ f_x = \mathrm{id}_{\mathcal{F}_x}$.

(2) An exact sequence $0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is called *locally split*, provided ι is locally split injective, i.e. the exact sequence of $\mathcal{O}_{X,x}$-modules $0 \longrightarrow \mathcal{F}_x \xrightarrow{\iota_x} \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$ splits for all $x \in X$.

(3) If $(X, \mathcal{O}_X)$ is a variety and $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ a morphism of vector bundles, we call $\operatorname{Im}\varphi$ *subbundle* of $\mathcal{G}$, provided the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \operatorname{Coker}\varphi \longrightarrow 0$$

is locally split.

The following result allows us to test whether a given coherent sheaf is a vector bundle. The result is given as an exercise in [Har77, (II.5.8)]. A proof can be found in [Ben16, (6.2.1)].

Theorem 7.2.1. *Let $(X, \mathcal{O}_X)$ be a connected variety.*

(1) The following statements are equivalent for a coherent $\mathcal{O}_X$-module $\mathcal{F}$.

(a) $\mathcal{F}$ is a vector bundle of rank n.

(b) $\dim_{\Bbbk}\colon X \longrightarrow \mathbb{Z}\,;\ x \mapsto \dim_{\Bbbk}\mathcal{F}(x)$ is constant with $\dim_{\Bbbk}(X) = \{n\}$.

(2) Let $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of vector bundles. The following statements are equivalent.

(a) The $\mathcal{O}_X$-module $\operatorname{Coker}\varphi$ is a vector bundle of rank $\operatorname{rk}(\operatorname{Coker}\varphi) = \operatorname{rk}(\mathcal{G}) - n$.

(b) For each $x \in X$ the induced $\Bbbk$-linear map $\varphi(x)\colon \mathcal{F}(x) \longrightarrow \mathcal{G}(x)$ on fibers has rank n.

(3) Let $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of vector bundles. The following statements are equivalent.

(a) φ is locally split injective.

(b) For each $x \in X$ the induced $\Bbbk$-linear map $\varphi(x)\colon \mathcal{F}(x) \longrightarrow \mathcal{G}(x)$ on fibers is injective.

Remark 7.2.2. Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ be a short exact sequence of vector bundles on a connected variety X and $x \in X$. Since $\mathcal{H}$ is a vector bundle, $\mathcal{H}_x$ is a free $\mathcal{O}_{X,x}$-module. Hence talking stalks at x yields a split exact sequence

$$0 \longrightarrow \mathcal{O}_{X,x}^{\operatorname{rk}(\mathcal{F})} \longrightarrow \mathcal{O}_{X,x}^{\operatorname{rk}(\mathcal{G})} \longrightarrow \mathcal{O}_{X,x}^{\operatorname{rk}(\mathcal{H})} \longrightarrow 0.$$

In particular, $\operatorname{rk}(\mathcal{G}) = \operatorname{rk}(\mathcal{F}) + \operatorname{rk}(\mathcal{H})$.

Corollary 7.2.3. *Let $(X, \mathcal{O}_X)$ be a connected variety and $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of vector bundles. The following statements are equivalent.*

(1) $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ is an isomorphism.

(2) The induced $\mathcal{O}_{X,x}$-linear map on stalks $\varphi_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is bijective for all $x \in X$.

(3) The induced map on fibers $\varphi(x)\colon \mathcal{F}(x) \longrightarrow \mathcal{G}(x)$ is bijective for all $x \in X$.

Proof. Note that (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) hold for arbitrary sheaves (cf. [GW10, (2.23)]). Assume that $\varphi(x)$ is bijective for all $x \in X$. Let $x \in X$. Since $\varphi(x)$ is surjective, we have $\mathcal{G}_x = \operatorname{im}\varphi_x + \mathfrak{m}_x\mathcal{G}_x$ and Nakayama's Lemma implies $\operatorname{im}\varphi_x = \mathcal{G}_x$. Moreover, Theorem 7.2.1(3) shows that φ_x is injective (even split injective) for all $x \in X$. □

We also record the following result that we will need in Appendix A.

Corollary 7.2.4. *Let $(X, \mathcal{O}_X)$ be a connected variety and $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$, $\psi\colon \mathcal{G} \longrightarrow \mathcal{H}$ be morphisms of vector bundles such that $\psi \circ \varphi = 0$. The following statements are equivalent.*

(1) $0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$ *is an exact sequence of $\mathcal{O}_X$-modules.*

(2) $0 \longrightarrow \mathcal{F}(x) \xrightarrow{\varphi(x)} \mathcal{G}(x) \xrightarrow{\psi(x)} \mathcal{H}(x) \longrightarrow 0$ *is exact for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$. Since $\mathcal{H}$ is a vector bundle, $\mathcal{H}_x$ is a free $\mathcal{O}_{X,x}$-module. Hence the exact sequence is locally split exact. Application of $- \otimes_{\mathcal{O}_{X,x}} \Bbbk$ yields an exact sequence on the level of fibers for all $x \in X$.

(2) $\Rightarrow$ (1). By Theorem 7.2.1(3) we have $\ker \varphi_x = 0$ for all $x \in X$. Application of Theorem 7.2.1(2) gives us an exact sequence of vector bundles

$$\delta\colon 0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\pi} \mathsf{Coker}\,\varphi \longrightarrow 0.$$

Since $\psi \circ \varphi = 0$, we obtain a morphism $\eta\colon \mathsf{Coker}\,\varphi \longrightarrow \mathcal{H}$ (cf. [GW10, (7.4)]) such that $\eta \circ \pi = \psi$. Let $x \in X$. Since $\psi(x)$ is surjective, $\eta(x)$ is surjective. Note that

$$\dim_{\Bbbk} \mathcal{H}(x) = \dim_{\Bbbk} \mathcal{G}(x) - \dim_{\Bbbk} \mathcal{F}(x) = \mathrm{rk}(\mathcal{G}) - \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathsf{Coker}\,\varphi) = \dim_{\Bbbk}(\mathsf{Coker}\,\varphi)(x).$$

We conclude that $\eta(x)\colon (\mathsf{Coker}\,\varphi)(x) \longrightarrow \mathcal{H}(x)$ is an isomorphism of $\Bbbk$-vector spaces and Corollary 7.2.3 implies that η is an isomorphism of vector bundles. Now $(\eta \circ \pi)_x = \psi_x$ implies $\ker \psi_x = \ker(\eta_x \circ \pi_x) = \ker \pi_x = \operatorname{im} \varphi_x$ and $\operatorname{im} \psi_x = \operatorname{im}(\eta_x \circ \pi_x) = \mathcal{H}_x$. Hence $0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$ is exact. □

8. Kronecker representations and Steiner bundles

Throughout this chapter, which is based on [BF24, §3], we let $r \geq 2$ and $1 \leq d < r$. We denote by $\mathrm{Gr}_d(A_r)$ the Grassmannian of d-dimensional subspaces of the arrow space A_r. Recall that $\mathrm{Gr}_d(A_r)$ is a projective variety of dimension $\dim \mathrm{Gr}_d(A_r) = d(r-d)$.

8.1. Preliminaries

Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_{\mathrm{Gr}_d(A_r)}$-modules. We let $\mathsf{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{G})$ be the sheaf of $\mathcal{O}_{\mathrm{Gr}_d(A_r)}$-homomorphisms from $\mathcal{F}$ to $\mathcal{G}$. By definition, we have $\mathsf{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{G})(\mathrm{Gr}_d(A_r)) = \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{G})$, the space of homomorphisms from $\mathcal{F}$ to $\mathcal{G}$, cf. [GW10, §7.4]. For $i \in \mathbb{N}_0$, we let

$$\mathrm{Ext}^i_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, -) : \mathrm{Coh}(\mathrm{Gr}_d(A_r)) \longrightarrow \mathrm{mod}\, \Bbbk$$

be the i-th right derived functor of $\mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, -) \cong \mathrm{Ext}^0_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, -)$. Moreover, we denote by $\mathcal{F}^\vee := \mathsf{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{O}_{\mathrm{Gr}_d(A_r)})$ the *dual* of $\mathcal{F}$. By [GW10, (7.4.6)] the functor

$$\mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, -) \colon\ \mathrm{Coh}(\mathrm{Gr}_d(A_r)) \longrightarrow \mathrm{mod}\, \Bbbk$$

and the *global section functor* (see also [Per08, (VII.4.6)])

$$\Gamma(\mathrm{Gr}_d(A_r), -) \colon\ \mathrm{Coh}(\mathrm{Gr}_d(A_r)) \longrightarrow \mathrm{mod}\, \Bbbk\ ;\ \mathcal{F} \mapsto \Gamma(\mathrm{Gr}_d(A_r), \mathcal{F}) := \mathcal{F}(\mathrm{Gr}_d(A_r))$$

are equivalent. Hence we obtain for each $i \in \mathbb{N}_0$ an equivalence between $\mathrm{Ext}^i_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, -)$ and the i-th right derived functor $\mathrm{H}^i(-)$ of the global section functor $\Gamma(\mathrm{Gr}_d(A_r), -)$. Note that $\mathrm{H}^i(\mathrm{Gr}_d(A_r), \mathsf{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{G})) \cong \mathrm{Ext}^i_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(\mathrm{Gr}_d(A_r))$ and $i \in \mathbb{N}_0$.

8.2. Relative projective representations and Steiner bundles

For $d \in \{1, \ldots, r-1\}$ we consider the *universal vector bundle* $\mathcal{U}_{(r,d)}$ of $\mathrm{Gr}_d(A_r)$. By definition, $\mathcal{U}_{(r,d)}$ is the locally free sheaf corresponding to the locally trivial linear fibration (E, p) (see Appendix A) over $\mathrm{Gr}_d(A_r)$, where

$$E := \{(\mathfrak{v}, a) \in \mathrm{Gr}_d(A_r) \times A_r \ \mid\ a \in \mathfrak{v}\} \ \text{ and } \ p : E \longrightarrow \mathrm{Gr}_d(A_r)\ ;\ (\mathfrak{v}, a) \mapsto \mathfrak{v}$$

denote the *incidence variety* and the canonical projection, respectively. Consequently,

$$\mathcal{U}_{(r,d)}(U) = \{s : U \longrightarrow E \text{ morphism} \mid\ p \circ s = \mathrm{id}_U\} \cong \{t : U \longrightarrow A_r \text{ morphism } \mid \forall\ \mathfrak{v} \in U \colon t(\mathfrak{v}) \in \mathfrak{v}\}$$

for every open subset $U \subseteq \mathrm{Gr}_d(A_r)$.

Given a pair (V_1, V_2) of $\Bbbk$-vector spaces, we put

$$\widetilde{V_1} := V_1 \otimes_\Bbbk \mathcal{U}_{(r,d)} \ \text{ and } \ \widetilde{V_2} := V_2 \otimes_\Bbbk \mathcal{O}_{\mathrm{Gr}_d(A_r)}.$$

Let $f_i : V_i \longrightarrow W_i$ $(i \in \{1,2\})$ be linear maps between $\Bbbk$-vector spaces. These give rise to morphisms $\tilde{f}_i : \widetilde{V_i} \longrightarrow \widetilde{W_i}$, where

$$\tilde{f}_1 \coloneqq f_1 \otimes \mathrm{id}_{\mathcal{U}_{(r,d)}} \quad \text{and} \quad \tilde{f}_2 \coloneqq f_2 \otimes \mathrm{id}_{\mathcal{O}_{\mathrm{Gr}_d(A_r)}}.$$

Let $M \in \mathrm{rep}(K_r)$. In analogy with [Ben16, §8.2], we consider for every open subset $U \subseteq \mathrm{Gr}_d(A_r)$ the morphism

$$\widetilde{\Theta}_{M,d}(U) : \widetilde{M_1}(U) \longrightarrow \widetilde{M_2}(U) \;\; ; \;\; m \otimes t \mapsto \sum_{i=1}^{r} \psi_M(\gamma_i \otimes m) \otimes \gamma_i^* \circ t$$

of $\mathcal{O}_{\mathrm{Gr}_d(A_r)}(U)$-modules.[1] Then $\widetilde{\Theta}_{M,d} : \widetilde{M_1} \longrightarrow \widetilde{M_2}$ is a morphism of vector bundles, and we define

$$\widetilde{\Theta}_d(M) \coloneqq \mathsf{Coker}\, \widetilde{\Theta}_{M,d} \in \mathrm{Coh}(\mathrm{Gr}_d(A_r)).$$

If $f : M \longrightarrow N$ is a morphism in $\mathrm{rep}(K_r)$ with components $f_i : M_i \longrightarrow N_i \quad (i \in \{1,2\})$, then

$$\tilde{f}_i : \widetilde{M_i} \longrightarrow \widetilde{N_i} \;\; ; \;\; m \otimes g \mapsto f_i(m) \otimes g \qquad (i \in \{1,2\})$$

are morphisms of vector bundles and there results a commutative diagram

$$\begin{array}{ccccccc}
\widetilde{M_1} & \xrightarrow{\widetilde{\Theta}_{M,d}} & \widetilde{M_2} & \longrightarrow & \widetilde{\Theta}_d(M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \tilde{f}_1} & & \downarrow{\scriptstyle \tilde{f}_2} & & & & \\
\widetilde{N_1} & \xrightarrow{\widetilde{\Theta}_{N,d}} & \widetilde{N_2} & \longrightarrow & \widetilde{\Theta}_d(N) & \longrightarrow & 0.
\end{array}$$

Consequently, there is a unique morphism

$$\widetilde{\Theta}_d(f) : \widetilde{\Theta}_d(M) \longrightarrow \widetilde{\Theta}_d(N),$$

which completes the diagram above. Thus, we obtain a functor

$$\widetilde{\Theta}_d : \mathrm{rep}(K_r) \longrightarrow \mathrm{Coh}(\mathrm{Gr}_d(A_r)).$$

For $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$, the standard isomorphisms $\mathrm{ev}_{1,\mathfrak{v}} : \mathcal{U}_{(r,d)}(\mathfrak{v}) \longrightarrow \mathfrak{v}$ and $\mathrm{ev}_{2,\mathfrak{v}} : \mathcal{O}_{\mathrm{Gr}_d(A_r)}(\mathfrak{v}) \longrightarrow \Bbbk$ (see Lemma A.3.2 and Example A.3.3) give us two commutative diagrams

$$\begin{array}{ccc}
\widetilde{V_1}(\mathfrak{v}) & \xrightarrow{\tilde{f}_1(\mathfrak{v})} & \widetilde{W_1}(\mathfrak{v}) \\
\downarrow{\scriptstyle \mathrm{id}_{V_1} \otimes \mathrm{ev}_{1,\mathfrak{v}}} & & \downarrow{\scriptstyle \mathrm{id}_{W_1} \otimes \mathrm{ev}_{1,\mathfrak{v}}} \\
V_1 \otimes_{\Bbbk} \mathfrak{v} & \xrightarrow{f_1 \otimes \mathrm{id}_{\mathfrak{v}}} & W_1 \otimes_{\Bbbk} \mathfrak{v}
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{V_2}(\mathfrak{v}) & \xrightarrow{\tilde{f}_2(\mathfrak{v})} & \widetilde{W_2}(\mathfrak{v}) \\
\downarrow{\scriptstyle \mathrm{id}_{V_2} \otimes \mathrm{ev}_{2,\mathfrak{v}}} & & \downarrow{\scriptstyle \mathrm{id}_{W_2} \otimes \mathrm{ev}_{2,\mathfrak{v}}} \\
V_2 \otimes_{\Bbbk} \Bbbk & \xrightarrow{f_2 \otimes \mathrm{id}_{\Bbbk}} & W_2 \otimes_{\Bbbk} \Bbbk
\end{array}$$

Definition. Let $M \in \mathrm{rep}(K_r)$. We say that M has *constant d-rank*, provided there is $\mathrm{rk}_d(M) \in \mathbb{N}_0$ such that $\mathrm{rk}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = \mathrm{rk}_d(M)$ for all $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$. We let $\mathrm{CR}(K_r, d)$ be the full subcategory of $\mathrm{rep}(K_r)$ whose objects have constant d-rank.

[1] Here $\{\gamma_1^*, \ldots, \gamma_r^*\} \subseteq A_r^*$ denotes the dual basis of $\{\gamma_1, \ldots, \gamma_r\}$.

Remark 8.2.1. (1) The category $\mathrm{CR}(K_r, d)$ is the analog of the category of modules of constant $(d, 1)$-radical rank that was considered in [CFP15, (4.1)].

(2) The definition of $\widetilde{\Theta}_d(M)$ does not depend on the choice of the dual bases $\{\gamma_1, \dots, \gamma_r\} \subseteq A_r$, $\{\gamma_1^*, \dots, \gamma_r^*\} \subseteq A_r^*$. In fact, one can define sheaves $\widetilde{M}_1'$ and $\widetilde{M}_2'$ via

$$\widetilde{M}_1'(U) \coloneqq \{s : U \longrightarrow A_r \otimes_{\Bbbk} M_1 \mid \forall\, \mathfrak{v} \in U\colon s(\mathfrak{v}) \subseteq \mathfrak{v} \otimes_{\Bbbk} M_1\} \subseteq \mathrm{Mor}(U, A_r \otimes_{\Bbbk} M_1)$$

and $\widetilde{M}_2'(U) \coloneqq \mathrm{Mor}(U, M_2)$ (cf. Example A.3.3). Setting $\widetilde{\Theta}'_{M,d}(U)(s) = \psi_M \circ s$ for all $s \in \widetilde{M}_1'(U)$, we obtain $\widetilde{\Theta}_d(M) \cong \mathsf{Coker}\, \widetilde{\Theta}'_{M,d}$.

In the following result we collect basic properties of $\widetilde{\Theta}_d$.

Lemma 8.2.2. *The following statements hold:*

(1) The functor $\widetilde{\Theta}_d$ is right exact.

(2) Let $M \in \mathrm{rep}(K_r)$. Then $M \in \mathrm{CR}(K_r, d)$ if and only if $\widetilde{\Theta}_d(M) \in \mathrm{Vect}(\mathrm{Gr}_d(A_r))$. In that case, we have $\mathrm{rk}(\widetilde{\Theta}_d(M)) = \dim_{\Bbbk} M_2 - \mathrm{rk}_d(M)$.

(3) We have $\ker \widetilde{\Theta}_d \cap \mathrm{CR}(K_r, d) = \{M \in \mathrm{rep}(K_r) \mid \mathrm{im}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = M_2$ for all $\mathfrak{v} \in \mathrm{Gr}_d(A_r)\}$.

Proof. (1) The right exactness of $\widetilde{\Theta}_d$ is a direct consequence of (the proof of) the Snake Lemma.

(2) Suppose that $M \in \mathrm{CR}(K_r, d)$. Given $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$, the standard isomorphisms $\mathrm{ev}_{1,\mathfrak{v}} : \mathcal{U}_{(r,d)}(\mathfrak{v}) \longrightarrow \mathfrak{v}$ and $\mathrm{ev}_{2,\mathfrak{v}} : \mathcal{O}_{\mathrm{Gr}_d(A_r)}(\mathfrak{v}) \longrightarrow \Bbbk$ (see Lemma A.3.2 and Example A.3.3) give the commutative diagram

$$\begin{array}{ccc} \widetilde{M}_1(\mathfrak{v}) & \xrightarrow{\widetilde{\Theta}_{M,d}(\mathfrak{v})} & \widetilde{M}_2(\mathfrak{v}) \\ \downarrow{\scriptstyle \mathrm{id}_{M_1} \otimes \mathrm{ev}_{1,\mathfrak{v}}} & & \downarrow{\scriptstyle \mathrm{id}_{M_2} \otimes \mathrm{ev}_{2,\mathfrak{v}}} \\ M_1 \otimes_{\Bbbk} \mathfrak{v} & \xrightarrow{\eta \circ \psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1} \circ \omega} & M_2 \otimes_{\Bbbk} \Bbbk, \end{array}$$

where ω flips the tensor factors and $\eta\colon M_2 \longrightarrow M_2 \otimes_{\Bbbk} \Bbbk\ ;\ m_2 \mapsto m_2 \otimes 1$. We conclude

$$\mathrm{rk}(\widetilde{\Theta}_{M,d}(\mathfrak{v})) = \mathrm{rk}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = \mathrm{rk}_d(M).$$

It now follows from Theorem 7.2.1 that $\widetilde{\Theta}_d(M) = \mathsf{Coker}\, \widetilde{\Theta}_{M,d}$ is a vector bundle of rank $\mathrm{rk}(\widetilde{\Theta}_d(M)) = \dim_{\Bbbk} M_2 - \mathrm{rk}_d(M)$.

Conversely, assume that $\widetilde{\Theta}_d(M)$ is a vector bundle. Then Theorem 7.2.1 implies that there is $n \in \mathbb{N}_0$ such that

$$\mathrm{rk}(\widetilde{\Theta}_{M,d}(\mathfrak{v})) = n \quad \text{for all } \mathfrak{v} \in \mathrm{Gr}_d(A_r).$$

Again, the commutative diagram above implies that $\mathrm{rk}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = n$ for all $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$, so that $M \in \mathrm{CR}(K_r, d)$.

(3) Let $N \in \mathrm{CR}(K_r, d)$. By definition, we have $N \in \{M \in \mathrm{rep}(K_r) \mid \mathrm{im}(\psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1}) = M_2$ for all $\mathfrak{v} \in \mathrm{Gr}_d(A_r)\}$ if and only if $\mathrm{rk}_d(M) = \dim_{\Bbbk} M_2$. In view of (2), this is equivalent to $\widetilde{\Theta}_d(M) = 0$.

□

Steiner bundles on projective space were first systematically studied by Dolgachev-Kapranov [DK93]. The following definition for Grassmannians is taken from [AM15].

Definition. A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathrm{Gr}_d(A_r))$ is referred to as a *Steiner bundle* if there exist vector spaces V_1, V_2 and an exact sequence

$$0 \longrightarrow V_1 \otimes_{\Bbbk} \mathcal{U}_{(r,d)} \longrightarrow V_2 \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(A_r)} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We denote by $\mathrm{StVect}(\mathrm{Gr}_d(A_r))$ the full subcategory of $\mathrm{Vect}(\mathrm{Gr}_d(A_r))$, whose objects are Steiner bundles.

Remark 8.2.3. In view of $\mathcal{U}_{(r,1)} \cong \mathcal{O}_{\mathbb{P}(A_r)}(-1)$ (cf. [EH16, §3.2.3]), we retrieve the original definition of [DK93].

The relationship between $\mathrm{rep}_{\mathrm{proj}}(K_r)$ and $\mathrm{StVect}(\mathrm{Gr}_d(A_r))$, which for $d = 1$ is implicit in Brambilla's work [Bra05; Bra08] on Steiner bundles (for $\Bbbk = \mathbb{C}$), was investigated by Jardim and Prata [JP15] for $\mathrm{char}(\Bbbk) = 0$ in the more general context of cokernel bundles. In our context, the category of "globally injective representations" defined in [JP15] coincides, thanks to Theorem 3.3.2, with $\mathrm{rep}_{\mathrm{proj}}(K_r, d)$.

Since we will employ the functors $\widetilde{\Theta}_d$ extensively, we give a detailed proof following [JP15, (3.5)] with proper modifications, thereby hopefully convincing the reader that their result does not necessitate any assumption on the characteristic of the base field in the case of cokernel bundles of type $(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)})$. To that end we require the following subsidiary result.

Proposition 8.2.4. *(1) The pair $(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)})$ is special exceptional, i.e.*

(i) $\dim_{\Bbbk} \mathrm{Ext}^n_{\mathrm{Gr}_d(A_r)}(\mathcal{U}_{(r,d)}, \mathcal{U}_{(r,d)}) = \delta_{n,0} = \dim_{\Bbbk} \mathrm{Ext}^n_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)})$ *for all* $n \in \mathbb{N}_0$.

(ii) $\dim_{\Bbbk} \mathrm{Ext}^n_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \mathcal{U}_{(r,d)}) = 0$ *for all* $n \in \mathbb{N}_0$.

(iii) $\dim_{\Bbbk} \mathrm{Ext}^n_{\mathrm{Gr}_d(A_r)}(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)}) = 0$ *for all* $n \in \mathbb{N}$.

(2) The $\Bbbk$-linear map

$$\psi \colon A_r^* \longrightarrow \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)}) \ ; \ \lambda \mapsto \psi_\lambda$$

is an isomorphism of $\Bbbk$-vector spaces, where

$$(\psi_\lambda)_U \colon \mathcal{U}_{(r,d)}(U) \longrightarrow \mathcal{O}_{\mathrm{Gr}_d(A_r)}(U) \ ; \ t \colon U \longrightarrow A_r \mapsto \lambda \circ t \colon U \longrightarrow \Bbbk.^{2}$$

(3) Let $\mathcal{F}, \mathcal{G} \in \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ be Steiner bundles. Then $\mathrm{Ext}^\ell_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{G}) = 0$ for all $\ell \geq 2$ and $\mathrm{Ext}^2_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{U}_{(r,d)}) = 0$.

Proof. (1) The vector bundles $\mathcal{O}_{\mathrm{Gr}_d(A_r)}$ and $\mathcal{U}_{(r,d)}$ are simple and $\mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \mathcal{U}_{(r,d)}) = 0$ (see Corollary A.3.7).

[2] Here we use the identification given in Example A.3.3.

Let $\mathcal{B}_{d,r-d}$ be the set of those partitions whose corresponding Young tableaux have at most d rows and at most $r-d$ columns. Given $\alpha \in \mathcal{B}_{d,r-d}$, we denote by α' its transpose partition. According to [BLV15, (1.3)][3], the vector bundle

$$\mathcal{T} \coloneqq \bigoplus_{\alpha \in \mathcal{B}_{d,r-d}} \bigwedge^{\alpha'} \mathcal{U}_{(r,d)}$$

is a tilting object in the bounded derived category $\mathsf{D}^b(\mathrm{Gr}_d(A_r))$ of $\mathrm{Coh}(\mathrm{Gr}_d(A_r))$. In particular,

$$0 = \mathrm{Hom}_{\mathsf{D}^b(\mathrm{Gr}_d(A_r))}(\mathcal{T}, \mathcal{T}[n]) \cong \mathrm{Ext}^n_{\mathrm{Gr}_d(A_r)}(\mathcal{T}, \mathcal{T})$$

for all $n \geq 1$. We consider the partitions $\alpha \coloneqq (0)$ and $\beta \coloneqq (1)$. Then $\alpha, \beta \in \mathcal{B}_{d,r-d}$, so that $\mathcal{O}_{\mathrm{Gr}_d(A_r)} = \bigwedge^{\alpha} \mathcal{U}_{(r,d)}$ and $\mathcal{U}_{(r,d)} = \bigwedge^{\beta} \mathcal{U}_{(r,d)}$ are direct summands of $\mathcal{T}$. Consequently, $\mathrm{Ext}^n_{\mathrm{Gr}_d(A_r)}(\mathcal{X}, \mathcal{Y}) = 0$ for $\mathcal{X}, \mathcal{Y} \in \{\mathcal{U}_{(r,d)}, \mathcal{O}_{\mathrm{Gr}_d(A_r)}\}$ and $n \geq 1$.

(2) See Corollary A.3.7 and Remark A.3.8 for a proof using the equivalence of categories

$$\mathsf{Ltf}(\mathrm{Gr}_d(A_r)) \cong \mathrm{Vect}(\mathrm{Gr}_d(A_r)).$$

A proof in the category of vector bundles on $\mathrm{Gr}_d(A_r)$ can be obtained by extending the arguments given in [Mar12, (1.1.12)] to fields of arbitrary characteristic.

(3) For $\mathcal{H} \in \{\mathcal{F}, \mathcal{G}\}$ we find $s_{\mathcal{H}}, t_{\mathcal{H}} \in \mathbb{N}$ together with a short exact sequence

$$0 \longrightarrow \mathcal{U}_{(r,d)}^{s_{\mathcal{H}}} \longrightarrow \mathcal{O}_{\mathrm{Gr}_d(A_r)}^{t_{\mathcal{H}}} \longrightarrow \mathcal{H} \longrightarrow 0.$$

Let $\ell \in \mathbb{N}$. We obtain the following diagram, which is exact at every point

$$\begin{array}{ccccc}
\mathrm{Ext}^{\ell}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}^{t_{\mathcal{F}}}_{\mathrm{Gr}_d(A_r)}, \mathcal{O}^{t_{\mathcal{G}}}_{\mathrm{Gr}_d(A_r)}) & & & & \mathrm{Ext}^{\ell+1}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}^{t_{\mathcal{F}}}_{\mathrm{Gr}_d(A_r)}, \mathcal{U}^{s_{\mathcal{G}}}_{(r,d)}) \\
\uparrow & & & & \uparrow \\
\mathrm{Ext}^{\ell}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{O}^{t_{\mathcal{G}}}_{\mathrm{Gr}_d(A_r)}) & \longrightarrow & \mathrm{Ext}^{\ell}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{G}) & \longrightarrow & \mathrm{Ext}^{\ell+1}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}, \mathcal{U}^{s_{\mathcal{G}}}_{(r,d)}) \\
\uparrow & & & & \uparrow \\
\mathrm{Ext}^{\ell-1}_{\mathrm{Gr}_d(A_r)}(\mathcal{U}^{s_{\mathcal{F}}}_{(r,d)}, \mathcal{O}^{t_{\mathcal{G}}}_{\mathrm{Gr}_d(A_r)}) & & & & \mathrm{Ext}^{\ell}_{\mathrm{Gr}_d(A_r)}(\mathcal{U}^{s_{\mathcal{F}}}_{(r,d)}, \mathcal{U}^{s_{\mathcal{G}}}_{(r,d)}).
\end{array}$$

The first statement follows immediately from (1). The second statement follows from (1) for $\ell = 1$ and $\mathcal{F} = \mathcal{G}$.

□

We consider the locally trivial linear fibration (E, p) over $\mathrm{Gr}_d(A_r)$ introduced at the beginning of Section 8.2 and the trivial fibration $\mathrm{pr}_{\mathrm{Gr}_d(A_r)} \colon \mathrm{Gr}_d(A_r) \times A_r \longrightarrow \mathrm{Gr}_d(A_r)$ of rank r. The map

$$\varphi \colon E \longrightarrow \mathrm{Gr}_d(A_r) \times A_r \ ; \ (\mathfrak{v}, a) \mapsto (\mathfrak{v}, a)$$

[3] Note that the authors of [BLV15] consider vector bundles on $\mathrm{Gr}_{r-d}(A_r^*)$. See also [Ful98, (14.6.5)].

is a morphism of fibrations and injective on fibers. Hence Proposition A.2.3 implies that we have a short exact sequence of locally trivial linear fibrations

$$0 \longrightarrow E \xrightarrow{\varphi} \mathrm{Gr}_d(A_r) \times A_r \longrightarrow \operatorname{coker} \varphi \longrightarrow 0.$$

Application of the exact functor $\mathcal{L}$ (see Corollary A.3.4 and Example A.3.3) gives us an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{U}_{(r,d)} \longrightarrow A_r \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(A_r)} \longrightarrow \mathcal{Q}_{(r,d)} \longrightarrow 0.$$

The vector bundle $\mathcal{Q}_{(r,d)}$ is called *universal quotient bundle* of $\mathrm{Gr}_d(A_r)$, cf. [EH16, §3.2.3]. The Chern polynomial[4] of $\mathcal{Q}_{(r,d)}$ is given by $c(\mathcal{Q}_{(r,d)}) = 1+\sigma_1+\sigma_2+\cdots+\sigma_{r-d}$ (see [EH16, §5.6.2]). In particular, $c_1(\mathcal{Q}_{(r,d)}) = \sigma_1$.

Theorem 8.2.5. *The following statements hold.*

(1) A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathrm{Gr}_d(A_r))$ is a Steiner bundle if and only if there is $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ such that $\mathcal{F} \cong \widetilde{\Theta}_d(M)$.

(2) The functor $\widetilde{\Theta}_d : \mathrm{rep}_{\mathrm{proj}}(K_r, d) \longrightarrow \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ is an equivalence.

(3) Let $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$. Then we have

$$\mathrm{rk}(\widetilde{\Theta}_d(M)) = \Delta_M(d) \text{ and } c_1(\widetilde{\Theta}_d(M)) = (\dim_{\Bbbk} M_1)\sigma_1.$$

Proof. Let $M \in \mathrm{CR}(K_r, d)$. The commutative diagram in the proof of Lemma 8.2.2 gives

$$(*) \quad \ker \widetilde{\Theta}_{M,d}(\mathfrak{v}) \cong \ker \psi_{M,\mathfrak{v}} \text{ for all } \mathfrak{v} \in \mathrm{Gr}_d(A_r).$$

(1) Suppose that $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ and consider the exact sequence

$$0 \longrightarrow \mathsf{Ker}\, \widetilde{\Theta}_{M,d} \longrightarrow \widetilde{M_1} \xrightarrow{\widetilde{\Theta}_{M,d}} \widetilde{M_2} \longrightarrow \widetilde{\Theta}_d(M) \longrightarrow 0.$$

We conclude using Theorem 3.3.2 that

$$\ker \widetilde{\Theta}_{M,d}(\mathfrak{v}) \overset{(*)}{\cong} \ker \psi_{M,\mathfrak{v}} = 0$$

for all $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$. According to Theorem 7.2.1(3), the morphism $\widetilde{\Theta}_{M,d}$ is locally split injective. In particular, $\ker(\widetilde{\Theta}_{M,d})_{\mathfrak{v}} = 0$ for all $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$, whence $\mathsf{Ker}\, \widetilde{\Theta}_{M,d} = 0$ by [GW10, (7.3.1)]. Consequently, $\widetilde{\Theta}_d(M)$ is a Steiner bundle.

Conversely, suppose that $\mathcal{F}$ is a Steiner bundle. Then there exists a pair (M_1, M_2) of vector spaces and an exact sequence

$$0 \longrightarrow \widetilde{M_1} \xrightarrow{\Psi} \widetilde{M_2} \longrightarrow \mathcal{F} \longrightarrow 0$$

of vector bundles. By Proposition 8.2.4(2) we have an isomorphism of $\Bbbk$-vector spaces

$$\mathrm{Hom}_{\Bbbk}(M_1, M_2) \otimes_{\Bbbk} A_r^* \longrightarrow \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\widetilde{M_1}, \widetilde{M_2})$$

[4] See Appendix B for details on the Chern polynomials on $\mathrm{Gr}_d(A_r)$.

sending $\zeta \otimes f$ to the map $m \otimes t \mapsto \zeta(m) \otimes f \circ t$. There thus exist $\Bbbk$-linear maps $M(\gamma_i) : M_1 \longrightarrow M_2$ such that for every $U \subseteq \mathrm{Gr}_d(A_r)$ open

$$\Psi_U(m \otimes t) = \sum_{i=1}^{r} M(\gamma_i)(m) \otimes \gamma_i^* \circ t = \widetilde{\Theta}_{M,d}(U)(m \otimes t) \qquad \text{for all } m \in M_1, t \in \mathcal{U}_{(r,d)}(U).$$

Letting $M \coloneqq (M_1, M_2, (M(\gamma_i))_{1 \leq i \leq r}) \in \mathrm{rep}(K_r)$, we see that $\mathcal{F} \cong \widetilde{\Theta}_d(M)$. By Theorem 7.2.1(2) we have for $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$

$$\begin{aligned} \dim_\Bbbk M_2 - \mathrm{rk}(\widetilde{\Theta}_{M,d}(\mathfrak{v})) = \mathrm{rk}(\widetilde{\Theta}_d(M)) = \mathrm{rk}(\mathcal{F}) &= \dim_\Bbbk M_2 - d \dim_\Bbbk M_1 \\ &= \dim_\Bbbk M_2 - \dim_\Bbbk \widetilde{M_1}(\mathfrak{v}), \end{aligned}$$

and conclude $0 = \ker \widetilde{\Theta}_{M,d}(\mathfrak{v}) \overset{(*)}{\cong} \ker \psi_{M,\mathfrak{v}}$. Consequently, $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$.

(2) In view of (1), it suffices to show that $\widetilde{\Theta}_d : \mathrm{rep}_{\mathrm{proj}}(K_r, d) \longrightarrow \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ is full and faithful. Let $M, N \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ be representations and $\varphi\colon \widetilde{\Theta}_d(M) \longrightarrow \widetilde{\Theta}_d(N)$ be a morphism of vector bundles. By (1) we find for $X \in \{M, N\}$ a short exact sequence

$$\mathbb{E}_X : 0 \longrightarrow \widetilde{X_1} \xrightarrow{\widetilde{\Theta}_{X,d}} \widetilde{X_2} \xrightarrow{\pi_X} \widetilde{\Theta}_d(X) \longrightarrow 0.$$

Application of $\mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\widetilde{M_2}, -)$ to $\mathbb{E}_N$ in conjunction with

$$\mathrm{Ext}^1_{\mathrm{Gr}_d(A_r)}(\widetilde{M_2}, \widetilde{N_1}) \cong (M_2 \otimes_\Bbbk N_1) \otimes_\Bbbk \mathrm{Ext}^1_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \mathcal{U}_{(r,d)}) = 0$$

(see Proposition 8.2.4) implies that

$$\pi_{N,*}\colon \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\widetilde{M_2}, \widetilde{N_2}) \longrightarrow \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\widetilde{M_2}, \widetilde{\Theta}_d(N)) \ ; \ f \mapsto \pi_N \circ f$$

is surjective and we find $\varphi_2\colon \widetilde{M_2} \longrightarrow \widetilde{N_2}$ such that

$$\varphi \circ \pi_M = \pi_N \circ \varphi_2.$$

There results a morphism $\varphi_1\colon \widetilde{M_1} \longrightarrow \widetilde{N_1}$ such that diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M_1} & \xrightarrow{\widetilde{\Theta}_{M,d}} & \widetilde{M_2} & \xrightarrow{\pi_M} & \widetilde{\Theta}_d(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \widetilde{N_1} & \xrightarrow{\widetilde{\Theta}_{N,d}} & \widetilde{N_2} & \xrightarrow{\pi_N} & \widetilde{\Theta}_d(N) & \longrightarrow & 0 \end{array}$$

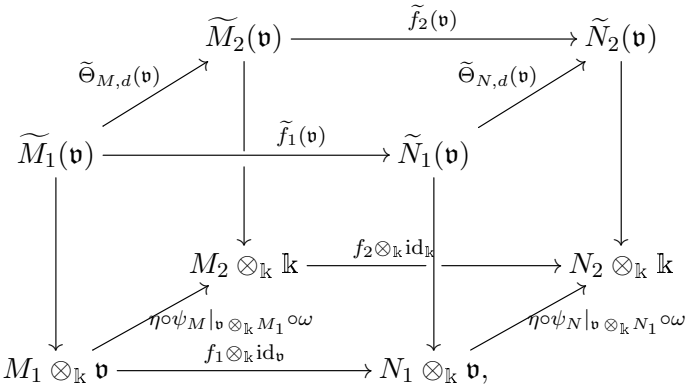
commutes. Since $\mathrm{End}_{\mathrm{Gr}_d(A_r)}(\mathcal{U}_{(r,d)}) \cong \Bbbk \cong \mathrm{End}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)})$ (see Proposition 8.2.4), we find $f_i \in \mathrm{Hom}_\Bbbk(M_i, N_i)$ for $i \in \{1, 2\}$ such that

$$f_1 \otimes \mathrm{id}_{\mathcal{U}_{(r,d)}} = \varphi_1 \text{ and } f_2 \otimes \mathrm{id}_{\mathcal{O}_{\mathrm{Gr}_d(A_r)}} = \varphi_2.$$

We obtain the commutative diagram

$$\begin{array}{ccc} M_1 \otimes_\Bbbk \mathcal{U}_{(r,d)} & \xrightarrow{\widetilde{\Theta}_{M,d}} & M_2 \otimes_\Bbbk \mathcal{O}_{\mathrm{Gr}_d(A_r)} \\ \downarrow{\scriptstyle f_1 \otimes \mathrm{id}_{\mathcal{U}_{(r,d)}}} & & \downarrow{\scriptstyle f_2 \otimes \mathrm{id}_{\mathcal{O}_{\mathrm{Gr}_d(A_r)}}} \\ N_1 \otimes_\Bbbk \mathcal{U}_{(r,d)} & \xrightarrow{\widetilde{\Theta}_{N,d}} & N_2 \otimes_\Bbbk \mathcal{O}_{\mathrm{Gr}_d(A_r)}. \end{array}$$

Let $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$. Going to the fibers yields the diagram



where all down facing arrows are evaluation isomorphisms (see Example A.3.3 and the diagram in the proof of Lemma 8.2.2). Using the previous considerations in the proof of Lemma 8.2.2 and in Section 8.2, a diagram chase shows the bottom face of this diagram is commutative. Hence

$$\psi_N \circ (\mathrm{id}_{\mathfrak{v}} \otimes f_1) = f_2 \circ \psi_M|_{\mathfrak{v} \otimes_{\Bbbk} M_1},$$

and $A_r = \bigcup_{\mathfrak{u} \in \mathrm{Gr}_d(A_r)} \mathfrak{u}$ implies that $\psi_N \circ (\mathrm{id}_{A_r} \otimes f_1) = f_2 \circ \psi_M$. Hence $(f_1, f_2) \in \mathrm{Hom}_{K_r}(M, N)$.

It remains to show that $\widetilde{\Theta}_d$ is faithful. Let $f \in \mathrm{Hom}_{K_r}(M, N)$ such that $\widetilde{\Theta}_d(f) = 0$. We apply $\mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, -)$ and obtain, using $\mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \mathcal{U}_{(r,d)}) = 0$ (see Proposition 8.2.4), a commutative diagram with exact rows

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \widetilde{M_2}) & \longrightarrow & \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \widetilde{\Theta}_d(M)) \\
\downarrow{\scriptstyle (\tilde{f}_1)_*} & & \downarrow{\scriptstyle (\tilde{f}_2)_*} & & \downarrow{\scriptstyle \widetilde{\Theta}_d(f)_* = 0} \\
0 & \longrightarrow & \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \widetilde{N_2}) & \xrightarrow{(\pi_N)_*} & \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, \widetilde{\Theta}_d(N)).
\end{array}$$

Hence $(\pi_N)_*$ is injective and $(\pi_N)_* \circ (\tilde{f}_2)_* = 0$ implies $0 = (\tilde{f}_2)_*$. For $X \in \{M_2, N_2\}$ we have an isomorphism of vector spaces $X \longrightarrow \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, X \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(A_r)})$, sending x to the map $t \mapsto x \otimes t$ and a commutative diagram

$$\begin{array}{ccc}
M_2 & \longrightarrow & \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, M_2 \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(A_r)}) \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle (\tilde{f}_2)_*} \\
N_2 & \longrightarrow & \mathrm{Hom}_{\mathrm{Gr}_d(A_r)}(\mathcal{O}_{\mathrm{Gr}_d(A_r)}, N_2 \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(A_r)}).
\end{array}$$

This shows $f_2 = 0$. Since $(f_1, f_2) \in \mathrm{Hom}_{K_r}(M, N)$, we have $0 = f_2 \circ \psi_M = \psi_N \circ (\mathrm{id}_{A_r} \otimes f_1)$ and $N \in \mathrm{rep}_{\mathrm{proj}}(K_r, d) \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ implies that the map $\psi_N|_{\Bbbk\gamma_1 \otimes_{\Bbbk} N_1}$ is injective. Let $m \in M_1$, then $0 = [\psi_N \circ (\mathrm{id}_{A_r} \otimes f_1)](\gamma_1 \otimes m) = \psi_N(\gamma_1 \otimes f_1(m))$ shows $f_1(m) = 0$ and therefore $f_1 = 0$.

(3) By Lemma 8.2.2 we have $\mathrm{rk}(\widetilde{\Theta}_d(M)) = \Delta_M(d)$. The exact sequence

$$0 \longrightarrow \mathcal{U}_{(r,d)} \longrightarrow A_r \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Gr}_d(A_r)} \longrightarrow \mathcal{Q}_{(r,d)} \longrightarrow 0$$

in conjunction with the Whitney sum formula (see Appendix B) yields

$$c_1(\mathcal{U}_{(r,d)}) = -c_1(\mathcal{Q}_{(r,d)}) = -\sigma_1.$$

By the same token, the defining sequence implies

$$c_1(\widetilde{\Theta}_d(M)) = -c_1(\widetilde{M_1}) = -(\dim_{\Bbbk} M_1)c_1(\mathcal{U}_{(r,d)}) = (\dim_{\Bbbk} M_1)\sigma_1.$$

□

Corollary 8.2.6. *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be an exact sequence in* $\operatorname{rep}(K_r)$ *such that* $C \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$*. Then the sequence* $0 \longrightarrow \widetilde{\Theta}_d(A) \longrightarrow \widetilde{\Theta}_d(B) \longrightarrow \widetilde{\Theta}_d(C) \longrightarrow 0$ *is exact.*

Proof. We have exact sequences $0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ in $\operatorname{rep}(K_r)$ for $i \in \{1, 2\}$. Tensoring with $\mathcal{U}_{(r,d)}$ and $\mathcal{O}_{\mathrm{Gr}_d(A_r)}$ yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{A_1} & \longrightarrow & \widetilde{B_1} & \longrightarrow & \widetilde{C_1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{\Theta}_{A,d}} & & \downarrow{\scriptstyle \widetilde{\Theta}_{B,d}} & & \downarrow{\scriptstyle \widetilde{\Theta}_{C,d}} & & \\ 0 & \longrightarrow & \widetilde{A_2} & \longrightarrow & \widetilde{B_2} & \longrightarrow & \widetilde{C_2} & \longrightarrow & 0 \end{array}$$

with exact rows. Since $C \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$, $\widetilde{\Theta}_d(M)$ is a Steiner bundle and $\operatorname{Ker} \widetilde{\Theta}_d(M) = 0$ (cf. Theorem 8.2.5(1)). Since $\widetilde{\Theta}_d$ is right exact (see Lemma 8.2.2), the assertion follows by applying the Snake Lemma. □

Recall from [Har77, (III.2.7)] that for a vector bundle $\mathcal{F} \in \operatorname{Vect}(\mathrm{Gr}_d(A_r))$ and $i > \dim \mathrm{Gr}_d(A_r) = d(r-d)$ we have $\mathrm{H}^i(\mathrm{Gr}_d(A_r), \mathcal{F}) = 0$. Hence

$$\chi(\mathcal{F}) = \sum_{i=0}^{d(r-d)} (-1)^i \dim_{\Bbbk} \mathrm{H}^i(\mathrm{Gr}_d(A_r), \mathcal{F})$$

is well-defined and called the *Euler characteristic* of $\mathcal{F}$.

Corollary 8.2.7. *For* $M, N \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ *the following statements hold.*

(1) We have $\dim_{\Bbbk} \operatorname{Ext}^i_{K_r}(M, N) = \dim_{\Bbbk} \operatorname{Ext}^i_{\mathrm{Gr}_d(A_r)}(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(N))$ *for all* $i \in \mathbb{N}_0$.

(2) We have $\chi(\mathsf{Hom}_{\mathrm{Gr}_d(A_r)}(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(N))) = \langle \underline{\dim}\, M, \underline{\dim}\, N \rangle_r$.

Proof. (1) Throughout the proof we suppress the index $\mathrm{Gr}_d(A_r)$ of the Ext^i-spaces. In view of Proposition 8.2.4(3), Theorem 8.2.5(2) and $\operatorname{rep}(K_r)$ being hereditary, it suffices to prove the statement for $i = 1$. By Proposition 8.2.4(1),(2) we have the exact sequences

$$0 \to \operatorname{Hom}(\widetilde{\Theta}_d(M), \widetilde{N_1}) \to \underbrace{\operatorname{Hom}(\widetilde{M_2}, \widetilde{N_1})}_{=0} \to \underbrace{\operatorname{Hom}(\widetilde{M_1}, \widetilde{N_1})}_{\cong M_1 \otimes_{\Bbbk} N_1} \to \operatorname{Ext}^1(\widetilde{\Theta}_d(M), \widetilde{N_1}) \to 0,$$

and

$$0 \to \operatorname{Hom}(\widetilde{\Theta}_d(M), \widetilde{N_2}) \to \underbrace{\operatorname{Hom}(\widetilde{M_2}, \widetilde{N_2})}_{\cong M_2 \otimes_{\Bbbk} N_2} \to \underbrace{\operatorname{Hom}(\widetilde{M_1}, \widetilde{N_2})}_{\cong (M_1 \otimes_{\Bbbk} N_2)^r} \to \operatorname{Ext}^1(\widetilde{\Theta}_d(M), \widetilde{N_2}) \to 0.$$

We conclude $\dim_{\Bbbk} \operatorname{Hom}(\widetilde{\Theta}_d(M), \widetilde{N_1}) = 0$, $\dim_{\Bbbk} \operatorname{Ext}^1(\widetilde{\Theta}_d(M), \widetilde{N_1}) = \dim_{\Bbbk} M_1 \otimes_{\Bbbk} N_1$ and $\dim_{\Bbbk} \operatorname{Hom}(\widetilde{\Theta}_d(M), \widetilde{N_2}) - \dim_{\Bbbk} \operatorname{Ext}^1(\widetilde{\Theta}_d(M), \widetilde{N_2}) = \dim_{\Bbbk} M_2 \otimes_{\Bbbk} N_2 - \dim_{\Bbbk}(M_1 \otimes_{\Bbbk} N_2)^r$. Another application of Proposition 8.2.4(3) gives us $\operatorname{Ext}^2(\widetilde{\Theta}_d(M), \widetilde{N_1}) = 0$ and therefore the exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}(\widetilde{\Theta}_d(M), \widetilde{N_1}) \longrightarrow \operatorname{Hom}(\widetilde{\Theta}_d(M), \widetilde{N_2}) \longrightarrow \operatorname{Hom}(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(N)) \\ \longrightarrow \operatorname{Ext}^1(\widetilde{\Theta}_d(M), \widetilde{N_1}) \longrightarrow \operatorname{Ext}^1(\widetilde{\Theta}_d(M), \widetilde{N_2}) \longrightarrow \operatorname{Ext}^1(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(N)) \longrightarrow 0. \end{aligned}$$

Using the Euler-Ringel form and Theorem 8.2.5 we conclude

$$\dim_{\Bbbk} \operatorname{Ext}^1(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(N)) = \dim_{\Bbbk} \operatorname{Hom}_{K_r}(M, N) - \langle \underline{\dim}\, M, \underline{\dim}\, N \rangle_r = \dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(M, N).$$

(2) This is a direct consequence of (1).

□

8.3. Exceptional Steiner bundles

In this section we shall classify exceptional Steiner bundles. Throughout, we assume that $r \geq 2$ and $d \in \{1, \ldots, r-1\}$. Our first result characterizes the representations corresponding to exceptional Steiner bundles.

Definition. A vector bundle $\mathcal{F} \in \operatorname{Vect}(\operatorname{Gr}_d(A_r))$ is called *exceptional*, provided $\operatorname{End}_{\operatorname{Gr}_d(A_r)}(\mathcal{F}) = \Bbbk \operatorname{id}_{\mathcal{F}}$ and $\operatorname{Ext}^n_{\operatorname{Gr}_d(A_r)}(\mathcal{F}, \mathcal{F}) = 0$ for all $n \in \mathbb{N}$.

Proposition 8.3.1. *Suppose that $r \geq 3$, and let $\mathcal{F} \in \operatorname{StVect}(\operatorname{Gr}_d(A_r))$ be a Steiner bundle. Then the following statements are equivalent:*

(1) $\mathcal{F}$ is exceptional.

(2) There is $M \in \operatorname{rep}(K_r)$ indecomposable preprojective such that $\mathcal{F} \cong \widetilde{\Theta}_d(M)$.

Proof. (1) $\Rightarrow$ (2). Theorem 8.2.5 provides $M \in \operatorname{rep}_{\operatorname{proj}}(K_r, d)$ such that $\widetilde{\Theta}_d(M) \cong \mathcal{F}$. In particular, M is a brick and hence indecomposable. In view of Theorem 4.2.2, the module M is not preinjective. Corollary 8.2.7 yields

$$\operatorname{Ext}^1_{K_r}(M, M) \cong \operatorname{Ext}^1_{\operatorname{Gr}_d(A_r)}(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(M)) = 0,$$

so that Theorem 1.5.1 in conjunction with the Euler-Ringel form (cf. Section 1.5) ensures that M is preprojective.

(2) $\Rightarrow$ (1). Recall from Theorem 4.2.2 that $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$. Since M is preprojective, an application of [ASS06, (VIII.2.7)] in conjunction with Corollary 8.2.7 implies

$$\operatorname{Ext}^1_{\operatorname{Gr}_d(A_r)}(\mathcal{F}, \mathcal{F}) \cong \operatorname{Ext}^1_{K_r}(M, M) = 0.$$

and $\operatorname{End}_{\operatorname{Gr}_d(A_r)}(\mathcal{F}) = \Bbbk \operatorname{id}_{\mathcal{F}}$. By Proposition 8.2.4 we have $\operatorname{Ext}^n_{\operatorname{Gr}_d(A_r)}(\mathcal{F}, \mathcal{F}) = 0$ for all $n \geq 2$. □

Recall from Section 1.5 that $(P_n(r))_{n\geq 0}$ denotes the family of indecomposable preprojective K_r-representations (with dimension vector $\underline{\dim}\, P_n(r) = (a_n(r), a_{n+1}(r))$). Let $n \in \mathbb{N}_0$. We define

$$\mathcal{E}_{n,d} \coloneqq \widetilde{\Theta}_d(P_n(r)) \in \operatorname{StVect}(\operatorname{Gr}_d(A_r)).$$

By Theorem 8.2.5 and Proposition 8.3.1, $\mathcal{E}_{n,d}$ is an exceptional Steiner bundle that satisfies

$$\operatorname{rk}(\mathcal{E}_{n,d}) = a_{n+1}(r) - da_n(r) \text{ and } c_1(\mathcal{E}_{n,d}) = a_n(r)\sigma_1.$$

Part (3) of our next result extends [Bra05, (2.1)], where the bundles involved were assumed to be generic. Part (4) shows that [Bra08, (6.3)] holds for Grassmannians over fields of arbitrary characteristic.

Theorem 8.3.2. *Suppose that $r \geq 3$. Then the following statements hold:*

(1) If $\mathcal{E} \in \operatorname{StVect}(\operatorname{Gr}_d(A_r))$ is an exceptional Steiner bundle, then $\mathcal{E} \cong \mathcal{E}_{n,d}$ for some $n \in \mathbb{N}_0$.

(2) An indecomposable Steiner bundle $\mathcal{F} \in \operatorname{StVect}(\operatorname{Gr}_d(A_r))$ is exceptional if and only if $\operatorname{rk}(\mathcal{F}) = a_{n+1}(r) - da_n(r)$ and $c_1(\mathcal{F}) = a_n(r)\sigma_1$ for some $n \in \mathbb{N}_0$.

(3) A non-zero Steiner bundle $\mathcal{F} \in \operatorname{StVect}(\operatorname{Gr}_d(A_r))$ satisfies $\operatorname{Ext}^1_{\operatorname{Gr}_d(A_r)}(\mathcal{F}, \mathcal{F}) = 0$ if and only if there exist $(a, b) \in \mathbb{N} \times \mathbb{N}_0$ and $n \in \mathbb{N}_0$ such that $\mathcal{F} \cong \mathcal{E}^a_{n,d} \oplus \mathcal{E}^b_{n+1,d}$.

(4) Let $(c, s) \in \mathbb{N}_0 \times \mathbb{N}$ such that $q_r(c, s + dc) > 1$, then there exist uniquely determined elements $n, a, b \in \mathbb{N}_0$ such that a generic Steiner bundle $\mathcal{F} \in \operatorname{StVect}(\operatorname{Gr}_d(A_r))$ of rank s and with first Chern class $c\sigma_1$ is isomorphic to $\mathcal{E}^a_{n,d} \oplus \mathcal{E}^b_{n+1,d}$.

Proof. (1) This follows immediately from Proposition 8.3.1, the definition of $\mathcal{E}_{n,d}$ and the above-mentioned classification of the indecomposable preprojective K_r-representations.

(2) In view of (1), exceptional Steiner bundles enjoy the stated properties. Conversely, let $\mathcal{F}$ be an indecomposable Steiner bundle such that $\operatorname{rk}(\mathcal{F}) = a_{n+1}(r) - da_n(r)$ and $c_1(\mathcal{F}) = a_n(r)\sigma_1$. According to Theorem 8.2.5, there is $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ indecomposable such that $\mathcal{F} \cong \widetilde{\Theta}_d(M)$ and $\underline{\dim}\, M = (a_n(r), a_{n+1}(r))$. Since $q_r(\underline{\dim}\, M) = q_r(a_n(r), a_{n+1}(r)) = 1$, Theorem 1.5.1 implies that there is a (up to isomorphism) unique indecomposable representation with dimension vector $(a_n(r), a_{n+1}(r))$. Hence $M \cong P_n(r)$ and $\mathcal{F} \cong \mathcal{E}_{n,d}$ is an exceptional Steiner bundle.

(3) Let $\mathcal{F} \in \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ be a non-zero Steiner bundle that satisfies $\mathrm{Ext}^1_{\mathrm{Gr}_d(A_r)}(\mathcal{F},\mathcal{F}) = 0$. Application of Theorem 8.2.5 implies the existence of $0 \neq M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ such that $\mathcal{F} \cong \widetilde{\Theta}_d(M)$ and Corollary 8.2.7 implies $\mathrm{Ext}^1_{K_r}(M,M) \cong \mathrm{Ext}^1_{\mathrm{Gr}_d(A_r)}(\mathcal{F},\mathcal{F}) = 0$. Let N be an indecomposable direct summand of M. By Theorem 4.2.2 we have $N \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$. As $\mathrm{Ext}^1_{K_r}(N,N) = 0$, [ASS06, (VIII.3.3)] implies that N is a brick. We conclude with Corollary 8.2.7 and Proposition 8.2.4(3) that $\widetilde{\Theta}_d(N)$ is exceptional and (1) implies the existence of $i \in \mathbb{N}_0$ such that $N \cong P_i(r)$. We fix $n \in \mathbb{N}_0$ minimal subject to $P_n(r)$ being a direct summand of M. Let $m > n$ and assume that $P_m(r)$ is a direct summand of M. We conclude with the Auslander-Reiten formula [ASS06, (IV.2.13)]

$$0 = \dim_{\Bbbk} \mathrm{Ext}^1_{K_r}(P_m(r), P_n(r)) = \dim_{\Bbbk} \mathrm{Hom}_{K_r}(P_{n+2}(r), P_m(r)).$$

Now Lemma 2.7.1 implies $n+2 > m$, i.e. $m = n+1$. The result now follows by applying $\widetilde{\Theta}_d$.

The other implication follows in a similar fashion.

(4) We show the statement on the level of representations and fix vector spaces V_1, V_2 of dimension c and $s+dc$, respectively. Note that we have $\dim_{\Bbbk} V_1 < \dim_{\Bbbk} V_2$. Now [Kac80, Thm 4 (b)] [5] ensures the existence of uniquely determined elements $a, b, n \in \mathbb{N}_0$ such that a generic representation in $N \in \mathrm{rep}(K_r; V_1, V_2)$ is of the form $N \cong aP_n(r) \oplus bP_{n+1}(r) \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$. By Theorem 8.2.5 we have $\mathrm{rk}(\widetilde{\Theta}_d(N)) = \Delta_{(V_1,V_2)}(d) = s$ and $c_1(\widetilde{\Theta}_d(N)) = c\sigma_1$.

□

Remark 8.3.3.

(1) The numerical data in (2) do not imply the indecomposability of $\mathcal{F}$: Let $E \in \mathrm{rep}(K_3)$ be indecomposable of dimension vector $\underline{\dim}\, E = (1,1)$. Since $\underline{\dim}\, \tau^{-1}_{K_3}(E) = (2,5)$, it follows from Theorem 4.2.1 that $\tau^{-1}_{K_3}(E) \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ and hence so is $M \coloneqq \tau^{-1}_{K_3}(E) \oplus P_1(3)$. We have $\underline{\dim}\, M = (3,8) = \underline{\dim}\, P_2(3)$, but $\mathcal{F} \coloneqq \widetilde{\Theta}_1(M)$ is not indecomposable.

(2) In view of Corollary 8.2.6, the functor $\widetilde{\Theta}_d|_{\mathrm{rep}_{\mathrm{proj}}(K_r,d)}$ is exact. For $i \geq 2$, the almost split sequence

$$0 \longrightarrow P_{i-2}(r) \longrightarrow rP_{i-1}(r) \longrightarrow P_i(r) \longrightarrow 0$$

thus induces an exact sequence

$$0 \longrightarrow \mathcal{E}^{\vee}_{i,d} \longrightarrow r\mathcal{E}^{\vee}_{i-1,d} \longrightarrow \mathcal{E}^{\vee}_{i-2,d} \longrightarrow 0$$

of vector bundles. For $d = 1$, this corresponds to the construction given in [Bra05, (2.1)] (with an extra term given by $\mathcal{E}_{-1,d} \coloneqq \mathcal{O}(-1)$). As pointed out in [Bra05, (2.3)], the sequences are thus related to the mutations studied in [GR87].

[5] Using the notation of [Kac80, Thm 4 (b)] we have $\beta_0 = \underline{\dim}\, I_0(r)$ and $\beta_{-i} = \underline{\dim}\, I_i(r)$, $\beta_i = \underline{\dim}\, P_{i-1}(r)$ for all $i \in \mathbb{N}$. Moreover, since $\mathrm{Ext}^1_{K_r}(P_i(r), P_j(r)) = 0$ for all $i,j \in \mathbb{N}_0$ with $|i-j| \leq 1$, [Kac82, Proposition 3] and Theorem 1.5.1 guarantee that [Kac80, Thm 4 (b)] holds in arbitrary characteristic.

8.4. Steiner bundles of small rank

By Theorem 8.2.5 we may view the first Chern class $c_1(\mathcal{F})$ of a Steiner bundle $\mathcal{F}$ as a number $c_1(\mathcal{F}) \in \mathbb{N}_0$. In the sequel, we will freely use this interpretation without further notice.

Next, we employ the functor $\widetilde{\Theta}_d$ to translate some results of Section 5.4 into the context of Steiner bundles.

Proposition 8.4.1. *Let $\mathcal{F} \in \mathrm{StVect}(\mathrm{Gr}_d(A_r))$ be a Steiner bundle.*

(1) Let $c \in \mathbb{N}_0$ and $s \in \mathbb{N}$.

(i) There exists a Steiner bundle of rank s and with first Chern class c if and only if we have $s \geq \min\{c(r-d), d(r-d)\}$.

(ii) There exists an indecomposable (a simple) Steiner bundle of rank s and with first Chern class c if and only if $s \geq \min\{c,d\}(r-d)$ and $q_r(c, s+dc) \leq 1$.

(2) If $c_1(\mathcal{F}) \leq d$ or $\mathrm{rk}(\mathcal{F}) < d(r-d)$, then $\mathcal{F} \cong \mathcal{O}_{\mathrm{Gr}_d(A_r)}^{\mathrm{rk}(\mathcal{F})-(r-d)c_1(\mathcal{F})} \oplus \mathcal{Q}_{(r,d)}^{c_1(\mathcal{F})}$.

(3) If $\mathrm{rk}(\mathcal{F}) = d(r-d)$, then $\mathcal{F}$ is either as in (2), or $\mathcal{F}$ is simple.

(4) If $\mathrm{rk}(\mathcal{F}) = d(r-d)$ and $c_1(\mathcal{F}) \geq d+1$, then $\mathcal{F}$ is simple.

Proof. Theorem 8.2.5 provides $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ such that $\widetilde{\Theta}_d(M) \cong \mathcal{F}$, $\mathrm{rk}(\mathcal{F}) = \Delta_M(d)$ and $c_1(\mathcal{F}) = \dim_{\Bbbk} M_1$.

(1) Statement (i) is a direct consequence of Theorem 5.4.1 and Theorem 8.2.5. Statement (ii) follows in the same fashion using Theorem 5.4.1, Theorem 8.2.5, Theorem 1.5.1 and Proposition 5.1.1.

(2) By assumption we have $\dim_{\Bbbk} M_1 \leq d$ or $\Delta_M(d) < d(r-d)$. By Corollary 5.4.2 and Corollary 5.4.6, the representation M is projective, whence $M \cong \Delta_M(r)P_0(r) \oplus (\dim_{\Bbbk} M_1)P_1(r)$. We have $\Delta_M(r) = \Delta_M(d) - (r-d)\dim_{\Bbbk} M_1 = \mathrm{rk}(\mathcal{F}) - (r-d)c_1(\mathcal{F})$. Moreover, $\widetilde{\Theta}_d(P_0(r)) \cong \mathcal{O}_{\mathrm{Gr}_d(A_r)}$. Since $\mathcal{Q}_{(r,d)}$ is a Steiner bundle, we find $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ such that $\mathcal{Q}_{(r,d)} \cong \widetilde{\Theta}_d(M)$. Recall from Section 8.2 that $\mathrm{rk}(\mathcal{Q}_{(r,d)}) = r-d$ and $c_1(\mathcal{Q}_{(r,d)}) = 1$. Application of Theorem 8.2.5 gives us $\underline{\dim}\, M = (1, r)$ and Corollary 5.4.6 implies $M \cong P_1(r)$.

(3) In view of Corollary 5.4.2, M is projective or a brick. In the former case, $\mathcal{F}$ is as in (2), in the latter case, Theorem 8.2.5 gives $\mathrm{End}_{\mathrm{Gr}_d(A_r)}(\mathcal{F}) \cong \mathrm{End}_{K_r}(M) \cong \Bbbk$, so that $\mathcal{F}$ is simple.

(4) This follows as in (3), using Corollary 5.4.2(2d,2b).

□

Remark 8.4.2. (1) Suppose that $\mathrm{char}(\Bbbk) = 0$. In this case, part (1) was proved in [AM15, (2.4)], using Porteous' formula.

(2) For more results on Steiner bundles of minimal rank, we refer the reader to [BF24, §2.3].

(3) Note that for $d = 1$ we have $\widetilde{\Theta}_1(P_1(r)) \cong \mathcal{Q}_{(r,1)} \cong \mathcal{T}_{\mathbb{P}(A_r)}(-1)$, where $\mathcal{T}_{\mathbb{P}(A_r)}$ denotes the *tangent bundle* (see [Ben16, §6.4]).

9. Homogeneous and uniform Steiner bundles

In this chapter we use the functor

$$\widetilde{\Theta}\colon \operatorname{rep}_{\mathrm{proj}}(K_r,1) \longrightarrow \operatorname{StVect}(\mathbb{P}(A_r))$$

established in Section 8.2 to show that several "wild" phenomena in $\operatorname{Vect}(\mathbb{P}^{r-1})$ already happen in the, at first glance, innocent-looking category of Steiner bundles on $\mathbb{P}^{r-1} \cong \mathbb{P}(A_r)$ for $r \geq 3$.

9.1. Restricting Steiner bundles and Kronecker representations

Let $M \in \operatorname{rep}_{\mathrm{proj}}(K_r,1)$ and $\alpha \in \operatorname{Inj}_{\Bbbk}(A_2,A_r)$. Then $\alpha^*(M) \in \operatorname{rep}_{\mathrm{proj}}(K_2,1)$ and [Wor13b, (4.3)] implies that $\alpha^*(M)$ is preprojective. Hence we find a (uniquely determined) decomposition

$$\alpha^*(M) \cong \bigoplus_{i\geq 0} b_i(\alpha) P_i(2).$$

Recall from Lemma 2.7.3 that we have $b_i(\alpha) = b_i(\beta)$ for all $\beta \in \operatorname{Inj}_{\Bbbk}(A_2,A_r)$ with $\operatorname{im}\alpha = \operatorname{im}\beta$ and obtain a well-defined decomposition

$$M|_{\mathfrak{v}} \coloneqq \bigoplus_{i\geq 0} b_i(\mathfrak{v}) P_i(2)$$

for all $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$ (cf. Lemma 2.7.3). By [BF24, (1.3.3)], there exists a (uniquely determined) sequence $(b_i(M))_{i\in\mathbb{N}_0} \in \mathbb{N}_0^{\mathbb{N}_0}$ such that

$$O_M \coloneqq \{\mathfrak{v} \in \operatorname{Gr}_2(A_r) \mid \forall i \in \mathbb{N}_0\colon b_i(\mathfrak{v}) = b_i(M)\}$$

is a dense open subset of $\operatorname{Gr}_2(A_r)$. We call

$$M_{\mathrm{gen}} \coloneqq \bigoplus_{i\geq 0} b_i(M) P_i(2) \in \operatorname{rep}(K_2)$$

the *generic decomposition* of M and call the closed set $\mathcal{J}_M \coloneqq \operatorname{Gr}_2(A_r) \setminus \mathcal{O}_M$ the set of *jumping lines* of M. We say that $M \in \operatorname{rep}_{\mathrm{proj}}(K_r,1)$ is *uniform*, provided $\mathcal{J}_M = \varnothing$. If M is uniform, we just write

$$M|_{K_2} = \bigoplus_{i\geq 0} b_i(M) P_i(2)$$

to indicate that M has generic decomposition $\bigoplus_{i\geq 0} b_i(M)P_i(2)$ with $\mathcal{J}_M = \varnothing$.

In general, it is difficult to compute the generic decomposition of a given representation. However, we have the following useful criterion.

Proposition 9.1.1. *[BF24, (2.5.5)] Let $0 \neq M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ and assume there are $\mathfrak{v}_0 \in \mathrm{Gr}_d(A_r)$, $c, d, n \in \mathbb{N}_0$ such that*

$$M|_{\mathfrak{v}} \cong cP_n(2) \oplus dP_{n+1}(2),$$

then $M_{\mathrm{gen}} = cP_n(2) \oplus dP_{n+1}(2)$.

The aforementioned considerations are motivated by the definition of homogeneous and uniform vector bundles on $\mathbb{P}^n$ given in [OSS80]. In the following we recall the basic definitions in the context of Steiner bundles on $\mathbb{P}(A_r)$ and explain the connection between uniform (homogeneous) representations and uniform (homogeneous) Steiner bundles.

Given $d \in \{1, \ldots, r\}$, a linear map $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ gives rise to an injective morphism $\hat{\alpha} : \mathbb{P}(A_d) \longrightarrow \mathbb{P}(A_r)$. There results the inverse image functor

$$\hat{\alpha}^* : \mathrm{Coh}(\mathbb{P}(A_r)) \longrightarrow \mathrm{Coh}(\mathbb{P}(A_d)),$$

which is known to be right exact (cf. [GW10, (7.11)]) that sends vector bundles to vector bundles. The additive functor $\hat{\alpha}^*$ commutes with direct sums and the map that sends a fraction $\frac{f}{g}$ of homogeneous polynomials to $\frac{f\circ\alpha}{g\circ\alpha}$ induces isomorphisms $\hat{\alpha}^*(\mathcal{O}_{\mathbb{P}(A_r)}(j)) \cong \mathcal{O}_{\mathbb{P}(A_d)}(j)$ for $j \in \mathbb{Z}$. Moreover, for $\gamma^* \in A_r^*$, the morphism

$$\gamma^* \cdot : \mathcal{O}_{\mathbb{P}(A_r)}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(A_r)} \ ; \ f \mapsto \gamma^* \circ f$$[1]

is being sent to

$$\hat{\alpha}^*(\gamma^* \cdot) : \mathcal{O}_{\mathbb{P}(A_d)}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(A_d)} \ ; \ g \mapsto (\gamma^* \circ \alpha) \circ g.$$

For ease of notation, we will write $\widetilde{\Theta} \coloneqq \widetilde{\Theta}_1 \colon \mathrm{rep}(K_r) \longrightarrow \mathrm{Coh}(\mathbb{P}(A_r))$ and $\widetilde{\Theta}_M \coloneqq \widetilde{\Theta}_{M,1}$.

Lemma 9.1.2. *Let $M \in \mathrm{rep}(K_r)$. For $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ we have*

$$\widetilde{\Theta}(\alpha^*(M)) \cong \hat{\alpha}^*(\widetilde{\Theta}(M))^{2}.$$

Proof. We define $\delta_i \coloneqq \alpha(\gamma_i) \in A_r$ for $i \in \{1, \ldots, d\}$ and choose $\delta_{d+1}, \ldots, \delta_r \in A_r$ such that $\{\delta_1, \ldots, \delta_r\}$ is a basis for A_r. For the corresponding dual bases we obtain

$$\delta_i^* \circ \alpha = \begin{cases} \gamma_i^*, & i \in \{1, \ldots, d\} \\ 0, & \text{else.} \end{cases}$$

We compute the morphism $\hat{\alpha}^*(\widetilde{\Theta}_M)$. Let $U \subseteq \mathbb{P}(A_d)$ be open, $g \in \mathcal{O}_{\mathbb{P}(A_d)}(-1)(U)$ and $m \in M$. In view of our foregoing observations, we obtain[3]

$$\begin{aligned} \hat{\alpha}^*(\widetilde{\Theta}_M)(m \otimes g) &= \sum_{i=1}^{r} M(\delta_i)(m) \otimes \hat{\alpha}^*(\delta_i^* \cdot)(g) = \sum_{i=1}^{d} M(\delta_i)(m) \otimes (\delta_i^* \circ \alpha) \circ g \\ &= \sum_{i=1}^{d} M(\alpha(\gamma_i))(m) \otimes \gamma_i^* \circ g = \widetilde{\Theta}_{\alpha^*(M)}(m \otimes g), \end{aligned}$$

[1] Recall from Section 8.2 and Remark 8.2.3 that elements $g \in \mathcal{O}_{\mathbb{P}(A_r)}(-1)(U)$ are morphisms $g \colon U \longrightarrow A_r$ such that $g(\mathfrak{u}) \in \mathfrak{u}$ for all $\mathfrak{u} \in U \subseteq \mathbb{P}(A_r) = \mathrm{Gr}_1(A_r)$.

[2] Note that $\widetilde{\Theta}$ has two different meanings.

[3] By Remark 8.2.1 the functor does not depend on the choice of bases. Here we consider $\widetilde{\Theta}_M$ with respect to $\{\delta_1, \ldots, \delta_r\}$ and $\{\delta_1^*, \ldots, \delta_r^*\}$.

so that

$$\hat{\alpha}^*(\widetilde{\Theta}_M) = \widetilde{\Theta}_{\alpha^*(M)}\,.$$

In view of the right exactness of $\hat{\alpha}^*$, this implies

$$\hat{\alpha}^*(\widetilde{\Theta}(M)) = \hat{\alpha}^*(\mathsf{Coker}\,\widetilde{\Theta}_M) \cong \mathsf{Coker}\,\hat{\alpha}^*(\widetilde{\Theta}_M) = \mathsf{Coker}\,\widetilde{\Theta}_{\alpha^*(M)} = \widetilde{\Theta}(\alpha^*(M)),$$

as desired. □

Let $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ and $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ be a representation such that $\widetilde{\Theta}(M) \cong \mathcal{F}$. Moreover, let $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$, $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ be such that $\operatorname{im}\alpha = \mathfrak{v}$. Assume that $\alpha^*(M)$ is preprojective. By Lemma 2.7.3 we have $\alpha^*(M) \cong \beta^*(M)$ for all $\beta \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ such that $\operatorname{im}\beta = \operatorname{im}\alpha$. In view of Lemma 9.1.2, we obtain

$$\hat{\alpha}^*(\widetilde{\Theta}(M)) \cong \hat{\beta}^*(\widetilde{\Theta}(M))$$

which allows us to define

$$\mathcal{F}|_{\mathfrak{v}} \coloneqq \hat{\alpha}^*(\widetilde{\Theta}(M)) \cong \widetilde{\Theta}(\alpha^*(M)) \in \mathrm{StVect}(\mathbb{P}(A_d)).$$

Since for $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$ the representation $\alpha^*(M) \in \mathrm{rep}_{\mathrm{proj}}(K_2, 1)$ is always preprojective (cf. [Wor13b, (4.3)]), we obtain:

Corollary 9.1.3. *Let $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ be a Steiner bundle.*

(1) There exists a (uniquely determined) sequence $(b_i(\mathcal{F}))_{i\in\mathbb{N}_0} \in \mathbb{N}_0^{\mathbb{N}_0}$ such that

$$O_{\mathcal{F}} \coloneqq \{\mathfrak{v} \in \mathrm{Gr}_2(A_r) \mid \forall i \in \mathbb{N}_0 \colon b_i(\mathfrak{v}) = b_i(\mathcal{F})\}$$

is a dense open subset of $\mathrm{Gr}_2(A_r)$.

(2) Let $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ such that $\widetilde{\Theta}(M) \cong \mathcal{F}$, $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$ and $M|_{\mathfrak{v}} = \bigoplus_{i\geq 0} b_i(\mathfrak{v})P_i(2)$, then

$$\mathcal{F}|_{\mathfrak{v}} \cong \bigoplus_{i\geq 0} b_i(\mathfrak{v})\mathcal{O}_{\mathbb{P}(A_2)}(i).$$

(3) We have $\widetilde{\Theta}(P_i(2)) \cong \mathcal{O}_{\mathbb{P}(A_2)}(i)$ for all $i \in \mathbb{N}_0$.

(4) Let $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ be such that $\widetilde{\Theta}(M) \cong \mathcal{F}$, then $O_{\mathcal{F}} = O_M$.

Proof. Let $i \in \mathbb{N}_0$, then Theorem 8.2.5 implies $\mathrm{rk}(\widetilde{\Theta}(P_i(2))) = 1$ and $c_1(\widetilde{\Theta}(P_i(2)))$. This already shows that (3) holds (cf. [Ben16, (6.3.1), (7.3.7)]).

We fix $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ such that $\widetilde{\Theta}(M) \cong \mathcal{F}$. For $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$ we obtain a decomposition $M|_{\mathfrak{v}} = \bigoplus_{i\geq 0} b_i(\mathfrak{v})P_i(2)$. Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$ be such that $\operatorname{im}\alpha = \mathfrak{v}$, then $\widetilde{\Theta}(M)|_{\mathfrak{v}} \cong \mathcal{F}|_{\mathfrak{v}}$ and Lemma 9.1.2 implies

$$\widetilde{\Theta}(M)|_{\mathfrak{v}} \cong \hat{\alpha}^*(\widetilde{\Theta}(M)) \cong \widetilde{\Theta}(\alpha^*(M)) \cong \bigoplus_{i\geq 0} b_i(\mathfrak{v})\,\widetilde{\Theta}(P_i(2)) \overset{(3)}{\cong} \bigoplus_{i\geq 0} b_i(\mathfrak{v})\mathcal{O}_{\mathbb{P}(A_2)}(i).$$

Now (1),(2) and (4) follow. □

Remark 9.1.4 (Restriction to projective subspaces). Let $\mathcal{F} \in \operatorname{Vect}(\mathbb{P}(A_r))$ be a vector bundle and $H \subseteq \mathbb{P}(A_r)$ be of the form $H = \operatorname{im}\hat{\alpha}$ for some $\alpha \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$ with $1 \leq d < r$. We consider the inclusion morphism $\iota\colon H \longrightarrow \mathbb{P}(A_r)$ and define the *restriction of $\mathcal{F}$ to H*

$$\mathcal{F}|_H \coloneqq \iota^*(\mathcal{F}),$$

where ι^* denotes the inverse image functor induced by ι. This bundle is not defined on $\mathbb{P}(A_d)$ but on $H \cong \mathbb{P}(A_d)$. However, the isomorphism $\alpha\colon A_d \longrightarrow \operatorname{im}\alpha$ induces an isomorphism of varieties $\mathbb{P}(A_d) \longrightarrow H$ and an exact equivalence of categories $\operatorname{Coh}(\mathbb{P}(A_d)) \longrightarrow \operatorname{Coh}(H)$ that sends $\hat{\alpha}^*(\mathcal{F})$ to $\mathcal{F}|_H$. In particular, $\hat{\alpha}^*(\mathcal{F})$ and $\mathcal{F}|_H$ have the same geometric properties.

Definition. Let $\mathcal{F} \in \operatorname{StVect}(\mathbb{P}(A_r))$ be a Steiner bundle and $\mathcal{F} \cong \widetilde{\Theta}(M)$ with $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$. We call

$$\jmath_{\mathcal{F}} \coloneqq \jmath_M$$

the set of *jumping lines* of $\mathcal{F}$. The bundle $\operatorname{StVect}(\mathbb{P}(A_r))$ is called *uniform*, provided $\jmath_{\mathcal{F}} = \varnothing$. As for uniform representations, we write in this case

$$\mathcal{F}|_{\mathbb{P}(A_2)} = \bigoplus_{i \geq 0} b_i(\mathcal{F})\mathcal{O}_{\mathbb{P}(A_2)}(i)$$

and call this decomposition the *splitting type* of $\mathcal{F}$.

The group $\operatorname{GL}(A_r)$ acts on $\mathbb{P}(A_r)$ via automorphisms and hence on $\operatorname{Coh}(\mathbb{P}(A_r))$: Given $g \in \operatorname{GL}(A_r)$ and $\mathcal{F} \in \operatorname{Coh}(\mathbb{P}(A_r))$, we define

$$(g^*.\mathcal{F})(U) \coloneqq \mathcal{F}(g^{-1}.U)$$

for every open subset $U \subseteq \mathbb{P}(A_r)$. By definition (see [GW10, §7.8]) we have

$$g^*.\mathcal{F} = \widehat{g^{-1}}^*(\mathcal{F})$$

for $g \in \operatorname{GL}(A_r) = \operatorname{Inj}_{\Bbbk}(A_r, A_r)$.

Definition. A coherent sheaf $\mathcal{F} \in \operatorname{Coh}(\mathbb{P}(A_r))$ is called *homogeneous*, provided $g^*.\mathcal{F} \cong \mathcal{F}$ for all $g \in \operatorname{GL}(A_r)$.

Since $\operatorname{GL}(A_r)$ acts transitively on $\operatorname{GL}(A_r)$, Theorem 7.2.1 implies that every homogeneous coherent sheaf is a vector bundle. Moreover, this implies that every homogeneous (Steiner) bundle is uniform.

Corollary 9.1.5. *Let $\mathcal{F}$ be a Steiner bundle and $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ such that $\widetilde{\Theta}(M) \cong \mathcal{F}$.*

(1) The representation M is uniform if and only if $\mathcal{F}$ is uniform.

(2) The representation M is homogeneous if and only if $\mathcal{F}$ is homogeneous.

Proof. (1) This follows from Corollary 9.1.3 since $\jmath_M = \jmath_{\mathcal{F}}$.

(2) Let $g \in \mathrm{GL}(A_r)$, then $g \in \mathrm{GL}(A_r) = \mathrm{Inj}_{\Bbbk}(A_r, A_r)$. In view of $g^*.\mathcal{F} = \widehat{g^{-1}}^*(\mathcal{F})$ for every coherent sheaf $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}(A_r))$, we conclude

$$\widetilde{\Theta}(g.M) = \widetilde{\Theta}((g^{-1})^*(M)) \overset{9.1.2}{\cong} \widehat{g^{-1}}^*(\widetilde{\Theta}(M)) \cong g^*.\widetilde{\Theta}(M).$$

□

Remark 9.1.6. Let $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}(A_r))$ be a homogeneous vector bundle (not necessarily a Steiner bundle). Let $2 \leq d < r$ and $\alpha, \beta \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$. Since $\mathrm{GL}(A_r)$ acts transitively on $\mathrm{Inj}_{\Bbbk}(A_d, A_r)$, we find $g \in \mathrm{GL}(A_r)$ such that $\alpha = g \circ \beta$ and conclude using [GW10, (7.8.8)] that

$$\hat{\alpha}^*(\mathcal{F}) \cong \hat{\alpha}^*(g^*.\mathcal{F}) = \hat{\alpha}^*(\widehat{g^{-1}}^*(\mathcal{F})) = (\widehat{g^{-1} \circ \alpha})^*(\mathcal{F}) = \hat{\beta}^*(\mathcal{F}).$$

Hence we may define

$$\mathcal{F}|_{\mathbb{P}(A_d)} = \hat{\alpha}^*(\mathcal{F}) \in \mathrm{Vect}(\mathbb{P}(A_d))$$

for some $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$. If $\mathcal{F} = \widetilde{\Theta}(M)$ is a Steiner bundle, this definition coincides with the definition given for uniform Steiner bundles (which includes homogeneous Steiner bundles).

9.2. Uniform and homogeneous Steiner bundles

Throughout this section, we assume that $r \geq 3$. Uniform and homogeneous vector bundles on $\mathbb{P}(A_r)$ are well-studied. Recently, Marchesi and Miró-Roig initiated the systematic study of uniform Steiner bundles on $\mathbb{P}(A_r)$ and their splitting types in [MM21].

Definition. Let $k \in \mathbb{N}_0$. We call $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ *uniform of k-type*, provided $\mathcal{F}$ is uniform and

$$\mathcal{F}|_{\mathbb{P}(A_2)} = \bigoplus_{i=0}^{k} b_i(\mathcal{F}) \mathcal{O}_{\mathbb{P}(A_2)}(i)$$

with $b_k(\mathcal{F}) \neq 0$.

Note that for every uniform bundle $\mathcal{F} \neq 0$ there exists a unique k such that $\mathcal{F}$ is of k-type. Steiner bundles of 0-type are just direct sums $\mathcal{O}_{\mathbb{P}(A_r)}$ since the first Chern class does not change under restriction. The following result gives a characterization of uniform Steiner bundles of 1-type and shows in conjunction with Theorem 4.2.2 that each regular component of K_r gives rise to infinitely many non-isomorphic uniform Steiner bundles.

Proposition 9.2.1. *Let $\mathcal{F}$ be a Steiner bundle on $\mathbb{P}(A_r)$ and $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ be such that $\mathcal{F} \cong \widetilde{\Theta}(M)$. The following statements are equivalent.*

(1) The bundle $\mathcal{F}$ is uniform of 1-type.

(2) The representation M is in $\mathrm{rep}_{\mathrm{proj}}(K_r, 2)$ and $\dim_{\Bbbk} M_1 \neq 0$.

In this case $\mathcal{F}|_{\mathbb{P}(A_2)} = (\dim_{\Bbbk} M_1)\mathcal{O}_{\mathbb{P}(A_2)}(1) \oplus \Delta_M(2)\mathcal{O}_{\mathbb{P}(A_2)}$.

Proof. (1) $\Rightarrow$ (2). We write $\mathcal{F}|_{\mathbb{P}(A_2)} = b_0(\mathcal{F})\mathcal{O}_{\mathbb{P}(A_2)} \oplus b_1(\mathcal{F})\mathcal{O}_{\mathbb{P}(A_2)}(1)$ and let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_2, A_r)$. We write $\alpha^*(M) = \bigoplus_{i\geq 0} b_i(\alpha)P_i(2)$. Application of Lemma 9.1.2 yields

$$\bigoplus_{i\geq 0} b_i(\alpha)\mathcal{O}_{\mathbb{P}(A_2)}(i) \cong \bigoplus_{i\geq 0} b_i(\alpha)\widetilde{\Theta}(P_i(2)) \cong \widetilde{\Theta}(\alpha^*(M)) \cong \hat{\alpha}^*(\widetilde{\Theta}(M)) \cong b_0(\mathcal{F})\mathcal{O}_{\mathbb{P}(A_2)} \oplus b_1(\mathcal{F})\mathcal{O}_{\mathbb{P}(A_2)}(1).$$

Hence $b_i(\alpha) = 0$ for $i > 1$ and $b_1(\alpha) \neq 0$. In conclusion, $\alpha^*(M)$ is projective with $\dim_{\Bbbk} M_1 \neq 0$, since $b_1(\alpha) \neq 0$. Now we apply Theorem 3.3.2.

(2) $\Rightarrow$ (1). Follows in the same fashion. □

Homogeneous bundles on $\mathbb{P}(A_r)$ are always uniform. In 1960 Schwarzenberger [Sch61b, (1.2)] raised the question, whether all uniform vector bundles on $\mathbb{P}(A_r) \cong \mathbb{P}^{r-1}$ are homogeneous. The first counterexample was given by Elencwajg [Ele79] in 1979. We begin this section by explaining how our result can be used to construct non-homogeneous Steiner bundles that are 1-uniform. We require the following statement. The proof may be found in [Far11, (2.2), (2.3)]) and Example 2.6.1.

Lemma 9.2.2. *The following statements hold.*

(1) Homogeneous representations are closed under direct sums and summands.

(2) An Auslander-Reiten component contains a homogeneous representation if and only if every representation in the component is homogeneous.

(3) All preprojective and preinjective representations are homogeneous.

By the following result, every non-homogeneous representation gives rise to infinitely many non-isomorphic Steiner bundles that are uniform of 1-type.

Proposition 9.2.3. *Let $r \geq 3$ and $\mathcal{C}$ be a regular component of K_r. The following statements are equivalent.*

(1) $\mathcal{C}$ contains a representation that is not homogeneous.

(2) For each $X \in (M_{\mathcal{C},2} \to)$ the Steiner bundle $\widetilde{\Theta}(X) \in \mathrm{StVect}(\mathbb{P}(A_r))$ is uniform of 1-type and not homogeneous.

Proof. (1) $\Rightarrow$ (2). By Lemma 9.2.2 every representation in $\mathcal{C}$ is not homogeneous. Let $X \in (M_{\mathcal{C},2} \to)$, then Corollary 9.1.5 and Proposition 9.2.1 imply that $\widetilde{\Theta}(M)$ is not homogeneous and uniform of 1-type.

(2) $\Rightarrow$ (1). This follows using Corollary 9.1.5. □

The following result produces uniform and non-homogeneous Steiner bundles by elementary operations.

Lemma 9.2.4. *Let $2 \leq d < r$ and $X \in \mathrm{rep}_{\mathrm{proj}}(K_d, 1)$ be indecomposable and not simple. Then the following statements hold.*

(1) *The representation* $X_{d,r}^- = (\sigma_{K_r}^{-1} \circ \inf)(X) \in \operatorname{rep}(K_r)$ *is indecomposable, quasi-simple, not homogeneous and* $X_{d,r}^- \in \operatorname{rep}_{\operatorname{proj}}(K_r, d-1) \setminus \operatorname{rep}_{\operatorname{proj}}(K_r, d)$.

(2) *For* $3 \leq d$ *and* $N \in (X_{d,r}^- \to)$ *the Steiner bundle* $\widetilde{\Theta}(N)$ *is not homogeneous and uniform of* 1*-type.*

Proof. (1) By Lemma 2.4.3 the representation $X_{d,r}^-$ is indecomposable, regular and [Bis18, (3.2.2)] implies that $X_{d,r}^-$ is quasi-simple. Since $\psi_{\inf(X)} \neq 0$ and $\psi_{\inf(X)}(\gamma_r \otimes -) = 0$, we conclude with Lemma 2.6.2 that $X_{d,r}^-$ can not be homogeneous. It remains to show that $X_{d,r}^- \in \operatorname{rep}_{\operatorname{proj}}(K_r, d-1) \setminus \operatorname{rep}_{\operatorname{proj}}(K_r, d)$. By Proposition 4.1.2 this is equivalent to $\inf(X) \cong \sigma_{K_r}(X_{d,r}^-) \in \operatorname{rep}_{\operatorname{esp}}(K_r, r-(d-1)) \setminus \operatorname{rep}_{\operatorname{esp}}(K_r, r-d)$. We have to show that

$$\{0\} = \bigcap_{b \in \mathfrak{v}} \ker b_{\inf(X)} \subseteq X_1$$

for every $\mathfrak{v} \in \operatorname{Gr}_{r-(d-1)}(A_r)$. Let $\mathfrak{v} \in \operatorname{Gr}_{r-(d-1)}(A_r)$, then we have $\{0\} \neq \mathfrak{v} \cap A_d$ and find $0 \neq a \in \mathfrak{v} \cap A_d$. Since $X \in \operatorname{rep}_{\operatorname{proj}}(K_d, 1)$, we have $\{0\} = ker a_X = \ker a_{\inf(X)}$ and conclude $\{0\} = \bigcap_{b \in \mathfrak{v}} \ker b_{\inf(X)}$. This shows $\inf(X) \in \operatorname{rep}_{\operatorname{esp}}(K_r, r-(d-1))$.

On the other hand, $A_d^\perp \coloneqq \bigoplus_{d+1}^r \Bbbk\gamma_i \in \operatorname{Gr}_{r-d}(A_r)$ satisfies

$$\bigcap_{b \in A_d^\perp} \ker b_{\inf(X)} = X_1 \neq \{0\}$$

and therefore $\inf(X) \notin \operatorname{rep}_{\operatorname{esp}}(K_r, r-d)$.

(2) By (1) the representation $X_{d,r}^-$ is not homogeneous and $d \geq 3$ implies $X_{d,r}^- \in \operatorname{rep}_{\operatorname{proj}}(K_r, d-1) \subseteq \operatorname{rep}_{\operatorname{proj}}(K_r, 2)$. Now we apply Proposition 9.2.3.

□

In [OSS80, (3.3.1)], the authors provide a notion that encompasses those of homogeneity and uniformity. In our notation, the corresponding property for Kronecker representations reads as follows:

Definition. Let $d \in \{1, \ldots, r\}$. We say that a representation $M \in \operatorname{rep}(K_r)$ is *d-homogeneous*, provided $\alpha^*(M) \cong \beta^*(M)$ for all $\alpha, \beta \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$.

Remark 9.2.5.

(1) Note that every d-homogeneous representation belongs to $\operatorname{CR}(K_r, d)$, where $\operatorname{CR}(K_r, r) \coloneqq \operatorname{rep}(K_r)$. For $d = 1$, we have in fact equality. Moreover, M is r-homogeneous if and only if it is homogeneous, while the uniform representations in $\operatorname{rep}_{\operatorname{proj}}(K_r, 1)$ correspond to those that are 2-homogeneous.

(2) If M is d-homogeneous, then we either have $\mathcal{V}(K_r, d)_M = \varnothing$ or $\mathcal{V}(K_r, d)_M = \operatorname{Gr}_d(A_r)$. (Here $\mathcal{V}(K_r, r)_M \coloneqq \varnothing$ in case M is projective and $\mathcal{V}(K_r, r)_M \coloneqq \{A_r\}$ otherwise.)

(3) If $\mathfrak{v} \notin \mathcal{V}(K_r, d)_M$, then
$$M|_{\mathfrak{v}} \cong \Delta_M(d)P_0(d) \oplus (\dim_{\Bbbk} M_1)P_1(d).$$
Consequently, every $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, d)$ is d-homogeneous.

(4) If M is homogeneous, then M is d-homogeneous for all $d \in \{1, \dots, r\}$.

Definition. Let $d \in \{0, \dots, r-1\}$. A vector bundle $\mathcal{F} \in \mathrm{Vect}(\mathbb{P}(A_r))$ is said to be *d-homogeneous*, provided
$$\hat{\alpha}^*(\mathcal{F}) \cong \hat{\beta}^*(\mathcal{F})$$
for all $\alpha, \beta \in \mathrm{Inj}_{\Bbbk}(A_{d+1}, A_r)$.

It follows directly from Lemma 9.1.2 that $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ is d-homogeneous if and only if $\widetilde{\Theta}(M)$ is $(d-1)$-homogeneous.

Lemma 9.2.6. *Let $1 \leq d < r$ and $M \in \mathrm{rep}_1(K_r)$.*

(1) We have
$$\ker \psi_{\mathrm{res}(\sigma_{K_r}^{-1}(M))} \cong \bigcap_{a \in A_d^{\perp}} \ker(a_M).$$

(2) Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ and $g \in \mathrm{GL}(A_r)$ be such that $g \circ \iota = \alpha$. We have
$$\ker \psi_{\alpha^*(\sigma_{K_r}^{-1}(M))} \cong \bigcap_{a \in g^{-\mathrm{tr}}(A_d^{\perp})} \ker(a_M).$$

Proof. (1) Recall that $A_d^{\perp} \coloneqq \bigoplus_{d+1}^{r} \Bbbk\gamma_i$. By definition (see Section 2.3) we have
$$\psi_{\sigma_{K_r}^{-1}(M)} \colon A_r \otimes_{\Bbbk} M_2 \longrightarrow \mathrm{coker}\, \eta_M \ ; \ a \otimes m_2 \mapsto a \otimes m_2 + \mathrm{im}\, \eta_M$$
and $\psi_{\mathrm{res}(\sigma_{K_r}^{-1}(M))} = \psi_{\sigma_{K_r}^{-1}(M)} \circ (\iota \otimes \mathrm{id}_{M_2})$. Direct computation now shows
$$\ker \psi_{\mathrm{res}(\sigma_{K_r}^{-1}(M))} = \eta_M\Big(\bigcap_{i=d+1}^{r} \ker((\gamma_i)_M)\Big).$$
Since $M \in \mathrm{rep}_1(K_r)$, η_M is injective and we conclude
$$\ker \psi_{\mathrm{res}(\sigma_{K_r}^{-1}(M))} \cong \bigcap_{i=d+1}^{r} \ker((\gamma_i)_M) = \bigcap_{a \in A_d^{\perp}} \ker(a_M).$$

(2) We have
$$\alpha^*(\sigma_{K_r}^{-1}(M)) \cong (g \circ \iota)^*(\sigma_{K_r}^{-1}(M)) \cong \iota^*(g^{-1}.\sigma_{K_r}^{-1}(M)) \overset{2.6.2}{\cong} \mathrm{res}(\sigma_{K_r}^{-1}(g^{\mathrm{tr}}.M))$$
Now (1) implies
$$\ker \psi_{\alpha^*(\sigma_{K_r}^{-1}(M))} \cong \ker \psi_{\mathrm{res}(\sigma_{K_r}^{-1}(g^{\mathrm{tr}}.M))} \cong \bigcap_{a \in A_d^{\perp}} \ker(a_{g^{\mathrm{tr}}.M}) = \bigcap_{a \in g^{-\mathrm{tr}}(A_d^{\perp})} \ker(a_M).$$

□

Proposition 9.2.7. *Let $2 \leq d < r$ and $X \in \operatorname{rep}_{\mathrm{proj}}(K_d, 1)$ be indecomposable and not simple. Then $X^-_{d,r} \in \operatorname{rep}_{\mathrm{proj}}(K_r, d-1)$ is $d-1$-homogeneous but not d-homogeneous.*

Proof. By Lemma 9.2.4 we have $X^-_{d,r} \in \operatorname{rep}_{\mathrm{proj}}(K_r, d-1) \setminus \operatorname{rep}_{\mathrm{proj}}(K_r, d)$. Hence $X^-_{d,r}$ is $d-1$-homogeneous and it suffices to show that there is $\mathfrak{u} \in \operatorname{Gr}_d(A_r)$ such that $X^-_{d,r}|_{\mathfrak{u}}$ is projective. As in the proof of Lemma 9.2.4, we consider $\inf(X)$. Let $\mathfrak{v} \in \operatorname{Gr}_{r-d}(A_r)$ such that $\mathfrak{v} \cap A_d \neq \{0\}$. Then

$$\{0\} = \bigcap_{a \in \mathfrak{v}} \ker a_{\inf(X)} \subseteq \inf(X)_1,$$

since $X \in \operatorname{rep}_{\mathrm{proj}}(K_d, 1)$. Since $\mathfrak{v}, A_d^\perp \in \operatorname{Gr}_{r-d}(A_r)$, we find $g \in \operatorname{GL}(A_r)$ such that $g^{-\mathrm{tr}}(A_d^\perp) = \mathfrak{v}$ and define $\alpha := g \circ \iota \in \operatorname{Inj}_{\Bbbk}(A_d, A_r)$. We conclude for $M := \inf(X) \in \operatorname{rep}_1(K_r)$

$$\{0\} = \bigcap_{a \in \mathfrak{v}} \ker a_M = \bigcap_{a \in g^{-\mathrm{tr}}(A_d^\perp)} \ker a_M \overset{9.2.6(2)}{\cong} \ker \psi_{\alpha^*(\sigma_{K_r}^{-1}(\inf(X)))} = \ker \psi_{\alpha^*(X^-_{d,r})}.$$

Let $\mathfrak{u} := \operatorname{im} \alpha \in \operatorname{Gr}_d(A_r)$. We apply Theorem 3.3.2 and conclude that $X^-_{d,r}|_{\mathfrak{u}} \cong \alpha^*(X^-_{d,r})$ is projective. □

9.3. Jumping lines and almost-uniform Steiner bundles

We have already seen that homogeneous vector bundles are always uniform. Recall that uniform vector bundles $\mathcal{F}$ are defined by the property that the set of jumping lines $\mathcal{J}_{\mathcal{F}}$ is empty. A natural generalization, that is due to Ellia [Ell17], is the following weaker property:

Definition. A vector bundle $\mathcal{F} \in \operatorname{Vect}(\mathbb{P}(A_r))$ is called *almost-uniform*, provided $\mathcal{J}_{\mathcal{F}}$ is non-empty and finite.

Using the functor $\widetilde{\Theta}\colon \operatorname{rep}_{\mathrm{proj}}(K_r, 1) \longrightarrow \operatorname{StVect}(\mathbb{P}(A_r))$ we are able to give a handy criterion to check for almost-uniform Steiner bundles.

Proposition 9.3.1. *Let $0 \neq M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$.*

(1) If there is $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$ such that $0 = \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M)$, then

$$M_{\mathrm{gen}} = \Delta_M(2) P_0(2) \oplus (\dim_{\Bbbk} M_1) P_1(2)$$

and

$$\mathcal{J}_{\widetilde{\Theta}(M)} = \mathcal{J}_M = \{\mathfrak{u} \in \operatorname{Gr}_2(A_r) \mid \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{u}), M) \neq 0\}.$$

(2) If $\{\mathfrak{u} \in \operatorname{Gr}_2(A_r) \mid \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{u}), M) \neq 0\}$ is finite and non-empty, then the Steiner bundle $\widetilde{\Theta}(M)$ is almost-uniform and

$$\mathcal{J}_{\widetilde{\Theta}(M)} = \{\mathfrak{u} \in \operatorname{Gr}_2(A_r) \mid \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{u}), M) \neq 0\}.$$

Proof. (1) Since $\operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M) = 0$, we conclude with Theorem 3.3.2 and Proposition 9.1.1 that

$$M_{\mathrm{gen}} = \Delta_M(2)P_0(2) \oplus (\dim_{\Bbbk} M_1)P_1(2)$$

and $O_M = \{\mathfrak{u} \in \mathrm{Gr}_2(A_r) \mid M|_{\mathfrak{u}} = \Delta_M(2)P_0(2) \oplus (\dim_{\Bbbk} M_1)P_1(2)\}$. Another application of Theorem 3.3.2 gives us $O_M = \{\mathfrak{u} \in \mathrm{Gr}_2(A_r) \mid \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{u}), M) = 0\}$. Hence

$$\mathcal{J}_M = \{\mathfrak{u} \in \mathrm{Gr}_2(A_r) \mid \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{u}), M) \neq 0\}.$$

By Corollary 9.1.3 we have $\mathcal{J}_{\widetilde{\Theta}(M)} = \mathcal{J}_M$.

(2) Since $\{\mathfrak{u} \in \mathrm{Gr}_2(A_r) \mid \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{u}), M) \neq 0\} \neq \varnothing$ is finite, there is $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$ such that $\operatorname{Hom}_{K_r}(P_1^-(\mathfrak{v}), M) = 0$. Now we apply (1).

□

A main result of [Ell17] is that for $n \in \mathbb{N}$ there exists an almost-uniform vector bundle on $\mathbb{P}^n$ of rank $2n-1$ with exactly one jumping line. The given examples [Ell17, (Proposition 16)] are precisely the Steiner bundles that one obtains from the representations $P_1^-(\mathfrak{v})$, $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$:

Proposition 9.3.2. *Let $r \geq 3$ and $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$. The Steiner bundle $\widetilde{\Theta}(P_1^-(\mathfrak{v}))$ satisfies*

(1) $\operatorname{rk}(\widetilde{\Theta}(P_1^-(\mathfrak{v}))) = 2(r-1)-1$, *and*

(2) $\mathcal{J}_{\widetilde{\Theta}(P_1^-(\mathfrak{v}))} = \{\mathfrak{v}\}$.

Proof. Let $\mathfrak{u} \in \mathrm{Gr}_2(A_r)$. Recall from Proposition 3.3.5 and Corollary 3.3.6 that $P_1^-(\mathfrak{u})$ is an elementary representation in $\mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ and $\underline{\dim}\, P_1^-(\mathfrak{u}) = (2, 2(r-1)+1)$. By Theorem 8.2.5 the Steiner bundle $\widetilde{\Theta}(P_1^-(\mathfrak{v})) \in \mathrm{StVect}(\mathbb{P}(A_r))$ has rank $2(r-1)-1$. By Theorem 2.6.7 we have $P_1^-(\mathfrak{u}) \cong P_1^-(\mathfrak{w})$ if and only if $\mathfrak{u} = \mathfrak{w}$. Since $\underline{\dim}\, P_1^-(\mathfrak{u}) = \underline{\dim}\, P_1^-(\mathfrak{w})$, we conclude with [KL96, (1.4)] that $\operatorname{Hom}_{K_r}(P_1^-(\mathfrak{u}), P_1^-(\mathfrak{v})) \neq 0$ if and only if $\mathfrak{u} = \mathfrak{v}$. Now we apply Proposition 9.3.1. □

Taking direct sums of the above Steiner bundles shows that every finite subset $\varnothing \neq X \subseteq \mathrm{Gr}_2(A_r)$ can be realized as a set of jumping lines. However, this construction does not give any insight into the "size" of the category consisting of Steiner bundles with X as the set of jumping lines. In the following we make use of a construction of Bongartz (see [Bon81],[ASS06, (VI.2.4)] and [Luk24]) and the process of simplification (see [KL91]) due to Ringel, to show that the category of Steiner bundles with set of jumping lines X corresponds to a wild subcategory in $\mathrm{rep}_{\mathrm{proj}}(K_r, 1)$. We are indebted to Frank Lukas for introducing us to Bongartz' construction (in the general context of hereditary algebras) and explaining his work to us.

Throughout this section we assume that $r \geq 3$ and fix representations $X_1, \ldots, X_n, Y \in \mathrm{rep}(K_r)$ such that $s_i \coloneqq \dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(Y, X_i) \neq 0$ for $i \in \{1, \ldots, n\}$. We stick to the notation introduced in [SY11, §III.3]. We fix short exact sequences

$$(\mathbb{E}_{X_i,j} \colon 0 \longrightarrow X_i \xrightarrow{f_{X_i,j}} E_{X_i,j} \xrightarrow{g_{X_i,j}} Y \longrightarrow 0)_{j=1}^{s_i}$$

such that $([\mathbb{E}_{X_i,j}])_{j=1}^{s_i}$ is a $\Bbbk$-basis of the space of extensions $\operatorname{ext}^1_{K_r}(Y,X_i) \cong \operatorname{Ext}^1_{K_r}(Y,X_i)$ for all $i \in \{1,\dots,n\}$. We define

$$f \coloneqq \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{s_i} f_{X_i,j}\colon \bigoplus_{i=1}^{n} X_i^{s_i} \longrightarrow \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{s_i} E_{X_i,j} \text{ and } g \coloneqq \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{s_i} E_{X_i,j} \longrightarrow \bigoplus_{i=1}^{n} Y^{s_i}$$

and obtain a short exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{n} X_i^{s_i} \xrightarrow{f} \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{s_i} E_{X_i,j} \xrightarrow{g} \bigoplus_{i=1}^{n} Y^{s_i} \longrightarrow 0.$$

We consider the diagonal embedding $\nabla\colon Y \longrightarrow \bigoplus_{i=1}^{n} Y^{s_i}$ and obtain a pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^{n} X_i^{s_i} & \xrightarrow{u} & E & \xrightarrow{v} & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \nabla} & & \\
0 & \longrightarrow & \bigoplus_{i=1}^{n} X_i^{s_i} & \xrightarrow{f} & \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{s_i} E_{X_i,j} & \xrightarrow{g} & \bigoplus_{i=1}^{n} Y^{s_i} & \longrightarrow & 0.
\end{array}$$

We are interested in the upper exact row

$$\mathbb{E}\colon 0 \longrightarrow \bigoplus_{i=1}^{n} X_i^{s_i} \xrightarrow{u} E \xrightarrow{v} Y \longrightarrow 0.$$

Let $\ell \in \{1,\dots,n\}$ and $1 \le j \le s_\ell$. We consider the linear projection

$$\pi_{X_\ell,j}\colon \bigoplus_{i=1}^{n} X_i^{s_i} \longrightarrow X_\ell \in \operatorname{Hom}_{K_r}(\bigoplus_{i=1}^{n} X_i^{s_i}, X_\ell)$$

sending $(a_1,\dots,a_{s_\ell}) \in X_\ell^{s_\ell}$ to a_j and $\pi_{X_\ell,j}(z) = 0$ for all $z \in X_i^{s_i}$ with $\ell \neq i$. By the same token, we define projections $p_{X_\ell,j}\colon \bigoplus_{i=1}^{n}\bigoplus_{k=1}^{s_i} E_{X_i,k} \longrightarrow E_{X_\ell,j}$ and $p_{Y_\ell,j}\colon \bigoplus_{i=1}^{n} Y^{s_i} \longrightarrow Y$. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^{n} X_i^{s_i} & \xrightarrow{u} & E & \xrightarrow{v} & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \nabla} & & \\
0 & \longrightarrow & \bigoplus_{i=1}^{n} X_i^{s_i} & \xrightarrow{f} & \bigoplus_{i=1}^{n}\bigoplus_{k=1}^{s_i} E_{X_i,k} & \xrightarrow{g} & \bigoplus_{i=1}^{n} Y^{s_i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{X_\ell,j}} & & \downarrow{\scriptstyle p_{X_\ell,j}} & & \downarrow{\scriptstyle p_{Y_\ell,j}} & & \\
0 & \longrightarrow & X_\ell & \xrightarrow{f_{X_\ell,j}} & E_{X_\ell,j} & \xrightarrow{g_{X_\ell,j}} & Y & \longrightarrow & 0
\end{array}$$

which gives us the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^{n} X_i^{s_i} & \xrightarrow{u} & E & \xrightarrow{v} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{X_\ell,j}} & & \downarrow{\scriptstyle p_{X_\ell,j}\circ w} & & \| & & \\
0 & \longrightarrow & X_\ell & \xrightarrow{f_{X_\ell,j}} & E_{X_\ell,j} & \xrightarrow{g_{X_\ell,j}} & Y & \longrightarrow & 0.
\end{array}$$

Hence $\mathbb{E}_{X_\ell,j}$ is obtained (up to isomorphism of exact sequences) as the pushout of $\mathbb{E}$ along $\pi_{X_\ell,j}$ (see [SY11, (I.12.21)]). In words of [SY11, (III.3 page 213)] this means $\pi_{X_\ell,j}\mathbb{E} \cong \mathbb{E}_{X_\ell,j}$. We denote by χ_{Y,X_ℓ} the isomorphism of $\Bbbk$-vector spaces

$$\chi_{Y,X_\ell}\colon \operatorname{ext}^1_{K_r}(Y,X_\ell) \longrightarrow \operatorname{Ext}^1_{K_r}(Y,X_\ell)$$

defined in [SY11, (III.3.6)]. We apply $\operatorname{Hom}_{K_r}(-,X_\ell)$ to

$$\mathbb{E}\colon 0 \longrightarrow \bigoplus_{i=1}^{n} X_i^{s_i} \xrightarrow{u} E \xrightarrow{v} Y \longrightarrow 0$$

and denote the connecting homomorphism by

$$\delta\colon \operatorname{Hom}_{K_r}(\bigoplus_{i=1}^{n} X_i^{s_i}, X_\ell) \longrightarrow \operatorname{Ext}^1_{K_r}(Y,X_\ell).$$

Now [SY17, (VII.3.3)] implies that $\delta(\pi_{X_\ell,j}) = \chi_{Y,X_\ell}([\pi_{X_\ell,j}\mathbb{E}]) = \chi_{Y,X_\ell}([\mathbb{E}_{X_\ell,j}])$. Since $([\mathbb{E}_{X_\ell,j}])_{j=1}^{s_\ell}$ is a $\Bbbk$-basis of $\operatorname{ext}^1_{K_r}(Y,X_\ell)$ and χ_{Y,X_ℓ} is an isomorphism of $\Bbbk$-vector spaces, we conclude:

Proposition 9.3.3 (Bongartz' construction)**.** *Let $1 \leq \ell \leq n$. The connecting homomorphism*

$$\delta\colon \operatorname{Hom}_{K_r}(\bigoplus_{i=1}^{n} X_i^{s_i}, X_\ell) \longrightarrow \operatorname{Ext}^1_{K_r}(Y,X_\ell)$$

obtained by applying $\operatorname{Hom}_{K_r}(-,X_\ell)$ to $\mathbb{E}$ is surjective.

Corollary 9.3.4. *In addition, assume that $X_1,\dots,X_n,Y$ are pairwise* Hom*-orthogonal.*

(1) If X_ℓ is a brick, then $\operatorname{Hom}_{K_r}(E,X_\ell) = 0$.

(2) If $X_1,\dots,X_n,Y$ are bricks, then E is a brick that satisfies $\operatorname{Hom}_{K_r}(X_i,E) \neq 0 = \operatorname{Hom}_{K_r}(Y,E)$ and $\operatorname{Hom}_{K_r}(Z,E) = 0$ for every representation Z that is Hom*-orthogonal to all $X_1,\dots,X_n,Y$.*

Proof. (1) We apply $\operatorname{Hom}_{K_r}(-,X_\ell)$ to $\mathbb{E}$ and obtain the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{K_r}(Y,X_\ell) \xrightarrow{v^*} \operatorname{Hom}_{K_r}(E,X_\ell) \longrightarrow \operatorname{Hom}_{K_r}(\bigoplus_{i=1}^{n} X_i^{s_i}, X_\ell) \xrightarrow{\delta} \operatorname{Ext}^1_{K_r}(Y,X_\ell) \longrightarrow 0.$$

Note that $\operatorname{Hom}_{K_r}(\bigoplus_{i=1}^n X_i^{s_i}, X_\ell) \cong \operatorname{Hom}_{K_r}(X_\ell^{s_\ell}, X_\ell)$ is an s_ℓ-dimensional space and therefore has the same dimension as $\operatorname{Ext}^1_{K_r}(Y,X_\ell)$. Since δ is surjective, δ is also injective and v^* is an isomorphism. The statement follows since $\operatorname{Hom}_{K_r}(Y,X_\ell) = 0$.

(2) We have a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_{K_r}(Y,Y) \longrightarrow \operatorname{Hom}_{K_r}(E,Y) \longrightarrow \operatorname{Hom}_{K_r}(\bigoplus_{i=1}^{n} X_i^{s_i}, Y) = 0$$

and conclude $\dim_{\Bbbk} \operatorname{Hom}_{K_r}(E,Y) = 1$. The short exact sequence

$$0 \longrightarrow \operatorname{Hom}_{K_r}(E, \bigoplus_{i=1}^{n} X_i^{s_i}) \longrightarrow \operatorname{Hom}_{K_r}(E,E) \longrightarrow \operatorname{Hom}_{K_r}(E,Y)$$

in conjunction with (1) implies that E is a brick.

Clearly, we have $\operatorname{Hom}_{K_r}(X_i,E) \neq 0$ for all $i \in \{1,\dots,n\}$. Let Z be Hom-orthogonal to every element in $\{X_1,\dots,X_n,Y\}$, then the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{K_r}(Z, \bigoplus_{i=1}^{n} X_i^{s_i}) \longrightarrow \operatorname{Hom}_{K_r}(Z,E) \longrightarrow \operatorname{Hom}_{K_r}(Z,Y)$$

gives us $\operatorname{Hom}_{K_r}(Z,E) = 0$. Finally, let $Z = Y$ and consider a morphism $h \in \operatorname{Hom}_{K_r}(Y,E)$. Then $h \circ v \in \operatorname{End}_{K_r}(E) = \Bbbk \operatorname{id}_E$. Since $\ker v \cong \bigoplus_{i=1}^{n} X_i^{s_i} \neq \{0\}$, we conclude $0 = h \circ v$ and the surjectivity of v gives $h = 0$.

□

Theorem 9.3.5. *Let $r \geq 3$, $1 < d < r$ and $\varnothing \neq X \subseteq \operatorname{Gr}_d(A_r)$ be a finite set. There exists a regular brick $E_X \in \operatorname{rep}(K_r)$ such that*

(1) $\mathcal{V}(K_r,d)_{E_X} = X$,

(2) $\mathcal{V}(K_r,d-1)_{E_X} = \varnothing$, *and*

(3) $\underline{\dim}\, E_X = (|X| \cdot (d(r-d)-1)+1) \cdot (d, rd-1)$.

Proof. Let $\mathfrak{u} \neq \mathfrak{w} \in \operatorname{Gr}_d(A_r)$. By Theorem 2.6.7 and Proposition 3.3.5 the representations $P_1^-(\mathfrak{u})$ and $P_1^-(\mathfrak{w})$ are non-isomorphic elementary representations that have the same dimension vector by construction. We apply [KL96, (1.4)] and conclude that $P_1^-(\mathfrak{u})$ and $P_1^-(\mathfrak{w})$ are Hom-orthogonal bricks. We write $X = \{\mathfrak{v}_1,\dots,\mathfrak{v}_n\}$ and fix $\mathfrak{u} \in \operatorname{Gr}_d(A_r) \setminus X$. We let $X_i \coloneqq P_1^-(\mathfrak{v}_i)$ for all $i \in \{1,\dots,n\}$ and $Y \coloneqq P_1^-(\mathfrak{u})$. Since X_i and Y are regular, we conclude with the Euler-Ringel form and Kac's Theorem

$$-\dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(Y,X_i) = q_r(\underline{\dim}\, Y) < 0,$$

hence $s_i \coloneqq \dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(Y,X_i) \neq 0$. The previous considerations give us a short exact sequence

$$\mathbb{E}\colon\ 0 \longrightarrow \bigoplus_{i=1}^{n} X_i^{s_i} \xrightarrow{f} E_X \xrightarrow{g} Y \longrightarrow 0$$

such that E_X enjoys the properties of the middle term considered in Corollary 9.3.4. In particular, E_X is a brick. Let $\mathfrak{w} \in \operatorname{Gr}_d(A_r) \setminus (X \cup \{\mathfrak{u}\})$, then $P_1^-(\mathfrak{w})$ is Hom-orthogonal to $X_1,\dots,X_n,Y$ and we conclude with Corollary 9.3.4(2) that

$$\operatorname{Gr}_d(A_r) \setminus (X \cup \{\mathfrak{u}\}) \subseteq \{\mathfrak{w} \in \operatorname{Gr}_d(A_r) \mid \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{w}),E_X) = 0\}.$$

Moreover, $\operatorname{Hom}_{K_r}(X_i,E_X) \neq 0 = \operatorname{Hom}_{K_r}(Y,E_X)$ for $i \in \{1,\dots,n\}$ implies

$$\operatorname{Gr}_d(A_r) \setminus X = \{\mathfrak{w} \in \operatorname{Gr}_d(A_r) \mid \operatorname{Hom}_{K_r}(P_1^-(\mathfrak{w}),E_X) = 0\}.$$

We conclude with Theorem 3.3.2 that

$$\mathcal{V}(K_r, d)_{E_X} = \{\mathfrak{m} \in \mathrm{Gr}_d(A_r) \mid \mathrm{Hom}_{K_r}(P_1^-(\mathfrak{m}), E_X) \neq 0\} = X.$$

Recall from Corollary 3.3.6 that $X_1, \ldots, X_n, Y$ are in $\mathrm{rep}_{\mathrm{proj}}(K_r, d-1)$. Since $\mathrm{rep}_{\mathrm{proj}}(K_r, d-1)$ is closed under extensions, we conclude $E_X \in \mathrm{rep}_{\mathrm{proj}}(K_r, d-1)$ and therefore $\mathcal{V}(K_r, d-1)_{E_X} = \varnothing$.

Finally, we compute the dimension vector of E_X. Let $i \in \{1, \ldots, n\}$. Since X_i and Y are Hom-orthogonal and $\underline{\dim}\, X_i = \underline{\dim}\, Y = \underline{\dim}\, P_1^-(\mathfrak{v}) = (d, rd-1)$, we obtain

$$\begin{aligned} s_i &= \dim_{\Bbbk} \mathrm{Ext}^1_{K_r}(Y, X_i) = -q_r(\underline{\dim}\, Y) = -q_r(\sigma_r(d, rd-1)) \\ &= -q_r(1, d) = d(r-d) - 1. \end{aligned}$$

Hence

$$\underline{\dim}\, E_X = |X| \cdot (d(r-d)-1) \cdot \underline{\dim}\, P_1^-(\mathfrak{v}) + \underline{\dim}\, P_1^-(\mathfrak{v}) = (|X| \cdot (d(r-d)-1) + 1) \cdot \underline{\dim}\, P_1^-(\mathfrak{v}).$$

□

Theorem 9.3.6. *Let $r \geq 3$ and $\varnothing \neq X \subseteq \mathrm{Gr}_2(A_r)$ be a finite subset. There exists an abelian and wild subcategory $\mathcal{D}$ of $\mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ such that for every $M \in \mathcal{D}$ the corresponding Steiner bundle $\widetilde{\Theta}(M)$ satisfies $\mathcal{J}_{\widetilde{\Theta}(M)} = X$. In particular, $\mathcal{J}_{\widetilde{\Theta}(M)}$ is almost-uniform.*

Proof. We fix the regular brick E_X constructed in Theorem 9.3.5 and consider the simplification category $\mathcal{E}(\{E_X\})$ (cf. [KL91, Section 1]). Since $E_X \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ and $\mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ is closed under extensions, we conclude $\mathcal{E}(\{E_X\}) \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$. Since E_X is regular, Kac's Theorem implies that $q_r(\underline{\dim}\, E_X) < 0$ and therefore

$$\dim_{\Bbbk} \mathrm{Ext}^1_{K_r}(E_X, E_X) = -q_r(\underline{\dim}\, E_X) + 1 \geq 2.$$

Now the Remark following [KL91, (1.4)] implies that $\mathcal{D} := \mathcal{E}(\{E_X\})$ is a wild subcategory of $\mathrm{rep}_{\mathrm{proj}}(K_r, 1)$. Let $0 \neq M \in \mathcal{E}(\{E_X\}) \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$. By definition we find $n \in \mathbb{N}$ and a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

such that $M_i / M_{i-1} \cong E_X$ for all $i \in \{1, \ldots, n\}$. By Theorem 9.3.5 we have $\mathcal{V}(K_r, 2)_{E_X} = X$ and conclude with Lemma 3.3.1 that $\mathcal{V}(K_r, 2)_M = X$. Now Theorem 3.3.2 implies that $\{\mathfrak{u} \in \mathrm{Gr}_2(A_r) \mid \mathrm{Hom}_{K_r}(P_1^-(\mathfrak{u}), M) \neq 0\} = \mathcal{V}(K_r, 2)_M = X$ is a non-empty finite set and it follows

$$\mathcal{J}_{\widetilde{\Theta}(M)} \overset{9.1.3}{=} \mathcal{J}_M \overset{9.3.1}{=} \{\mathfrak{u} \in \mathrm{Gr}_2(A_r) \mid \mathrm{Hom}_{K_r}(P_1^-(\mathfrak{u}), M) \neq 0\} = X.$$

□

9.4. k-type uniform Steiner bundles

In this section we take a closer look at the k-type of uniform Steiner bundles for $r \geq 3$.

Definition. Let $\mathcal{F}$ be a uniform Steiner bundle on $\mathbb{P}(A_r)$ and

$$\mathcal{F}|_{\mathbb{P}(A_2)} = \bigoplus_{i \geq 0} b_i \mathcal{O}_{\mathbb{P}(A_2)}(i).$$

We call $\operatorname{supp}(\mathcal{F}) := \{i \in \mathbb{N}_0 \mid b_i \neq 0\}$ the *support* of $\mathcal{F}$ and say that $\mathcal{F}$ has *connected splitting type*, provided $\operatorname{supp}(\mathcal{F})$ is an interval in $\mathbb{N}_0$.

The systematic study of splitting types of uniform Steiner bundles has been initiated by Marchesi and Miró-Roig in [MM21]. All uniform Steiner bundles $\mathcal{F}$ of k-type constructed in loc. cit. satisfy $\operatorname{supp}(\mathcal{F}) = \{0, 1, \ldots, k\}$. In particular, they all have connected splitting type. In this section we prove the existence of indecomposable k-uniform Steiner bundles that do not have a connected splitting type and show that there exists no upper bound for the "size of gaps" in the splitting type.

Theorem 9.4.1. *Let $n \in \mathbb{N}_0$.*

(1) There exists a simple and uniform Steiner bundle of $n+1$-type such that $\operatorname{supp}(\mathcal{F}) = \{n, n+1\}$.

(2) For $n \geq 2$, there exists an indecomposable and uniform Steiner bundle $\mathcal{F}$ of $n+1$-type such that $\operatorname{supp}(\mathcal{F}) = \{0, 1, n, n+1\}$. In particular, there is no upper bound for the size of the gaps for indecomposable uniform Steiner bundles.

Proof. (1) For $n = 0$ the statement follows from Theorem 3.3.2. Let $n \geq 1$. We apply Corollary 5.5.4(b) and find a brick $M \in \operatorname{rep}_{\text{proj}}(K_r, 1)$ such that $M|_{\mathfrak{v}}$ is independent of $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$ and given by

$$M|_{\mathfrak{v}} \cong [(n+2)(n+1)(r-2)]P_n(2) \oplus [n(n+1)(r-2)]P_{n+1}(2).$$

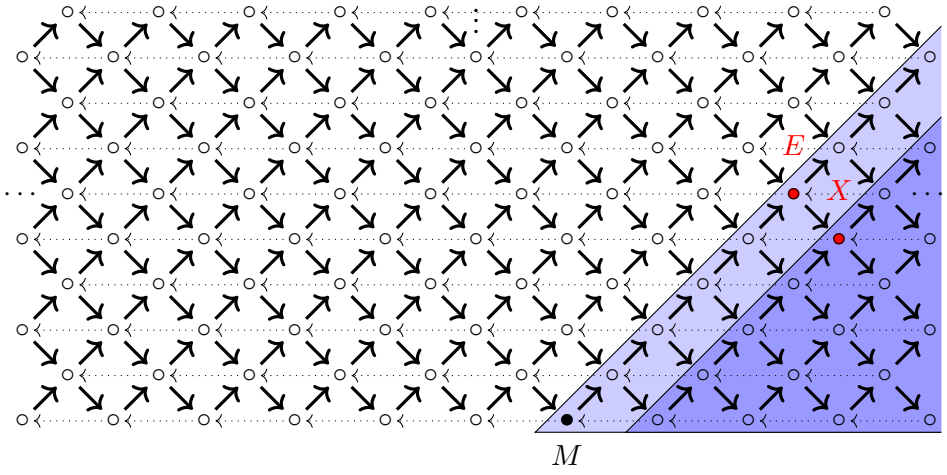
We set $\mathcal{F} := \widetilde{\Theta}(M) \in \operatorname{StVect}(\mathbb{P}(A_r))$ and conclude for $\mathfrak{v} \in \operatorname{Gr}_2(A_r)$

$$\mathcal{F}|_{\mathfrak{v}} \overset{9.1.3}{\cong} [(n+2)(n+1)(r-2)]\mathcal{O}_{\mathbb{P}(A_2)}(n) \oplus [n(n+1)(r-2)]\mathcal{O}_{\mathbb{P}(A_2)}(n+1)$$

with $(n+2)(n+1)(r-2) \neq 0 \neq n(n+1)(r-2)$.

(2) We consider M as in the proof of (1). Then $M \in \operatorname{rep}_{\text{proj}}(K_r, 1) \setminus \operatorname{rep}_{\text{proj}}(K_r, 2)$ is regular (see Theorem 3.3.2 and Theorem 4.2.2) and quasi-simple since M is a brick (see [Ker94, (9.2), (9.4)]).

We choose $\ell \in \mathbb{N}_{\geq 2}$ and set $E := M_{[\ell]}$ as well as $X := \tau_{K_r}^{-1}(M_{[\ell-1]})$. The situation may be illustrated as follows:



We have a short exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow X \longrightarrow 0.$$

By Theorem 4.2.2 we have $X \in \mathrm{rep}_{\mathrm{proj}}(K_r, r-1) \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, 2)$ and Proposition 9.2.1 implies that $\widetilde{\Theta}(X)$ is of 1-type with $\mathrm{supp}(\widetilde{\Theta}(X)) = \{0, 1\}$. Let $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$, then $X|_{\mathfrak{v}}$ is projective and the exact sequence

$$0 \longrightarrow M|_{\mathfrak{v}} \longrightarrow E|_{\mathfrak{v}} \longrightarrow X_{\mathfrak{v}} \longrightarrow 0$$

splits. We define $\mathcal{F} \coloneqq \widetilde{\Theta}(E)$. Twofold application of Corollary 9.1.3 yields

$$\mathcal{F}|_{\mathfrak{v}} \cong \widetilde{\Theta}(E|_{\mathfrak{v}}) \cong \widetilde{\Theta}(M|_{\mathfrak{v}}) \oplus \widetilde{\Theta}(X|_{\mathfrak{v}}) \cong \widetilde{\Theta}(M)|_{\mathfrak{v}} \oplus \widetilde{\Theta}(X)|_{\mathfrak{v}}.$$

Since $\widetilde{\Theta}(M)$ and $\widetilde{\Theta}(X)$ are uniform, we conclude that $\mathcal{F}$ is uniform and

$$\mathrm{supp}(\mathcal{F}) = \mathrm{supp}(\widetilde{\Theta}(M)) \cup \mathrm{supp}(\widetilde{\Theta}(X)) = \{0, 1, n, n+1\}.$$

□

Remark 9.4.2. Let $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ be as constructed in Corollary 5.5.4(a). We have $\underline{\dim}\, M = ((n+1)x, (n+2)(x+r-2))$ with $x \geq (2n^2+3n)(r-2)$. By Proposition 9.2.1 and Theorem 8.2.5 $\widetilde{\Theta}(M)$ is a uniform Steiner bundle of $(n+1)$-type. We have

$$\begin{aligned}\mathrm{rk}(\widetilde{\Theta}(M)) &= (n+2)(x+r-2) - (n+1)x = (n+1)(x+r-2) + (x+r-2) - (n+1)x \\ &= (n+1)(r-2) + (x+r-2) = (n+2)(r-2) + x\end{aligned}$$

and $c_1(\widetilde{\Theta}(M)) = (n+1)x$. In conclusion,

$$\mathrm{rk}(\widetilde{\Theta}(M))(n+1) - c_1(\widetilde{\Theta}(M)) = (n+1)(n+2)(r-2)$$

and

$$\mathrm{rk}(\widetilde{\Theta}(M)) = \frac{c_1(\widetilde{\Theta}(M))}{n+1} + (n+2)(r-2).$$

We also record the following results.

Proposition 9.4.3. *Let $c, s, n \in \mathbb{N}$. The following statements hold.*

(1) If $c \leq n[s-(n+1)(r-2)]$ and $s+c \geq (n+1)(r-1)+1$, then a general[4] Steiner bundle $\mathcal{F}$ of rank s and with first Chern class c satisfies that for each $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$ every direct summand $\mathcal{O}_{\mathbb{P}(A_2)}(i)$ of $\mathcal{F}|_{\mathfrak{v}}$ satisfies $i \leq n$.

(2) If $c \geq n[(n+1)(r-2)+s]$ and $s \geq r-1$, then a general Steiner bundle $\mathcal{F}$ of rank s and with first Chern class c satisfies that for each $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$ every direct summand $\mathcal{O}_{\mathbb{P}(A_2)}(i)$ of $\mathcal{F}|_{\mathfrak{v}}$ satisfies $i \geq n$.

(3) Let $1 \leq d < r$. If $(d-1)c - s < d(r-d)$, $s \geq r-1$ and $\mathcal{F}$ is a Steiner bundle of rank s and with first Chern class c, then there exists $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ such that $\mathcal{F}|_{\mathfrak{v}}$ has $\mathcal{O}_{\mathbb{P}(A_d)}$ as a direct summand.

Proof. We consider a pair of vector spaces such that $\dim_{\Bbbk} V_1 = c$ and $\dim_{\Bbbk} V_2 = s+c$.

(1) We have

$$\dim_{\Bbbk} V_2 = s + c \geq (n+1)(r-1)+1,$$

and

$$\begin{aligned} n\dim_{\Bbbk} V_2 - (n+1)\dim_{\Bbbk} V_1 &= n(s+c) - (n+1)c \\ &= ns - c \geq ns - n[s-(n+1)(r-2)] \\ &= n(n+1)(r-2). \end{aligned}$$

By Lemma 5.5.5(1) we find a non-empty open subset $\mathcal{O} \subseteq \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ of $\mathrm{rep}(K_r; V_1, V_2)$ such that for each $M \in \mathcal{O}$ and every $\mathfrak{v} \in \mathrm{Gr}_2(A_r)$, we have $M|_{\mathfrak{v}} \in \mathrm{add}(P_0(2), \ldots, P_n(2))$. We conclude with Corollary 9.1.3 that $\widetilde{\Theta}(M)|_{\mathfrak{v}} \in \mathrm{add}(\mathcal{O}_{\mathbb{P}(A_2)}, \ldots, \mathcal{O}_{\mathbb{P}(A_2)}(n))$.

(2) We have

$$\begin{aligned} (n+1)\dim_{\Bbbk} V_1 - n\dim_{\Bbbk} V_2 &= (n+1)c - n(c+s) \\ &= c - ns \geq n[(n+1)(r-2)+s] - ns \\ &= n(n+1)(r-2) \end{aligned}$$

and $\dim_{\Bbbk} V_2 - \dim_{\Bbbk} V_1 = s \geq r-1$. The statement follows as in (1) using Lemma 5.5.5(2).

(3) Let $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ with $\underline{\dim}\, M = (c, s+c) \in \mathbb{N}^2$ and $\widetilde{\Theta}(M) \cong \mathcal{F}$. Then $d\dim_{\Bbbk} M_1 - \dim_{\Bbbk} M_2 = (d-1)c - s < d(r-d)$. We conclude with Theorem 5.3.2 that $D_{K_r}(M) \notin \mathrm{rep}_{\mathrm{esp}}(K_r, d)$. This means that there is $\mathfrak{v} \in \mathrm{Gr}_d(A_r)$ such that

$$\psi_M|_{\mathfrak{v}\otimes_{\Bbbk} M_1} \colon \mathfrak{v} \otimes_{\Bbbk} M_1 \longrightarrow M_2$$

is not surjective. Let $\alpha \in \mathrm{Inj}_{\Bbbk}(A_d, A_r)$ be such that $\mathrm{im}\,\alpha = \mathfrak{v}$, then $\alpha^*(M)$ has $P_0(d)$ as a direct summand. Since $\widetilde{\Theta}$ commutes with direct sums, $\mathcal{O}_{\mathbb{P}(A_d)}$ is a direct summand of $\mathcal{F}|_{\mathfrak{v}}$.

□

[4] See Lemma 5.5.5 for the definition.

9.5. Homogeneous Steiner bundles of small rank

In this section we classify homogeneous Steiner bundles of small rank in the case that the field is of characteristic 0.

Lemma 9.5.1. *Let $r \geq 3$. Let $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ be a Steiner bundle and $a \in \mathbb{Z} \setminus \{0\}$, then $\mathcal{F}(a) \notin \mathrm{StVect}(\mathbb{P}(A_r))$.*

Proof. We assume that $\mathcal{F}$ and $\mathcal{F}(a) \in \mathrm{StVect}(\mathbb{P}(A_r))$ are Steiner bundles with $a \geq 1$. We find $M, N \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ such that $\widetilde{\Theta}_1(M) \cong \mathcal{F}$ and $\widetilde{\Theta}_1(N) \cong \mathcal{F}(a)$ and conclude with [Ben16, (7.5.1)] and Theorem 8.2.5 that

$$\dim_{\Bbbk} N_1 = c_1(\mathcal{F}(a)) = c_1(\mathcal{F}) + a \operatorname{rk}(\mathcal{F}) = \dim_{\Bbbk} M_1 + a\Delta_M.$$

Moreover, $\operatorname{rk}(\mathcal{F}) = \operatorname{rk}(\mathcal{F}(a))$ implies that $\Delta_M = \Delta_N$. We conclude $\underline{\dim}\, N = \underline{\dim}\, M + (a\Delta_M, a\Delta_M)$. Since application of $\mathcal{O}_{\mathbb{P}(A_r)}(a) \otimes_{\mathbb{P}(A_r)} -$ induces an auto-equivalence on $\mathrm{Coh}(\mathbb{P}(A_r))$, two-fold application of Corollary 8.2.7(2) yields

$$q_r(\underline{\dim}\, M) = \chi(\mathsf{Hom}_{\mathrm{Gr}_d(A_r)}(\widetilde{\Theta}_d(M), \widetilde{\Theta}_d(M))) = \chi(\mathsf{Hom}_{\mathrm{Gr}_d(A_r)}(\widetilde{\Theta}_d(N), \widetilde{\Theta}_d(N))) = q_r(\underline{\dim}\, N).$$

Hence we obtain for $b \coloneqq a\Delta_M$ and $(x, y) \coloneqq \underline{\dim}\, M$:

$$\begin{aligned} 0 &= q_r(\underline{\dim}\, M) - q_r(\underline{\dim}\, M + (b, b)) = rb^2 - 2b^2 + brx + bry - 2bx - 2by \\ &= b(rb - 2b + rx + ry - 2x - 2y) = b(b + x + y)(r - 2) \\ &= a\Delta_M(a\Delta_M + \dim_{\Bbbk} M_1 + \dim_{\Bbbk} M_2)(r - 2) \neq 0. \end{aligned}$$

This is a contradiction. □

Remark 9.5.2. Note that we need the assumption $r \geq 3$, since $\mathcal{O}_{\mathbb{P}(A_2)}(a)$ is a Steiner bundle for all $a \geq 0$.

By [BE82] the assumptions of the following Lemma are satisfied when $\Bbbk$ is of characteristic zero.

Lemma 9.5.3. *Let $r \geq 3$. Assume that the only homogeneous vector bundles on $\mathbb{P}(A_r) \cong \mathbb{P}^{r-1}$ of rank at most $2r - 3$ are direct sums of the following ones:*

$$S^2\mathcal{T}_{\mathbb{P}(A_3)}(a), \bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)}(b), \mathcal{O}_{\mathbb{P}(A_r)}(c), \mathcal{T}_{\mathbb{P}(A_r)}(d), \Omega^1_{\mathbb{P}(A_r)}(e).$$

Let $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_r))$ be an indecomposable homogeneous Steiner bundle of rank at most $2r - 3$. Then one the following two cases holds:

(1) $r = 3$ and $\mathcal{F} \in \{S^2\mathcal{T}_{\mathbb{P}(A_r)}(-2), \mathcal{O}_{\mathbb{P}(A_r)}, \mathcal{T}_{\mathbb{P}(A_r)}(-1)\}$.

(2) $r \geq 4$ and $\mathcal{F} \in \{\mathcal{O}_{\mathbb{P}(A_r)}, \mathcal{T}_{\mathbb{P}(A_r)}(-1)\}$.

In particular, $\operatorname{rk}(\mathcal{F}) \in \{1, 3, r - 1\}$ and $\operatorname{rk}(\mathcal{F}) = 3$ only happens for $r = 3$ or $r = 4$.

Proof. Clearly, $\mathcal{O}_{\mathbb{P}(A_r)}$ and $\mathcal{T}_{\mathbb{P}(A_r)}(-1)$ (see [Ben16, (6.4.2)]) are Steiner bundles. In view of the assumption and Lemma 9.5.1, it remains to consider the vector bundles

$$S^2\mathcal{T}_{\mathbb{P}(A_3)}(a), \bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)}(b) \text{ and } \Omega^1_{\mathbb{P}(A_r)}(e).$$

By Theorem 8.2.5 and Corollary 9.1.5 we find $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ homogeneous such that $\mathcal{F} \cong \widetilde{\Theta}(M)$ with $\operatorname{rk}(\mathcal{F}) = \Delta_M(1)$. The assumption that $\mathcal{F} \cong \Omega_{\mathbb{P}(A_r)}(e)$ is a Steiner bundle gives us $\Delta_M(1) = \operatorname{rk}(\Omega_{\mathbb{P}(A_r)}(e)) = r - 1$ and M is of minimal type. Since $\Omega_{\mathbb{P}(A_r)} = \mathcal{T}^\vee_{\mathbb{P}(A_r)}$ is exceptional, we conclude with Proposition 8.3.1 that $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ is preprojective and of minimal type. Recall from the proof of Theorem 4.2.2 that for all $\ell \in \mathbb{N}$ we have

$$\Delta_{\sigma^{-\ell}_{K_r}(P_2(r))} > \Delta_{P_2(r)} = r^2 - r - 1 > r - 1.$$

Hence $M \cong P_1(r)$, since $M_1 \neq 0$ and $\mathcal{F} \cong \widetilde{\Theta}_1(M) = \mathcal{T}_{\mathbb{P}(A_r)}(-1)$, a contradiction.

Before we continue with the remaining two cases, we need some preparations:

(i) We have $\mathcal{T}_{\mathbb{P}(A_r)}|_{\mathbb{P}(A_s)} = (r-s)\mathcal{O}_{\mathbb{P}(A_s)}(1) \oplus \mathcal{T}_{\mathbb{P}(A_s)}$ for all $2 \leq s < r$: We have $\mathcal{T}_{\mathbb{P}(A_r)}(-1) \cong \widetilde{\Theta}(P_1(r))$ with $P_1(r) \in \operatorname{rep}_{\mathrm{proj}}(K_r, r-1)$ (cf. Remark 8.4.2 and Theorem 4.2.2). Since $P_1(r)|_{K_s} \cong (r-s)P_0(s) \oplus P_1(s)$, we have

$$\begin{aligned}\mathcal{T}_{\mathbb{P}(A_r)}(-1)|_{\mathbb{P}(A_s)} &\cong \widetilde{\Theta}(P_1(r))|_{\mathbb{P}(A_s)} \overset{9.1.2}{\cong} \widetilde{\Theta}(P_1(r)|_{K_s}) \cong (r-s)\widetilde{\Theta}(P_0(s)) \oplus \widetilde{\Theta}(P_1(s)) \\ &\cong (r-s)\mathcal{O}_{\mathbb{P}(A_s)} \oplus \mathcal{T}_{\mathbb{P}(A_s)}(-1).\end{aligned}$$

Now the statement follows from [GW10, (7.8.8)].

(ii) We have $c_1(S^2\mathcal{T}_{\mathbb{P}(A_3)}(a)) = 9 + 3a$: Thanks to [GW10, (11.1.6)], we have an isomorphism $(S^2\mathcal{T}_{\mathbb{P}(A_3)})|_{\mathbb{P}(A_2)} \cong S^2(\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)})$[5]. By (i) we have $\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)} \cong \mathcal{O}_{\mathbb{P}(A_2)}(1) \oplus \mathcal{T}_{\mathbb{P}(A_2)}$. Moreover, $\operatorname{rk}(\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)}) = 2$ and $3 = c_1(\mathcal{T}_{\mathbb{P}(A_3)}) = c_1(\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)})$ imply $\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)} \cong \mathcal{O}_{\mathbb{P}(A_2)}(1) \oplus \mathcal{O}_{\mathbb{P}(A_2)}(2)$.

There results an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(A_2)}(1) \otimes_{\mathbb{P}(A_2)} \mathcal{O}_{\mathbb{P}(A_2)}(2) \xrightarrow{\iota} \mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)} \otimes_{\mathbb{P}(A_2)} \mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)} \to S^2(\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)}) \to 0,$$

where $\iota(f \otimes g) = f \otimes g - g \otimes f$. Since the middle term of the exact sequence is isomorphic to

$$\mathcal{O}_{\mathbb{P}(A_2)}(2) \oplus (\mathcal{O}_{\mathbb{P}(A_2)}(1) \otimes_{\mathbb{P}(A_2)} \mathcal{O}_{\mathbb{P}(A_2)}(2)) \oplus (\mathcal{O}_{\mathbb{P}(A_2)}(2) \otimes_{\mathbb{P}(A_2)} \mathcal{O}_{\mathbb{P}(A_2)}(1)) \oplus \mathcal{O}_{\mathbb{P}(A_2)}(4),$$

we have $c_1(\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)} \otimes_{\mathbb{P}(A_2)} \mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)}) = 12$ and conclude

$$c_1(S^2\mathcal{T}_{\mathbb{P}(A_3)}) = c_1(S^2\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)}) = c_1(\mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)} \otimes_{\mathbb{P}(A_2)} \mathcal{T}_{\mathbb{P}(A_3)}|_{\mathbb{P}(A_2)}) - 3 = 12 - 3 = 9.$$

Finally, we conclude with [Ben16, (7.5.1)] that $c_1(S^2\mathcal{T}_{\mathbb{P}(A_3)})(a) = 9 + 3a$.

(iii) We have $c_1(\bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)}(b)) = 15 + 6b$: Recall that $\operatorname{rk}(\bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)}) = \binom{4}{2} = 6$, while [Ben16, §6.10] provides an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(A_5)} \longrightarrow \mathcal{O}(1)^5_{\mathbb{P}(A_5)} \longrightarrow \mathcal{O}(2)^{\binom{5}{2}} \longrightarrow \bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)} \longrightarrow 0.$$

[5] Cf. Remark 9.1.6.

The Whitney sum formula gives us

$$c_1(\bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)}) = 20 - 5 = 15$$

and application of [Ben16, (7.5.1)] implies $c_1(\bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)}(b)) = 15 + 6b$.

Now we suppose that $\mathcal{F} \cong S^2\mathcal{T}_{\mathbb{P}(A_3)}(a)$. Then we have $\operatorname{rk}(\mathcal{F}) = 3$. We consider $R \coloneqq K[X_1, X_2, X_3]$ with its natural $\mathbb{N}_0$-grading and let $N = (R_1, R_2, (X_i \cdot)_{1 \leq i \leq 3}) \in \operatorname{rep}(K_3)$. The representation is indecomposable, homogeneous and belongs to $\operatorname{rep}_{\operatorname{proj}}(K_r, 1)$. Consequently, $\widetilde{\Theta}(N) \in \operatorname{StVect}(\mathbb{P}(A_3))$ is an indecomposable and homogeneous Steiner bundle of rank 3. Thus, our general assumption provides $b \in \mathbb{Z}$ such that $\widetilde{\Theta}(N) \cong S^2\mathcal{T}_{\mathbb{P}(A_3)}(b)$. Application of Lemma 9.5.1 implies $a = b$ and Theorem 8.2.5 gives us $M \cong N$, so that $3 = \dim_{\Bbbk} R_1 = \dim_{\Bbbk} N_1 = \dim_{\Bbbk} M_1 = c_1(\mathcal{F}) \overset{(ii)}{=} 9 + 3a$. As a result, $a = -2$.

It remains to show that $\bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)}(b)$ is not a Steiner bundle. Assume to the contrary that $\mathcal{F} \cong \bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)}(b)$ is a (homogeneous) Steiner bundle. Then its restriction $\mathcal{F}|_{\mathbb{P}(A_4)}$ is a homogeneous Steiner bundle and by [GW10, (7.8.8), (7.48)] we have

$$\mathcal{F}|_{\mathbb{P}(A_4)} \cong (\bigwedge^2 \mathcal{T}_{\mathbb{P}(A_5)})|_{\mathbb{P}(A_4)}(b) \cong \bigwedge^2 (\mathcal{T}_{\mathbb{P}(A_5)}|_{\mathbb{P}(A_4)})(b).$$

Recall from (i) that $\mathcal{T}_{\mathbb{P}(A_5)}|_{\mathbb{P}(A_4)} \cong \mathcal{T}_{\mathbb{P}(A_4)} \oplus \mathcal{O}_{\mathbb{P}(A_4)}(1)$, which implies that

$$\bigwedge^2 (\mathcal{T}_{\mathbb{P}(A_5)}|_{\mathbb{P}(A_4)}) \cong (\bigwedge^2 \mathcal{T}_{\mathbb{P}(A_4)}) \oplus \mathcal{T}_{\mathbb{P}(A_4)}(1)$$

is a direct sum of homogeneous bundles on $\mathbb{P}(A_4)$. We conclude

$$\mathcal{F}|_{\mathbb{P}(A_4)} \cong (\bigwedge^2 \mathcal{T}_{\mathbb{P}(A_4)})(b) \oplus \mathcal{T}_{\mathbb{P}(A_4)}(b+1)$$

is a direct sum of homogeneous Steiner bundles and both direct summands have rank $3 \leq 2 \cdot 4 - 3$. For $r = 4$ we have proven above that the only indecomposable homogeneous Steiner bundles are $\mathcal{O}_{\mathbb{P}(A_4)}$ and $\mathcal{T}_{\mathbb{P}(A_4)}(-1)$. Hence we find $m, n \in \mathbb{N}_0$ such that

$$\mathcal{F}|_{\mathbb{P}(A_4)} \cong \bigwedge^2 (\mathcal{T}_{\mathbb{P}(A_5)}|_{\mathbb{P}(A_4)})(b) \cong n\mathcal{O}_{\mathbb{P}(A_4)} \oplus m\mathcal{T}_{\mathbb{P}(A_4)}(-1).$$

Note that $m \in \{0, 1, 2\}$ since $\operatorname{rk}(\mathcal{F}) = 6$ and $\operatorname{rk}(\mathcal{T}_{\mathbb{P}(A_4)}) = 3$. Consequently,

$$m = c_1(\mathcal{F}|_{\mathbb{P}(A_4)}) = c_1(\mathcal{F}) \overset{(iii)}{=} 15 + 6b,$$

which is not possible. □

Theorem 9.5.4. *Assume that the assumptions of Lemma 9.5.3 hold. Let $M \in \operatorname{rep}_{\operatorname{proj}}(K_r, r-1)$ be an indecomposable non-simple representation with $\dim_{\Bbbk} M_1 > r$.*

(1) If $r \geq 4$, $\Delta_M(r-1) \leq 2r - 3$, then M is not homogeneous.

(2) *If $r = 3$ and $\Delta_M(r-1) \leq 2r-4 = 2$, then M is not homogeneous.*

Proof. We assume that M is homogeneous. We apply Proposition 4.1.2 and conclude $N := \sigma_{K_r}(M) \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$. Since $M \in \mathrm{rep}_1(K_r)$, the indecomposable representation N has dimension vector $(r \dim_{\Bbbk} M_1 - \dim_{\Bbbk} M_2, \dim_{\Bbbk} M_1)$. Since M is homogeneous, the same is true for N and $\mathcal{F} := \widetilde{\Theta}(N)$ (see Lemma 2.6.2 and Corollary 9.1.5). We have $\mathrm{rk}(\mathcal{F}) = \Delta_N = \Delta_M(r-1) \leq 2r-3$. In view of Lemma 9.5.3, it suffices to consider the following cases.

(i) If $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}(A_r)}$, we have $N \cong P_0(r)$ and conclude $M \cong P_1(r)$. This is a contradiction since $\dim_{\Bbbk}(P_1(r))_1 = 1 < r$.

(ii) If $\mathcal{F} \cong \mathcal{T}_{\mathbb{P}(A_r)}(-1)$, we get $N \cong P_1(r)$ (see Remark 8.4.2) and $M \cong P_2(r)$. This is a contradiction since $\dim_{\Bbbk}(P_2(r))_1 = a_2(r) = r < \dim_{\Bbbk} M_1$.

(iii) If $r = 3$ and $\mathcal{F} \cong S^2\mathcal{T}_{\mathbb{P}(A_3)}(-2)$, then we have $\underline{\dim}\, N = (3,6)$ and $\underline{\dim}\, M = (6,15)$. We have $\Delta_M(r-1) = 15 - 12 = 3$. This is a contradiction to the assumption in (2).

□

Let $n, c \in \mathbb{N}_{\geq 2}$. In [MM21] the authors study uniform Steiner bundles that are not homogeneous over an algebraically closed field of characteristic 0 and show that each 1-type uniform Steiner bundle $\mathcal{E}$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^c \to \mathcal{O}_{\mathbb{P}^n}^{r+c} \to \mathcal{E} \to 0$$

satisfies $c + 2(n-1) \leq r$ (note that this result follows from Proposition 9.2.1 and Theorem 5.4.1). Then they show that this lower bound is sharp by constructing for all $n \geq 2$ and all $c \geq 2$ a family of Steiner bundles $U_{n,c} \in \mathrm{StVect}(\mathbb{P}^n)$ with and $c_1(U_{n,c}) = c$, $\mathrm{rk}(U_{n,c}) = c + 2(n-1)$. Moreover, they show that each $U_{n,c}$ has the following properties (see [MM21, (4.6)]):

(i) $U_{n,c}$ is uniform of 1-type.

(ii) $U_{n,c}|_{\mathbb{P}^2} \cong U_{2,c} \oplus \mathcal{O}_{\mathbb{P}^2}^{2n-4}$ for a suitable projective plane $\mathbb{P}^2 \subseteq \mathbb{P}^n$.

They then devote section 4 to prove that all $U_{2,c}$ are not homogeneous for $c \geq 4$ and conclude with $(i), (ii)$ that each $U_{n,c}$ is a uniform but not homogeneous Steiner bundle of 1-type.

We give an alternate proof for the last statement that does not rely on the structure of the bundles $U_{2,c}$ but uses Theorem 9.5.4.

Theorem 9.5.5. *Assume that the assumptions of Lemma 9.5.3 hold. Let $c \geq 4$ and $\mathcal{F} \in \mathrm{StVect}(\mathbb{P}(A_3))$ be a uniform Steiner bundle of 1-type with $c_1(\mathcal{F}) = c$, $\mathrm{rk}(\mathcal{F}) = c + 2$. Then $\mathcal{F}$ is not homogeneous.*

Proof. We write $\mathcal{F} = \widetilde{\Theta}(M)$ with $M \in \mathrm{rep}_{\mathrm{proj}}(K_3, 2)$. In view of Corollary 9.1.5, it suffices to show that M is not homogeneous. We have $\underline{\dim}\, M = (c, 2c+2)$ and $\Delta_M(2) = 2c + 2 - 2c = 2 = 2(3-2)$, i.e. M is of minimal type. Since $\dim_{\Bbbk} M_1 = c \geq 4 > 3 = 2 + 1$, we conclude with Corollary 5.4.2 that M is not projective. Another application of Corollary 5.4.2 implies that M is indecomposable. Since $\dim_{\Bbbk} M_1 > r$ and $\Delta_M(2) = 2$, we can apply Theorem 9.5.4 and conclude that M is not homogeneous. □

Remark 9.5.6. Theorem 9.5.5 may also be proven using Van de Ven's Theorem [OSS80, (2.2.2)] for $\mathbb{P}^2(\mathbb{C})$: We assume that M is homogeneous. As before, one checks that M is indecomposable and regular. Hence $\underline{\dim}\,\sigma_{K_r}(M) \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ is indecomposable and homogeneous with dimension vector $(c-2, c)$. Therefore, the Steiner bundle $\widetilde{\Theta}(\sigma_{K_r}(M))$ is an indecomposable uniform Steiner bundle of rank 2 and with first Chern class $c-2$. Hence [OSS80, (2.2.2)] implies that $\mathcal{E}$ is isomorphic to some a-twist of the tangent bundle $\mathcal{T}_{\mathbb{P}^2}$, i.e. $\widetilde{\Theta}(\sigma_{K_r}(M)) \cong \mathcal{T}_{\mathbb{P}^2}(a)$. Using Lemma 9.5.1 one obtains $a = -1$. Hence $\sigma_{K_r}(M) \cong P_1(r)$ and $M \cong P_2(3)$. This is a contradiction since $\underline{\dim}\, P_2(3) = (3, 8) \neq (c, 2c+2)$ as $c \geq 4$.

10. Stability

In this chapter we relate the stability of Steiner bundles on $\mathbb{P}(A_r)$ to that of the corresponding representations of $\mathrm{rep}_{\mathrm{proj}}(K_r, 1)$. Except for some minor changes in notation and references, this chapter has been taken from [BF24, §6].

10.1. Preliminaries

Throughout this section, we assume that $r \geq 3$. Let $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}(A_r))$ be a torsion-free sheaf. By definition, the $\mathcal{O}_{\mathbb{P}(A_r),x}$-module $\mathcal{F}_x$ is torsion-free for every $x \in \mathbb{P}(A_r)$. If $\mathcal{F} \neq 0$, then $\mathrm{rk}(\mathcal{F}) \neq 0$ and we denote by

$$\mu(\mathcal{F}) \coloneqq \frac{c_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$$

the *slope* of $\mathcal{F}$. Recall that $\mathcal{F}$ is called *semistable* (*stable*), provided

$$\mu(\mathcal{G}) \leq \mu(\mathcal{F}) \quad (\mu(\mathcal{G}) < \mu(\mathcal{F}))$$

for every proper subsheaf $0 \subsetneq \mathcal{G} \subsetneq \mathcal{F}$ (such that $\mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$).

Let $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$. As before, write $\widetilde{\Theta}(M) \coloneqq \widetilde{\Theta}_1(M)$ and recall the defining exact sequence

$$0 \longrightarrow \widetilde{M_1} \xrightarrow{\widetilde{\Theta}_M} \widetilde{M_2} \longrightarrow \widetilde{\Theta}(M) \longrightarrow 0,$$

where $\widetilde{M_i} = M_i \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}(A_r)}(i-2)$ for $i \in \{1, 2\}$. We continue to write $\Delta_M \coloneqq \Delta_M(1)$, so that Theorem 5.4.1 yields $\Delta_M \geq r - 1$ in case $M_1 \neq 0$.

We begin by recalling the See-Saw Lemma:

Lemma 10.1.1. *Let $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ be a short exact sequence of non-zero torsion-free coherent sheaves on $\mathbb{P}(A_r)$. Then we have $\mu(\mathcal{E}) \leq \mu(\mathcal{F}) \leq \mu(\mathcal{G})$ or $\mu(\mathcal{G}) \leq \mu(\mathcal{F}) \leq \mu(\mathcal{E})$. Moreover, $\mu(\mathcal{H}) = \mu(\mathcal{F})$ for $\mathcal{H} \in \{\mathcal{E}, \mathcal{G}\}$ implies $\mu(\mathcal{E}) = \mu(\mathcal{F}) = \mu(\mathcal{G})$.*

Proof. The Whitney sum formula (see Theorem B.2.1) implies

$$\mu(\mathcal{F}) = \frac{\mathrm{rk}(\mathcal{E})\mu(\mathcal{E}) + \mathrm{rk}(\mathcal{G})\mu(\mathcal{G})}{\mathrm{rk}(\mathcal{F})}.$$

If $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ and $\mu(\mathcal{F}) < \mu(\mathcal{G})$, the foregoing formula yields

$$\mu(\mathcal{F}) = \frac{\mathrm{rk}(\mathcal{E}) + \mathrm{rk}(\mathcal{G})}{\mathrm{rk}(\mathcal{F})}\mu(\mathcal{F}) < \frac{\mathrm{rk}(\mathcal{E})\mu(\mathcal{E}) + \mathrm{rk}(\mathcal{G})\mu(\mathcal{G})}{\mathrm{rk}(\mathcal{F})} = \mu(\mathcal{F}),$$

a contradiction. Thus, $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$, and the other assertions follow analogously. □

10.2. Stability for representations and Steiner bundles

The main result of this section recasts [Kar93, (5.1)], which was formulated for $\Bbbk = \mathbb{C}$, into our setup.

Definition. For $0 \neq M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$, we define

$$\mu(M) \coloneqq \frac{\dim_{\Bbbk} M_1}{\Delta_M}.$$

We say that M is *stable*, provided $\mu(N) < \mu(M)$ for every subrepresentation $0 \subsetneq N \subsetneq M$. We say that M is *semistable*, provided $\mu(N) \leq \mu(M)$ for every subrepresentation $0 \subsetneq N \subsetneq M$.

Note that the See-Saw Lemma also holds for short exact sequences

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$$

with $M, M', M'' \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ non-zero.

We identify the Chow ring $A(\mathbb{P}(A_r))$ with the truncated polynomial ring $\mathbb{Z}[X]/(X^r)$ (see Appendix B). By Theorem 8.2.5 we have

$$\mu(M) = \mu(\widetilde{\Theta}(M))$$

for every non-zero $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$. In particular, $\mu(\widetilde{\Theta}(M)) \geq 0$ for every such M.

Proposition 10.2.1. *Let $0 \neq M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ be a representation. The following statements hold:*

(1) If $\widetilde{\Theta}(M)$ is semistable, then M is semistable.

(2) If $\widetilde{\Theta}(M)$ is stable, then M is stable.

Proof. Let $0 \subsetneq N \subsetneq M$ be a proper subrepresentation with canonical inclusion $\iota : N \hookrightarrow M$. There results a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{N_1} & \xrightarrow{\widetilde{\Theta}_N} & \widetilde{N_2} & \longrightarrow & \widetilde{\Theta}(N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{\iota}_1} & & \downarrow{\scriptstyle \tilde{\iota}_2} & & \downarrow{\scriptstyle \widetilde{\Theta}(\iota)} & & \\ 0 & \longrightarrow & \widetilde{M_1} & \xrightarrow{\widetilde{\Theta}_M} & \widetilde{M_2} & \longrightarrow & \widetilde{\Theta}(M) & \longrightarrow & 0 \end{array}$$

with exact rows. In view of $\operatorname{Coker} \tilde{\iota}_1 \cong \widetilde{(M/N)_1}$, the Snake Lemma provides a monomorphism $\operatorname{Ker} \widetilde{\Theta}(\iota) \hookrightarrow \widetilde{(M/N)_1}$. By definition, $\widetilde{(M/N)_1} \cong (M_1/N_1) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}(A_r)}(-1)$ is semistable (see [OSS80, (1.2.4)]) and with slope $\mu(\widetilde{(M/N)_1}) = -1$ whenever $\widetilde{(M/N)_1} \neq 0$.

Since $\operatorname{Coh}(\mathbb{P}(A_r))$ is an abelian category (see [GW10, §7.19]), there is an exact sequence (see also [GW10, (7.3.2)])

$$(*) \qquad 0 \longrightarrow \operatorname{Ker} \widetilde{\Theta}(\iota) \longrightarrow \widetilde{\Theta}(N) \longrightarrow \operatorname{Im} \widetilde{\Theta}(\iota) \longrightarrow 0$$

and $\widetilde{\Theta}$ being faithful entails $\operatorname{Im} \widetilde{\Theta}(\iota) \neq 0$.

(1) Suppose that $\widetilde{\Theta}(M)$ is semistable. If $\operatorname{Ker}\widetilde{\Theta}(\iota) = 0$, we have $\mu(N) = \mu(\widetilde{\Theta}(N)) = \mu(\operatorname{Im}\widetilde{\Theta}(\iota)) \leq \mu(\widetilde{\Theta}(M)) = \mu(M)$. Alternatively, we have $0 \neq \operatorname{Ker}\widetilde{\Theta}(\iota) \hookrightarrow \widetilde{(M/N)}_1$ and conclude $N_1 \neq M_1$. Hence the above considerations show $\mu(\operatorname{Ker}\widetilde{\Theta}(\iota)) \leqq \mu(\widetilde{M/N}) = -1 < \mu(\widetilde{\Theta}(N))$ and Lemma 10.1.1 in conjunction with $(*)$ implies $\mu(N) = \mu(\widetilde{\Theta}(N)) < \mu(\operatorname{Im}\widetilde{\Theta}(\iota)) \leq \mu(\widetilde{\Theta}(M)) = \mu(M)$. Hence M is semistable.

(2) Suppose that $\widetilde{\Theta}(M)$ is stable. Then M is indecomposable by the See-Saw Lemma and the assumption $N_1 = M_1$ either implies $M = P_0(r) = N$, or $N_2 \subseteq M_2 = \operatorname{im}\psi_M = \operatorname{im}\psi_N \subseteq N_2$, a contradiction to $M \neq N$. Hence we may assume that $N_1 \subsetneq M_1$. If $\operatorname{rk}(\widetilde{\Theta}(N)) \geq \operatorname{rk}(\widetilde{\Theta}(M))$, we obtain $\Delta_N \geq \Delta_M$, whence $\mu(N) = \frac{\dim_{\Bbbk} N_1}{\Delta_N} < \frac{\dim_{\Bbbk} M_1}{\Delta_M} = \mu(M)$. We may therefore assume that $\operatorname{rk}(\widetilde{\Theta}(N)) < \operatorname{rk}(\widetilde{\Theta}(M))$.

If $\operatorname{Ker}\widetilde{\Theta}(\iota) = 0$, we have $\operatorname{rk}(\operatorname{Im}\widetilde{\Theta}(\iota)) = \operatorname{rk}(\widetilde{\Theta}(N)) < \operatorname{rk}(\widetilde{\Theta}(M))$ as well as $\mu(N) = \mu(\widetilde{\Theta}(N)) = \mu(\operatorname{Im}\widetilde{\Theta}(\iota)) < \mu(\widetilde{\Theta}(M)) = \mu(M)$. Alternatively, we may proceed as in (1).

□

Remark 10.2.2. There is a slope μ, defined for non-zero, torsion-free coherent sheaves of $\operatorname{Gr}_d(A_r)$, such that

$$\mu(\widetilde{\Theta}_d(M)) = \mu(M)$$

for all $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d) \setminus \{0\}$. In view of [Bis04], the bundle $\mathcal{U}_{(r,d)}$ is semistable relative to this slope, so that the foregoing result may be generalized to this context.

Lemma 10.2.3. *Let $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, d)$ be non-projective and of minimal type. Then M is stable.*

Proof. Let $N \subsetneq M$ be a proper subrepresentation. We shall show that $\mu(N) < \mu(M)$. Decomposing

$$N = \bigoplus_{i=1}^{n} N_i$$

into its indecomposable constituents, Lemma 10.1.1 and an easy induction on n imply $\mu(N) \leq \max_{1\leq i\leq n}\mu(N_i)$. Hence we may assume N to be indecomposable.

If N is projective, then we have $N = (\dim_{\Bbbk} N_1)P_0(r) \oplus \Delta_N(r)P_1(r)$ and conclude $\mu(N) \leq \frac{1}{r-1}$. Assuming $\mu(N) \geq \mu(M)$, we have $\Delta_M \geq (r-1)\dim_{\Bbbk} M_1$ and conclude $d(r-d)+(d-1)\dim_{\Bbbk} M_1 = \Delta_M \geq (r-1)\dim_{\Bbbk} M_1$, so that $(r-d)\dim_{\Bbbk} M_1 \leq d(r-d)$. Hence $\dim_{\Bbbk} M_1 \leq d$, and Corollary 5.4.6 shows that M is projective, a contradiction.

We may thus assume that N is not projective, so that Corollary 5.4.2(1) implies $\Delta_N(d) \geq \Delta_M(d)$. In view of Corollary 5.4.2(2), the non-projective representation M is indecomposable. The assumption $N_1 = M_1$ thus entails $N_2 = \psi_M(A_r \otimes_{\Bbbk} N_1) = \psi_M(A_r \otimes_{\Bbbk} M_1) = M_2$, a contradiction. Consequently,

$$\begin{aligned}
\dim_{\Bbbk} N_1\Delta_M &= (\dim_{\Bbbk} N_1)(\Delta_M(d) + (d-1)\dim_{\Bbbk} M_1) \\
&= \dim_{\Bbbk} N_1\Delta_M(d) + (d-1)\dim_{\Bbbk} M_1 \dim_{\Bbbk} N_1 \\
&< \dim_{\Bbbk} M_1\Delta_N(d) + (d-1)\dim_{\Bbbk} M_1 \dim_{\Bbbk} N_1 \\
&= (\dim_{\Bbbk} M_1)(\Delta_N(d) + (d-1)\dim_{\Bbbk} N_1) \\
&= \dim_{\Bbbk} M_1\Delta_N,
\end{aligned}$$

whence $\mu(N) < \mu(M)$. As a result, M is stable. □

Remark 10.2.4. For $d = 1$ and $\mathrm{char}(\Bbbk) = 0$, the foregoing result is a consequence of Proposition 10.2.1(2) and [BS92, (2.7)].

10.3. The distribution of slopes

For $N \in \mathrm{rep}(K_r)$ with $N_1 \neq 0$ we put

$$\nu(N) := \frac{\dim_\Bbbk N_2}{\dim_\Bbbk N_1}$$

and note that

$$\mu(M) = \frac{1}{\nu(M) - 1}$$

for every $M \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ such that $\dim_\Bbbk M_1 \neq 0$. Given $r \geq 3$, recall from Section 1.5 that $L_r := \frac{r+\sqrt{r^2-4}}{2}$. We have $\frac{1}{L_r} = \frac{r-\sqrt{r^2-4}}{2}$ and $r - L_r = \frac{1}{L_r}$.

Proposition 10.3.1. *Let $M, N \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ be indecomposable with $M \ncong P_0(r)$.*

(1) We have $\lim_{n\to\infty} \nu(\sigma_{K_r}^{-n}(M)) = L_r$.

(2) The sequence $(\mu(\sigma_{K_r}^{-n}(M)))_{n\in\mathbb{N}_0}$ of slopes converges to $\frac{1}{L_r - 1}$.

(3) If M is regular, then the sequence $(\mu(\sigma_{K_r}^{-n}(M)))_{n\in\mathbb{N}_0}$ is strictly decreasing.

(4) If M is regular and N is preprojective, then $\mu(N) < \frac{1}{L_r - 1} < \mu(M)$.

Proof. We write $(x_n, y_n) := \underline{\dim}\, \sigma_{K_r}^{-n}(M)$ for all $n \in \mathbb{N}_0$ and recall that

$$\underline{\dim}\, \sigma_{K_r}^{-1}(X) = (\dim_\Bbbk X_2, r \dim_\Bbbk X_2 - \dim_\Bbbk X_1)$$

for each representation X without $S(1)$ as a direct summand. Theorem 4.2.2 yields $\sigma_{K_r}^{-n}(M) \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ and $\sigma_{K_r}^{-n}(M)$ is regular or preprojective with $x_n \neq 0$. For $n \in \mathbb{N}_0$ we set

$$d_n(M) := \nu(\sigma_{K_r}^{-n}(M))$$

and note that $\sigma_{K_r}^{-n}(M) \in \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ implies $d_n(M) > 1$ (see Theorem 5.4.1).

(1) We first consider the case where M is preprojective. Kac's Theorem implies

$$0 < q_r(\underline{\dim}\, \sigma_{K_r}^{-n}(M)) = q_r(x_n, y_n) = x_n^2 + y_n^2 - r x_n y_n,$$

whence

$$(*) \qquad d_n(M) = \frac{y_n}{x_n} > r - \frac{x_n}{y_n} = \frac{r y_n - x_n}{y_n} = d_{n+1}(M) > 1.$$

Now let M be regular and recall from Section 1.5 that $f_r := X^2 - rX + 1 \in \mathbb{R}[X]$ has roots $L_r, \frac{1}{L_r}$. As $r \geq 3$, Kac's Theorem yields $0 > x_n^2 + y_n^2 - r x_n y_n$, so that $f(d_n(M)) < 0$ and $d_n(M) < L_r$ (see also Corollary 1.5.2). Moreover, the computation of $(*)$ implies $d_n(M) < d_{n+1}(M) < L_r$.

We have shown that the sequence $(d_n(M))_{n\in\mathbb{N}_0}$ converges in both cases and the identity

$$d_{n+1}(M) = \frac{ry_n - x_n}{y_n} = r - \frac{1}{d_n(M)}$$

implies that $d := \lim_{n\to\infty} d_n(M)$ is a root of f. Since $\frac{1}{L_r} < 1 < d_n(M)$ for all $n \in \mathbb{N}_0$, it follows that $d = L_r$.

(2) We have $\mu(\sigma_{K_r}^{-n}(M)) = \frac{1}{d_n(M)-1}$. Since $d_n(M) > 1$ for all $n \in \mathbb{N}_0$, we conclude with (1) that $(\mu(\sigma_{K_r}^{-1}(M)))_{n\in\mathbb{N}_0}$ converges to $\frac{1}{L_r-1}$.

(3) Let M be regular. As noted in the proof of (1), the sequence $(d_n(M))_{n\in\mathbb{N}_0}$ is strictly increasing, so that the identity $\mu(\sigma_{K_r}^{-n}(M)) = \frac{1}{d_n(M)-1}$ implies the assertion.

(4) Since M is regular, we have $1 < d_0(M) < L_r$ and therefore $\mu(M) = \frac{1}{d_0(M)-1} > \frac{1}{L_r-1}$. If $N = P_0(r)$, then $\mu(N) = 0$. Alternatively, $d_0(N)$ is well-defined and $q_r(\underline{\dim}\, N) > 0$ yields $d_0(N) > L_r$ (see Corollary 1.5.2). This shows $\mu(N) = \frac{1}{d_0(N)-1} < \frac{1}{L_r-1}$.

□

We continue by recording three important consequences.

Corollary 10.3.2. *Let $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ be indecomposable.*

(1) The representation M is (semi)stable if and only if $\sigma_{K_r}^{-1}(M) \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ is (semi)stable.

(2) The following statements are equivalent:

(1) All representations in $\{\sigma_{K_r}^{z}(M) \mid z \in \mathbb{Z}\} \cap \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ are (semi)stable[1].

(2) The representation M is (semi)stable.

(3) If M is regular and semistable, then M is quasi-simple.

Proof. Let $Z \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$. Recall that we have $\sigma_{K_r}^{-1}(Z) \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ and

$$\underline{\dim}\, \sigma_{K_r}^{-1}(Z) = (\dim_{\Bbbk} Z_2, r\dim_{\Bbbk} Z_2 - \dim_{\Bbbk} Z_1).$$

Hence we obtain for $U, V \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$:

$$(*) \quad \mu(\sigma_{K_r}^{-1}(U)) \leq (<)\mu(\sigma_{K_r}^{-1}(V)) \Leftrightarrow \dim_{\Bbbk} U_1 \dim_{\Bbbk} V_2 \leq (<) \dim_{\Bbbk} V_1 \dim_{\Bbbk} U_2 \Leftrightarrow \mu(U) \leq (<)\mu(V).$$

(1) Since $\mu(P_0(r)) = 0 < \mu(P_1(r))$ and $\sigma_{K_r}^{-1}(P_\ell(r)) = P_{\ell+1}(r))$ for all $\ell \in \mathbb{N}_0$, we obtain $\mu(P_\ell(r)) < \mu(P_{\ell+1}(r))$ for all $\ell \in \mathbb{N}_0$. Clearly, $P_0(r)$ is stable. Now we fix $\ell \in \mathbb{N}$ and let $0 \subsetneq X \subsetneq P_\ell(r)$ be a subrepresentation. We fix a direct summand $Y \subseteq X$ of maximal slope. As in the proof of Lemma 10.2.3 one checks that $\mu(Y) \geq \mu(X)$. Since the only proper indecomposable subrepresentations of $P_\ell(r)$ are (up to isomorphism) $P_0(r), \ldots, P_{\ell-1}(r)$, we conclude $\mu(P_\ell(r)) > \mu(P_{\ell-1}(r)) \geq \mu(Y) \geq \mu(X)$. Hence every preprojective indecomposable representation is stable and we only have to consider the case that M is regular.

[1] For $M = P_\ell(r)$ we only consider $z \in \mathbb{Z}_{\geq -\ell}$.

At first we assume that M is stable. Let $0 \subsetneq X \subsetneq N := \sigma_{K_r}^{-1}(M)$, and assume that X is indecomposable. If X is preprojective, then Proposition 10.3.1(4) implies $\mu(X) < \mu(N)$. Alternatively, X is regular. Since $\sigma_{K_r}\colon \operatorname{rep}_2(K_r) \longrightarrow \operatorname{rep}_1(K_r)$ is exact, there is a proper inclusion $0 \neq \sigma_{K_r}(X) \hookrightarrow M$. As M is stable, we obtain $\mu(\sigma_{K_r}(X)) < \mu(M)$ and $(*)$ implies $\mu(X) \leq \mu(N)$. If X is not indecomposable, we consider an indecomposable summand $Y \subseteq X$ of maximal slope and argue as in the preprojective case. This shows that $\sigma_{K_r}^{-1}(M)$ is stable.

Now we assume that $\sigma_{K_r}^{-1}(M)$ is stable and let $0 \subsetneq X \subsetneq M$ be a subrepresentation of maximal slope. We show that $\mu(X) < \mu(M)$. If $M/X \in \operatorname{add}(I_0(r))$, then we find $a \in \mathbb{N}$ such that $\underline{\dim}\, X = (\dim_{\Bbbk} M_1 - a, \dim_{\Bbbk} M_2)$ and conclude with $\dim_{\Bbbk} M_2 \neq 0$ that

$$\mu(X) = \frac{\dim_{\Bbbk} M_1 - a}{\Delta_M + a} < \frac{\dim_{\Bbbk} M_1}{\Delta_M} = \mu(M).$$

Hence we may assume that $M/X = Q \oplus aI_0(r)$ with $a \in \mathbb{N}_0$ and $Q \in \operatorname{rep}_1(K_r)$ non-zero. We find a subrepresentation $0 \subsetneq Y \subsetneq M$ with $\underline{\dim}\, Y = \underline{\dim}\, X + (a, 0)$ and maximality of $\mu(X)$ implies

$$\frac{\dim_{\Bbbk} X_1 + a}{\Delta_X - a} = \mu(Y) \leq \mu(X) = \frac{\dim_{\Bbbk} X_1}{\Delta_X}.$$

This can only happen if $a = 0$. Hence we have a short exact sequence

$$0 \longrightarrow X \longrightarrow M \longrightarrow Q \longrightarrow 0$$

in $\operatorname{rep}_1(K_r)$. Since $\sigma_{K_r}^{-1} : \operatorname{rep}_1(K_r) \longrightarrow \operatorname{rep}_2(K_r)$ is exact (cf. Section 2.3), we obtain a proper inclusion $0 \neq \sigma_{K_r}^{-1}(X) \hookrightarrow \sigma_{K_r}^{-1}(M)$ and $\sigma_{K_r}^{-1}(M)$ being stable in conjunction with $(*)$ implies that $\mu(X) < \mu(M)$. Hence M is stable. The arguments concerning semistability are analogous.

(2) This is a direct consequence of (1).

(3) We write $M = N_{[\ell]}$ with quasi-socle N and assume that $\ell \geq 2$. General AR-theory provides a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow \tau_{K_r}^{-1}(N_{[\ell-1]}) \longrightarrow 0$$

in $\operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ and the See-Saw Lemma for representations gives us $\mu(N) \leq \mu(M) \leq \mu(\tau_{K_r}^{-1}(N_{[\ell-1]}))$. Moreover, $N_{[\ell-1]}$ being a subrepresentation of $M = N_{[\ell]}$ and Proposition 10.3.1(3) imply

$$\mu(M) \leq \mu(\tau_{K_r}^{-1}(N_{[\ell-1]})) < \mu(N_{[\ell-1]}) \leq \mu(M),$$

a contradiction. Hence $\ell = 1$ and M is quasi-simple.

□

Remark 10.3.3. (1) Owing to Corollary 10.3.2, a cone of a regular component that is contained in $\operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ either contains no (semi)stable representations or all of its quasi-simple vertices are (semi)stable. Moreover, all indecomposable preprojective representations are stable.

(2) Let $r \geq 3$ and B_1, B_2 be non-isomorphic indecomposable representations with dimension vector $(1,1)$. Clearly, B_1, B_2 are elementary representations and by [KL96, (1.4)] B_1 and B_2 are Hom-orthogonal. We conclude $\dim_{\Bbbk} \operatorname{Ext}^1_{K_r}(B_2, B_1) = \dim_{\Bbbk} \operatorname{Hom}_{K_r}(B_2, B_1) - q_r(1,1) = r - 2 > 0$. In particular, we find a non-split short exact sequence

$$(*) \quad 0 \longrightarrow B_1 \longrightarrow N \longrightarrow B_2 \longrightarrow 0$$

and conclude with [ASS06, (VIII.2.8)] $\dim_{\Bbbk} \operatorname{End}_{K_r}(N) < \dim_{\Bbbk} \operatorname{End}_{K_r}(B_1 \oplus B_2) = 2$. Hence N is a brick with dimension vector $(2,2)$ and regular by Kac's Theorem. Since regular bricks for K_r are quasi-simple (see [Ker94, (9.2), (9.4)]), we conclude that every quasi-simple vertex in the regular component $\mathcal{C}$ of N is a brick.

There exists $n \in \mathbb{N}$ such that $\tau_{K_r}^{-n}(B_1), \tau_{K_r}^{-n}(B_2), \tau_{K_r}^{-n}(N) \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$. Recall that $\tau_{K_r}^{-n}$ is an exact endo-functor of the category of regular representations. In view of $(*)$, there results an exact sequence

$$0 \longrightarrow \tau_{K_r}^{-n}(B_1) \longrightarrow \tau_{K_r}^{-n}(N) \longrightarrow \tau_{K_r}^{-n}(B_2) \longrightarrow 0,$$

whose (nonzero) extreme terms have equal dimension vectors and hence equal slopes. The See-Saw Lemma now shows that $\tau_{K_r}^{-n}(N)$ is not stable. By Corollary 10.3.2(2), $\mathcal{C} \cap \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ does not contain any stable vertices.

Proposition 10.3.4. *Let $E \in \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$ be elementary. Then E is stable.*

Proof. Let $0 \subsetneq X \subsetneq E$ be a proper indecomposable subrepresentation. In view of Proposition 10.3.1, we may assume that X is regular. Thanks to Proposition 6.1.1, the factor module E/X is preinjective. Consequently, $\Delta_{E/X} < 0$, so that $\Delta_E = \Delta_X + \Delta_{E/X} < \Delta_X$. As $\dim_{\Bbbk} X_1 \neq 0$, we obtain

$$\mu(X) < \frac{\dim_{\Bbbk} X_1}{\Delta_E} \leq \mu(E).$$

As in the proof of Corollary 10.3.2, this implies the assertion. □

10.4. Filtrations

As before, we assume that $r \geq 3$. Recall that the invertible map

$$\sigma_r \colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \,;\, (x,y) \mapsto (rx - y, x)$$

describes the σ_{K_r}-orbits on the level of the Grothendieck group $K_0(\operatorname{rep}(K_r))$: For $M \in \operatorname{rep}_2(K_r)$ we have $\underline{\dim}\, \sigma_{K_r}(M) = \sigma_r(\underline{\dim}\, M)$, while $\underline{\dim}\, \sigma_{K_r}^{-1}(N) = \sigma_r^{-1}(\underline{\dim}\, N)$ for $N \in \operatorname{rep}_1(K_r)$. We denote by $\lfloor\ \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$ the floor function.

Lemma 10.4.1. *Let $(a,b) \in \mathbb{N}^2 \setminus \{0\}$ be such that $q_r(a,b) < 0$ and $b = \lfloor aL_r \rfloor$. Let $M \in \operatorname{rep}(K_r)$ be an indecomposable representation with $\underline{\dim}\, M = \sigma_r^{-\ell}(a,b)$ for some $\ell \in \mathbb{N}$. Then M is regular and there is a filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

such that for each $i \in \{1, \ldots, n\}$ the filtration factor M_i/M_{i-1} is elementary and belongs to $\operatorname{rep}_{\mathrm{proj}}(K_r, 1)$.

Proof. As $q_r(\underline{\dim}\, M) = q_r(a,b) < 0$, Kac's Theorem ensures that the representation M is regular. We define $N \coloneqq \sigma_{K_r}^{\ell}(M)$ and note that $\underline{\dim}\, N = (a,b)$. Thanks to [Bis23, (5.2(1))] there is a filtration

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{n-1} \subsetneq N_n = N$$

such that for each $j \in \{1,\dots,n\}$ the filtration factor $E^j \coloneqq N_j/N_{j-1}$ is elementary with the property that there is at most one $i \in \{1,\dots,n\}$ such that E^i is not in $\operatorname{rep}_{\mathrm{proj}}(K_r,1)$. If this is the case, there is $\mathfrak{v} \in \operatorname{Gr}_1(A_r)$ such that $E^i \cong P_1^-(\mathfrak{v})$ (see Corollary 3.3.6). We have $\underline{\dim}\, P_1^-(\mathfrak{v}) = (1, r-1)$ and therefore $\underline{\dim}\, \sigma_{K_r}^{-1}(E^i) = (r-1, r^2-r-1)$. The representation $X \coloneqq \sigma_{K_r}^{-1}(E^i)$ satisfies

$$q_r(\underline{\dim}\, X) + \dim_{\Bbbk} X_2 - \dim_{\Bbbk} X_1 = (r-2)(r-1) \geq 1$$

and Theorem 4.2.1 ensures that $X \in \operatorname{rep}_{\mathrm{proj}}(K_r,1)$. Since $\sigma_{K_r}^{-1}$ is exact on the category of regular representations and $\operatorname{rep}_{\mathrm{proj}}(K_r,1)$ is closed under $\sigma_{K_r}^{-1}$ (see Theorem 4.2.2), we conclude that

$$0 = \sigma_{K_r}^{-\ell}(N_0) \subsetneq \sigma_{K_r}^{-\ell}(N_1) \subsetneq \cdots \subsetneq \sigma_{K_r}^{-\ell}(N_{n-1}) \subsetneq \sigma_{K_r}^{-\ell}(N_n) = M$$

is the desired filtration. □

Recall the definition of $\Delta_{(a,b)}(d)$ from Section 3.3.

Lemma 10.4.2. *Let $(a,b) \in \mathbb{N}^2 \setminus \{0\}$ be such that $q_r(a,b) < 0$ and $q_r(a,b) + \Delta_{(a,b)}(r-1) \geq 1$. For $\ell \in \mathbb{N}_0$, we let $(a_\ell, b_\ell) \coloneqq \sigma_r^{-\ell}(a,b)$. Then the following statements hold:*

(1) We have $q_r(a_\ell, b_\ell) + \Delta_{(a_\ell,b_\ell)}(r-1) \geq 1$.

(2) Let M be an indecomposable representation with dimension vector (a_ℓ, b_ℓ). Then M has a filtration such that each filtration factor is an elementary representation belonging to $\operatorname{rep}_{\mathrm{proj}}(K_r,1)$. In particular, $M \in \operatorname{rep}_{\mathrm{proj}}(K_r,1)$.

Proof. Let $(x,y) \in \mathbb{N}^2$. Direct computation shows that $\Delta_{\sigma_r^{-1}(x,y)}(r-1) = \Delta_{(x,y)}$.

(1) We prove the statement by induction on $\ell \geq 0$, the case $\ell = 0$ being trivial. For $\ell > 0$ we obtain

$$\begin{aligned} q_r(a_\ell, b_\ell) + \Delta_{(a_\ell,b_\ell)}(r-1) &= q_r(a_\ell, b_\ell) + \Delta_{\sigma_r^{-1}(a_{\ell-1},b_{\ell-1})}(r-1) \\ &= q_r(a_{\ell-1}, b_{\ell-1}) + \Delta_{(a_{\ell-1},b_{\ell-1})} \\ &> q_r(a_{\ell-1}, b_{\ell-1}) + \Delta_{(a_{\ell-1},b_{\ell-1})}(r-1) \geq 1. \end{aligned}$$

(2) In view of Lemma 10.4.1, it suffices to show that $(x,y) \coloneqq \sigma_r(a_\ell, b_\ell)$ satisfies $q_r(x,y) < 0$ and $y = \lfloor xL_r \rfloor$. Clearly, we have $q_r(x,y) = q_r(a_\ell, b_\ell) = q_r(a,b) < 0$. By definition, we obtain

$$q_r(x,y) + y - x = q_r(a_\ell, b_\ell) + \Delta_{\sigma_r(a_\ell,b_\ell)} = q_r(a_\ell, b_\ell) + \Delta_{(a_\ell,b_\ell)}(r-1),$$

so that (1) implies $q_r(x,y) + y - x \geq 1$. As $q_r(x,y) < 0$, we have $x < y$ and [Bis23, (4.2)] yields $y = \lfloor xL_r \rfloor$. □

Corollary 10.4.3. *Let $\mathcal{C} \subseteq \Gamma(K_r)$ be a regular component and assume that $X \in \mathcal{C}$ is quasi-simple such that $q_r(\underline{\dim}\, X) + \Delta_X(r-1) \geq 1$. Then every representation in $(X \to)$ possesses a filtration whose filtration factors are elementary and belong to $\mathrm{rep}_{\mathrm{proj}}(K_r, 1)$.*

Proof. Let $M \in (X \to)$, then each of the filtration factors $M_{[i]}/M_{[i-1]}$ of the quasi-composition series of M is isomorphic to $\tau_{K_r}^{-\ell_i}(X) \cong \sigma_{K_r}^{-2\ell_i}(X)$ for some $\ell_i \in \mathbb{N}_0$. By Lemma 10.4.2, each of these representations possesses a filtration such that each filtration factor is elementary and in $\mathrm{rep}_{\mathrm{proj}}(K_r, 1)$. □

Remark 10.4.4. Let $M \in \mathrm{rep}(K_r)$ be a regular indecomposable and quasi-simple representation. As in the proof of Theorem 4.2.2 we find N in the τ_{K_r}-orbit of M such that

$$q_r(\underline{\dim}\, N) + \Delta_N \geq 1$$

We have

$$\Delta_{\sigma_{K_r}^{-1}(N)}(r-1) = \dim_{\Bbbk} N_2 - \dim_{\Bbbk} N_1 = \Delta_N$$

and conclude

$$q_r(\underline{\dim}\, \sigma_{K_r}^{-1}(N)) + \Delta_{\sigma_{K_r}^{-1}(N)}(r-1) \geq 1$$

It follows that every regular AR-component $\mathcal{C} \subseteq \Gamma(K_r)$ contains a quasi-simple representation X such that $q_r(\underline{\dim}\, X) + \Delta_X(r-1) \geq 1$.

10.5. The Harder-Narasimhan filtration

Definition. Let $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}(A_r))$ be a torsion-free coherent sheaf. There exists a uniquely determined filtration, the *Harder-Narasimhan filtration* (cf. [Lan08, Section 1.2]),

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{t-1} \subsetneq \mathcal{F}_t = \mathcal{F}$$

that satisfies the following two properties:

(1) Each quotient $\mathcal{F}^i := \mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable ($i \in \{1, \ldots, t\}$).

(2) We have $\mu(\mathcal{F}^1) > \mu(\mathcal{F}^2) > \cdots > \mu(\mathcal{F}^{t-1}) > \mu(\mathcal{F}^t)$.

Remark 10.5.1. By induction on $t \geq 2$ and Lemma 10.1.1 it follows that $\mu(\mathcal{F}_{t-1}) > \mu(\mathcal{F}_t) > \mu(\mathcal{F}^t)$. Hence the sheaf $\mathcal{F}$ is semistable if and only if $t = 1$.

Lemma 10.5.2. *Let $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}(A_r))$ be a non-zero, torsion-free sheaf that is not semistable. There exists a non-zero subsheaf $\mathcal{U} \subsetneq \mathcal{F}$ with the following properties:*

(a) $\mathcal{F}/\mathcal{U}$ is semistable,

(b) $\mu(\mathcal{U}) > \mu(\mathcal{F})$,

(c) $\operatorname{Hom}_{\mathbb{P}(A_r)}(\mathcal{S}, \mathcal{F}/\mathcal{U}) = 0$ *for each semistable sheaf* $\mathcal{S} \in \operatorname{Coh}(\mathbb{P}(A_r))$ *with* $\mu(\mathcal{F}) \leq \mu(\mathcal{S})$.

Proof. We consider the Harder-Narasimhan filtration

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{t-1} \subsetneq \mathcal{F}_t = \mathcal{F}$$

with $t \geq 2$ and set $\mathcal{U} \coloneqq \mathcal{F}_{t-1}$. Then (a) holds by definition and (b) follows from Remark 10.5.1. Moreover, Remark 10.5.1 also implies

$$\mu(\mathcal{F}/\mathcal{U}) = \mu(\mathcal{F}^t) < \mu(\mathcal{F}) \leq \mu(\mathcal{S})$$

for each semistable sheaf $\mathcal{S} \in \operatorname{Coh}(\mathbb{P}(A_r))$ with $\mu(\mathcal{F}) \leq \mu(\mathcal{S})$. Let $f : \mathcal{S} \longrightarrow \mathcal{F}/\mathcal{U}$ be a morphism for such a sheaf $\mathcal{S}$. As $\mathcal{F}/\mathcal{U}$ and $\mathcal{S}$ are semistable, the assumption $f \neq 0$ in conjunction with Lemma 10.1.1 implies

$$\mu(\mathcal{F}/\mathcal{U}) \geq \mu(\mathsf{Im}\, f) \geq \mu(\mathcal{S}),$$

which contradicts $\mu(\mathcal{F}/\mathcal{U}) < \mu(\mathcal{S})$. □

10.6. The Beilinson Spectral Sequence

In the following two sections we write $\mathbb{P}^{r-1}$ for $\mathbb{P}(A_r)$. For $\mathcal{F} \in \operatorname{Coh}(\mathbb{P}^{r-1})$ and $n \in \mathbb{Z}$, we write $\mathcal{F}(n)$ for the n-th Serre twist of $\mathcal{F}$.

We denote by $\Omega \coloneqq \Omega_{\mathbb{P}^{r-1}}$ the cotangent bundle of $\mathbb{P}^{r-1}$ (see [Ben16, (6.10)]), and, for $m \in \mathbb{N}_0$, we let $\Omega^m \coloneqq \bigwedge^m \Omega$ be the m-th exterior power of Ω, where $\Omega^0 \coloneqq \mathcal{O}_{\mathbb{P}^{r-1}}$.

Lemma 10.6.1. *Let* $-r < b \leq 0$ *and* $i \in \mathbb{N}_0$. *For* $m \in \mathbb{N}_0$ *we have*

$$\mathrm{H}^i(\mathbb{P}^{r-1}, \Omega^m(m+b)) \cong \begin{cases} \Bbbk & \text{if } i = m = -b \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $i \geq r$ there is nothing to show (see [Ben16, (3.10.5)]). Hence we assume that $0 \leq i \leq r-1$. By Bott's formula (see [Ben16, (6.10.1)]) the only cases to consider are

- $i = 0$, $0 \leq m < r$ and $m + b > m$: This can not happen since $b \leq 0$.
- $m + b = 0$ and $m = i$: Then $\dim_{\Bbbk} \mathrm{H}^i(\mathbb{P}^{r-1}, \Omega^m(m+b)) = 1$.
- $i = r-1$, $0 \leq m < r$, $m + b \leq m - r$: This can not happen since $b > -r$. □

Corollary 10.6.2. *(cf. [DK93, (3.2)]) Let* $\mathcal{F} \in \operatorname{Coh}(\mathbb{P}^{r-1})$ *be a coherent sheaf,* $-(r-1) < d \leq 0$ *and*

$$0 \longrightarrow \widetilde{V_1}(d) \longrightarrow \widetilde{V_2}(d) \longrightarrow \mathcal{F} \longrightarrow 0$$

be a short exact sequence[2]. *If* $\mathrm{H}^i(\mathbb{P}^{r-1}, \mathcal{F} \otimes_{\mathbb{P}^{r-1}} \Omega^m(m)) \neq 0$, *then* $i = -d$ *and* $m \in \{-d, -(d-1)\}$,

[2]Note that $\widetilde{V_1}(d) = (V_1 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^{r-1}}(-1))(d) = V_1 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^{r-1}}(d-1)$ and $\widetilde{V_2}(d) = V_2 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^{r-1}}(d)$.

Proof. Upon tensoring the exact sequence over $\mathcal{O}_{\mathbb{P}^{r-1}}$ with the vector bundle $\Omega^m(m)$, we obtain the exact sequence

$$0 \longrightarrow V_1 \otimes_{\Bbbk} \Omega^m(m+d-1) \longrightarrow V_2 \otimes_{\Bbbk} \Omega^m(m+d) \longrightarrow \mathcal{F} \otimes_{\mathbb{P}^{r-1}} \Omega^m(m) \longrightarrow 0.$$

The long exact sequence in cohomology provides an exact sequence

$$V_2 \otimes_{\Bbbk} \mathrm{H}^i(\mathbb{P}^{r-1}, \Omega^m(m+d)) \longrightarrow \mathrm{H}^i(\mathbb{P}^{r-1}, \mathcal{F} \otimes_{\mathbb{P}^{r-1}} \Omega^m(m)) \longrightarrow V_1 \otimes_{\Bbbk} \mathrm{H}^{i+1}(\mathbb{P}^{r-1}, \Omega^m(m+d-1)).$$

Since $-r < d-1, d \leq 0$, Lemma 10.6.1 shows that the assumption $\mathrm{H}^i(\mathbb{P}^{r-1}, \mathcal{F} \otimes_{\mathbb{P}^{r-1}} \Omega^m(m)) \neq 0$ entails either $i = m = -d$ or $i+1 = m = -(d-1)$. Hence $(i,m) = (-d,-d)$ or $(i,m) = (-d,-(d-1))$. □

Theorem 10.6.3. *(see [Huy06, (8.28)]) Let $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^{r-1})$ be a coherent sheaf. There is a second quadrant spectral sequence $E = E_{\mathcal{F}}$ with first page*

$$E_1^{pq} = \mathrm{H}^q(\mathbb{P}^{r-1}, \mathcal{F} \otimes_{\mathbb{P}^{r-1}} \Omega^{-p}(-p)) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^{r-1}}(p)$$

for $(-p,q) \in \{0,\dots,r-1\}^2$ and zero otherwise, which converges to

$$E^{p+q} \Rightarrow \begin{cases} \mathcal{F}, \text{ for } p+q=0 \\ 0, \text{ for } p+q \neq 0. \end{cases}$$

Moreover, this sequence is natural in $\mathcal{F}$.

Definition. Given $-(r-1) < d \leq 0$ and $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^{r-1})$, we denote the differential $d_1^{d-1,-d}$ on the first page of the spectral sequence $E_{\mathcal{F}}$ by

$$\alpha_{\mathcal{F},d} : E_1^{d-1,-d} \longrightarrow E_1^{d,-d}.$$

Corollary 10.6.4. *Let $\mathcal{G}$ be a Steiner bundle and $-(r-1) < d \leq 0$. There exists a short exact sequence*

$$0 \longrightarrow E_1^{d-1,-d} \xrightarrow{\alpha_{\mathcal{G}(d),d}} E_1^{d,-d} \longrightarrow \mathcal{G}(d) \longrightarrow 0.$$

Proof. By Corollary 10.6.2 we know that the first page of the spectral sequence $E_{\mathcal{G}(d)}$ has at most two non-zero entries $E_1^{d-1,-d} = \mathrm{H}^{-d}(\mathbb{P}^{r-1}, \mathcal{G}(d) \otimes_{\mathbb{P}^{r-1}} \Omega^{-(d-1)}(-(d-1))) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^{r-1}}(d-1)$ and $E_1^{d,-d} = \mathrm{H}^{-d}(\mathbb{P}^{r-1}, \mathcal{G}(d) \otimes_{\mathbb{P}^{r-1}} \Omega^{-d}(-d)) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^{r-1}}(d)$. In particular, $E_{\mathcal{G}(d)}$ collapses at E_2, and we have $E_\infty = E_2$. Since $E_1^{d+1,-d} = 0 = E_1^{d-2,-d}$, we obtain

$$E_2^{d,-d} = \mathsf{Ker}\, d_1^{d,-d} / \mathsf{Im}\, \alpha_{\mathcal{G}(d),d} = E_1^{d,-d} / \mathsf{Im}\, \alpha_{\mathcal{G}(d),d} = \mathsf{Coker}\, \alpha_{\mathcal{G}(d),d} \text{ and } E_2^{d-1,-d} = \mathsf{Ker}\, \alpha_{\mathcal{G}(d),d}$$

while all other terms vanish. Now Theorem 10.6.3 implies $\mathsf{Ker}\, \alpha_{\mathcal{G}(d),d} = 0$ and $\mathcal{G}(d) \cong \mathsf{Coker}\, \alpha_{\mathcal{G}(d),d}$. There results the short exact sequence

$$0 \longrightarrow E_1^{d-1,-d} \xrightarrow{\alpha_{\mathcal{G}(d),d}} E_1^{d,-d} \longrightarrow \mathcal{G}(d) \longrightarrow 0.$$ □

10.7. Stability for 3-Kronecker representations and Steiner bundles

In terms of its methods, this section is based on those set forth in [Dré87, (IV.3)] (see also [Dré99, (5.1.2)]).

Lemma 10.7.1. *Let $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^{r-1})$ be torsion-free, $b \in \mathbb{Z}$. Then $\mathcal{F}$ is (semi)stable if and only if the same is true for $\mathcal{F}(b)$.*

Proof. Note that $\mathcal{O}_{\mathbb{P}^{r-1}}(b) \otimes_{\mathbb{P}^{r-1}} -$ induces an exact auto-equivalence on the category of coherent sheaves. Given a torsion-free coherent sheaf $\mathcal{G} \neq 0$, the sheaf $\mathcal{G}(b)$ is locally isomorphic to $\mathcal{G}$ and hence also torsion-free and of the same rank. We conclude with [Ben16, (7.5.1)]

$$\mu(\mathcal{O}_{\mathbb{P}^{r-1}}(b) \otimes_{\mathbb{P}^{r-1}} \mathcal{G}) = \mu(\mathcal{G}(b)) = \frac{c_1(\mathcal{G}(b))}{\mathrm{rk}(\mathcal{G})} = \frac{c_1(\mathcal{G}) + b\,\mathrm{rk}(\mathcal{G})}{\mathrm{rk}(\mathcal{G})} = \mu(\mathcal{G}) + b.$$

Hence the equivalence respects the order of slopes. □

Proposition 10.7.2. *Let $M \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$. Then the following statements hold:*

(1) If $\Delta_M(2) \geq 0$ and M is semistable, then $\widetilde{\Theta}(M)$ is semistable.

(2) If $\Delta_M(2) > 0$ and M is stable, then $\widetilde{\Theta}(M)$ is stable.

Proof. We begin the proof by listing several properties that both cases have in common.

Given $\mathcal{U} \in \mathrm{Coh}(\mathbb{P}^2)$, we consider the differentials

$$\alpha_{\mathcal{U},-1}\colon \mathrm{H}^1(\mathbb{P}^2, \mathcal{U} \otimes_{\mathbb{P}^2} \Omega^2(2)) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow \mathrm{H}^1(\mathbb{P}^2, \mathcal{U} \otimes_{\mathbb{P}^2} \Omega^1(1)) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^2}(-1)$$

and

$$\delta_{\mathcal{U},-1}\colon \mathrm{H}^2(\mathbb{P}^2, \mathcal{U} \otimes_{\mathbb{P}^2} \Omega^2(2)) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow \mathrm{H}^2(\mathbb{P}^2, \mathcal{U} \otimes_{\mathbb{P}^2} \Omega^1(1)) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^2}(-1)$$

from the first page of the Beilinson spectral sequence $E_{\mathcal{U}}$ (see Theorem 10.6.3).

Let $\mathcal{F} \coloneqq \widetilde{\Theta}(M)(-1)$ and $\mathcal{U} \subsetneq \mathcal{F}$ be a subsheaf. The inclusion $\mathcal{U} \hookrightarrow \mathcal{F}$ induces $\Bbbk$-linear maps

$$\tau_i : \mathrm{H}^1(\mathbb{P}^2, \mathcal{U} \otimes_{\mathbb{P}^2} \Omega^i(i)) \longrightarrow \mathrm{H}^1(\mathbb{P}^2, \mathcal{F} \otimes_{\mathbb{P}^2} \Omega^i(i))$$

for $i \in \{1, 2\}$.

Throughout, we assume that $\mathcal{U} \subsetneq \mathcal{F}$ is a non-zero subsheaf such that

$$(\star) \qquad \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(\ell), \mathcal{F}/\mathcal{U}) = 0 \ \text{ for all } \ \ell \geq 0.$$

We proceed in steps:

(a) *τ_1 is injective.*

Recall that $\mathrm{H}^0(\mathbb{P}^2, \mathcal{G})$ is naturally isomorphic to the global section of $\mathcal{G}$ for every $\mathcal{O}_{\mathbb{P}^2}$-module (see [Ben16, §3.10]). Hence the isomorphism $\Omega^1(1)^\vee \cong \mathcal{T}_{\mathbb{P}^2}(-1)$ and [Ben16, (6.1.3)] imply

$$\mathrm{H}^0(\mathbb{P}^2, (\mathcal{F}/\mathcal{U}) \otimes_{\mathbb{P}^2} \Omega^1(1)) \cong \mathrm{H}^0(\mathbb{P}^2, (\Omega^1(1)^\vee)^\vee \otimes_{\mathbb{P}^2} (\mathcal{F}/\mathcal{U})) \cong \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{F}/\mathcal{U}).$$

Application of the left exact functor $\operatorname{Hom}_{\mathbb{P}^2}(-,\mathcal{F}/\mathcal{U})$ to the Euler sequence $0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \longrightarrow \mathcal{T}_{\mathbb{P}^2}(-1) \longrightarrow 0$ yields $\dim_{\Bbbk} \operatorname{Hom}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2}(-1),\mathcal{F}/\mathcal{U}) \leq \dim_{\Bbbk} \operatorname{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}^3,\mathcal{F}/\mathcal{U}) \overset{(\star)}{=} 0$. Hence we have $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2},(\mathcal{F}/\mathcal{U})\otimes_{\mathbb{P}^2}\Omega^1(1)) = 0$. The long exact sequence in cohomology yields the exactness of

$$\mathrm{H}^0(\mathbb{P}^2,(\mathcal{F}/\mathcal{U})\otimes_{\mathbb{P}^2}\Omega^1(1)) \longrightarrow \mathrm{H}^1(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^1(1)) \xrightarrow{\tau_1} \mathrm{H}^1(\mathbb{P}^2,\mathcal{F}\otimes_{\mathbb{P}^2}\Omega^1(1)).$$

Consequently, τ_1 is injective. ◇

(b) *τ_2 is injective.*

Upon tensoring the sequence $0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{U} \longrightarrow 0$ with the vector bundle $\Omega^2(2)$ the long exact sequence in cohomology yields the exactness of

$$\mathrm{H}^0(\mathbb{P}^2,(\mathcal{F}/\mathcal{U})\otimes_{\mathbb{P}^2}\Omega^2(2)) \longrightarrow \mathrm{H}^1(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^2(2)) \xrightarrow{\tau_2} \mathrm{H}^1(\mathbb{P}^2,\mathcal{F}\otimes_{\mathbb{P}^2}\Omega^2(2)).$$

By [Ben16, §6.10], we have $\Omega^2(2) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. The choice of $\mathcal{U}$ implies

$$\mathrm{H}^0(\mathbb{P}^2,(\mathcal{F}/\mathcal{U})\otimes_{\mathbb{P}^2}\Omega^2(2)) \cong \operatorname{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1),\mathcal{F}/\mathcal{U}) \overset{(\star)}{=} 0,$$

so that τ_2 is injective. ◇

(c) $\operatorname{Ker}\alpha_{\mathcal{U},-1} = 0$.

Corollary 10.6.4[3] and the naturality of the Beilinson spectral sequence yield the following commutative diagram

$$(\dagger) \qquad \begin{array}{ccc} \mathrm{H}^1(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^2(2))\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{\alpha_{\mathcal{U},-1}} & \mathrm{H}^1(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^1(1))\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}(-1) \\ \downarrow{\scriptstyle \tau_2\otimes\mathrm{id}_{\mathcal{O}_{\mathbb{P}^2}(-2)}} & & \downarrow{\scriptstyle \tau_1\otimes\mathrm{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)}} \\ 0 \longrightarrow \mathrm{H}^1(\mathbb{P}^2,\mathcal{F}\otimes_{\mathbb{P}^2}\Omega^2(2))\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{\alpha_{\mathcal{F},-1}} & \mathrm{H}^1(\mathbb{P}^2,\mathcal{F}\otimes_{\mathbb{P}^2}\Omega^1(1))\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}(-1), \end{array}$$

whose lower row is exact. In view of (b), the morphism $\tau_2 \otimes \mathrm{id}_{\mathcal{O}_{\mathbb{P}^2}(-2)}$ is a monomorphism, so that $\operatorname{Ker}\alpha_{\mathcal{U},-1} = 0$. ◇

(d) *There is an exact sequence*

$$0 \longrightarrow \operatorname{Coker}\alpha_{\mathcal{U},-1} \longrightarrow \mathcal{U} \longrightarrow \operatorname{Ker}\delta_{\mathcal{U},-1} \longrightarrow 0$$

of torsion-free coherent sheaves.

Owing to Corollary 10.6.2, we have

$$\dim_{\Bbbk} \mathrm{H}^0(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^m(m)) \leq \dim_{\Bbbk} \mathrm{H}^0(\mathbb{P}^2,\mathcal{F}\otimes_{\mathbb{P}^2}\Omega^m(m)) = 0$$

for $m \in \{0,1,2\}$ as well as $\mathrm{H}^1(\mathbb{P}^2,\mathcal{F}) = 0$. Since $0 \overset{(\star)}{=} \operatorname{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2},\mathcal{F}/\mathcal{U}) = \mathrm{H}^0(\mathbb{P}^2,\mathcal{F}/\mathcal{U})$, we conclude with the exactness of

$$\mathrm{H}^0(\mathbb{P}^2,\mathcal{F}/\mathcal{U}) \longrightarrow \mathrm{H}^1(\mathbb{P}^2,\mathcal{U}) \longrightarrow \mathrm{H}^1(\mathbb{P}^2,\mathcal{F})$$

that $\mathrm{H}^1(\mathbb{P}^2,\mathcal{U}) = 0$. Consequently, the first page of the Beilinson spectral sequence $E_{\mathcal{U}}$ reads

$\mathrm{H}^2(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^2(2))\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}(-2)$	$\mathrm{H}^2(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^1(1))\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}(-1)$	$\mathrm{H}^2(\mathbb{P}^2,\mathcal{U})\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}$
$\mathrm{H}^1(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^2(2))\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}(-2)$	$\mathrm{H}^1(\mathbb{P}^2,\mathcal{U}\otimes_{\mathbb{P}^2}\Omega^1(1))\otimes_{\Bbbk}\mathcal{O}_{\mathbb{P}^2}(-1)$	0
0	0	0

[3]Note that we apply Corollary 10.6.4 to the Steiner bundle $\mathcal{G} = \widetilde{\Theta}(M)$ with $d = -1$.

with all other terms being zero. In view of (c), the second page thus is of the following form:

$\mathsf{Ker}\,\delta_{\mathcal{U},-1}$	$*$	$*$
0	$\mathsf{Coker}\,\alpha_{\mathcal{U},-1}$	0
0	0	0

Theorem 10.6.3 in conjunction with [CE99, (XV.5.11)][4] now provides an exact sequence

$$0 \longrightarrow \mathsf{Coker}\,\alpha_{\mathcal{U},-1} \longrightarrow \mathcal{U} \longrightarrow \mathsf{Ker}\,\delta_{\mathcal{U},-1} \longrightarrow 0$$

of torsion-free coherent sheaves. ◇

(e) *If* $\mu(\mathcal{U}) \geq (>)\mu(\mathcal{F})$, *then* $\mathsf{Coker}\,\alpha_{\mathcal{U},-1} \neq 0$ *and* $\mu(\mathsf{Coker}\,\alpha_{\mathcal{U},-1}) \geq (>)\mu(\mathcal{F})$.
Suppose that $\mathsf{Ker}\,\delta_{\mathcal{U},-1} \neq 0$. Since $\mathcal{O}_{\mathbb{P}^2}(-2)$ is semistable, we conclude with $\mathsf{Ker}\,\delta_{\mathcal{U},-1} \subseteq \mathrm{H}^2(\mathbb{P}^2, \mathcal{U} \otimes_{\mathbb{P}^2} \Omega^2(2)) \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^2}(-2)$, [OSS80, (1.2.4)], $\frac{\dim_{\Bbbk} M_1}{\Delta_M} \geq 0$ and [Ben16, (7.5.1)] that

$$\mu(\mathsf{Ker}\,\delta_{\mathcal{U},-1}) \leq \mu(\mathcal{O}_{\mathbb{P}^2}(-2)) = -2 < -1 + \frac{\dim_{\Bbbk} M_1}{\Delta_M} = \mu(\mathcal{F}) \leq \mu(\mathcal{U}).$$

In particular, $\mathsf{Ker}\,\delta_{\mathcal{U},-1} \ncong \mathcal{U}$ and therefore (d) forces $\mathsf{Coker}\,\alpha_{\mathcal{U},-1} \neq 0$. Now (d) in conjunction with Lemma 10.1.1 implies $\mu(\mathsf{Coker}\,\alpha_{\mathcal{U},-1}) > \mu(\mathcal{U}) \geq \mu(\mathcal{F})$.
Alternatively, $\mathsf{Ker}\,\delta_{\mathcal{U},-1} = 0$, so that $\mathsf{Coker}\,\alpha_{\mathcal{U},-1} \cong \mathcal{U}$ and $\mu(\mathsf{Coker}\,\alpha_{\mathcal{U},-1}) = \mu(\mathcal{U}) \geq (>)\mu(\mathcal{F})$. ◇

(f) *There exists a subrepresentation* $N \subseteq M$ *of* M *such that* $\widetilde{\Theta}(N) \cong \mathsf{Coker}\,\alpha_{\mathcal{U},-1}(1)$.
Setting $V_i := \mathrm{H}^1(\mathbb{P}^2, \mathcal{U} \otimes_{\mathbb{P}^2} \Omega^{3-i}(3-i))$ and $W_i := \mathrm{H}^1(\mathbb{P}^2, \mathcal{F} \otimes_{\mathbb{P}^2} \Omega^{3-i}(3-i))$ for $i \in \{1,2\}$, application of $- \otimes_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2}(1)$ to the diagram (†) in conjunction with (c) and Corollary 10.6.4 yields

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{V_1} & \xrightarrow{\alpha_{\mathcal{U},-1}(1)\widetilde{}} & \widetilde{V_2} & \longrightarrow & \mathsf{Coker}\,\alpha_{\mathcal{U},-1}(1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{\tau}_2} & & \downarrow{\scriptstyle \tilde{\tau}_1} & & \downarrow & & \\
0 & \longrightarrow & \widetilde{W_1} & \xrightarrow{\alpha_{\mathcal{F},-1}(1)\widetilde{}} & \widetilde{W_2} & \longrightarrow & \widetilde{\Theta}(M) & \longrightarrow & 0,
\end{array}$$

where we have used the notation introduced in Section 8.2. Let $x \in \mathbb{P}^2$. By (b) and the diagram at the beginning of Section 8.2, the map $\tilde{\tau}_2(x)$ is injective and since $\widetilde{\Theta}(M)$ is a vector bundle, Theorem 7.2.1(2) shows that this also holds for $\alpha_{\mathcal{F},-1}(1)(x)$. Passing to fibers in the diagram above, we see that $\alpha_{\mathcal{U},-1}(1)(x)$ is injective, and another application of Theorem 7.2.1(2) ensures that $\mathsf{Coker}\,\alpha_{\mathcal{U},-1}(1)$ is a vector bundle, hence a Steiner bundle.
The proof of Theorem 8.2.5(1) provides $\Bbbk$-linear maps

$$V(\gamma_i) : V_1 \longrightarrow V_2 \;\text{ and }\; W(\gamma_i) : W_1 \longrightarrow W_2 \qquad (i \in \{1, \ldots, r\})$$

such that

$$\widetilde{\Theta}_V = \alpha_{\mathcal{U},-1}(1) \;\text{ and }\; \widetilde{\Theta}_W = \alpha_{\mathcal{F},-1}(1),$$

where $X := (X_1, X_2, (X(\gamma_i))_{1 \leq i \leq r})$ for $X \in \{V, W\}$ with $V, W \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ such that $\widetilde{\Theta}(W) \cong \widetilde{\Theta}(M)$ and $\widetilde{\Theta}(V) \cong \mathsf{Coker}\,\alpha_{\mathcal{U},-1}(1)$. Theorem 8.2.5(2) implies $M \cong W$ and the diagram above yields

$$\tilde{\tau}_1 \circ \widetilde{\Theta}_V = \widetilde{\Theta}_W \circ \tilde{\tau}_2.$$

[4] One sets $r = 2 = q'$, $q = 1$ and $n = 0$.

As in the proof of Theorem 8.2.5 we conclude

$$\psi_W \circ (\mathrm{id}_{A_3} \otimes \tau_2) = \tau_1 \circ \psi_V,$$

so that $(\tau_2, \tau_1) : V \longrightarrow W$ is a morphism, which, by virtue of (a) and (b), is injective. Thus, V is isomorphic to a non-zero subrepresentation N of M and $\widetilde{\Theta}(N) \cong \widetilde{\Theta}(N) \cong \mathsf{Coker}\, \alpha_{\mathcal{U},-1}(1)$. $\diamond$

(1) Suppose that $\widetilde{\Theta}(M)$ is not semistable. By Lemma 10.7.1 the vector bundle $\mathcal{F} \coloneqq \widetilde{\Theta}(M)(-1)$ is not semistable. For a representation $0 \neq N \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ with $N_1 = 0$, the bundle $\widetilde{\Theta}(N)(-1) \cong N_2 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}^2}(-1)$ is semistable (see [OSS80, (1.2.4)]). In view of $\Delta_M(2) \geq 0$, we thus have $0 < 2 \dim_{\Bbbk} M_1 \leq \dim_{\Bbbk} M_2$ as well as $\mu(\mathcal{F}) = \mu(\widetilde{\Theta}(M)) - 1 = \frac{\dim_{\Bbbk} M_1}{\Delta_M} - 1 \leq 0$. Lemma 10.5.2 provides $\mathcal{U} \subsetneq \mathcal{F}$ non-zero such that $\mathcal{F}/\mathcal{U}$ is semistable with

(i) $\mu(\mathcal{U}) > \mu(\mathcal{F})$, and

(ii) $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(\ell), \mathcal{F}/\mathcal{U}) = 0$ for all $\ell \geq 0$.

In particular, $(\star)$ is satisfied. Thanks to (e) and (f) there exists a subrepresentation $N \subseteq M$ such that $\widetilde{\Theta}(N) \cong \mathsf{Coker}\, \alpha_{\mathcal{U},-1}(1) \neq 0$. In particular, $N \neq 0$. Moreover, (f) implies

$$\mu(N) = \mu(\mathsf{Coker}\, \alpha_{\mathcal{U},-1}(1)) > \mu(\widetilde{\Theta}(M)) = \mu(M),$$

which contradicts M being semistable.

(2) As before, we set $\mathcal{F} \coloneqq \widetilde{\Theta}(M)(-1)$. Since the assumptions of (1) are satisfied, $\mathcal{F}$ is semistable. Suppose that $\mathcal{F}$ is not stable. Proceeding as in (1), we first observe that M is indecomposable (even a brick), so that we may assume that $M_1 \neq 0$. We apply [OSS80, (1.2.2)][5] in conjunction with Lemma 10.1.1 to obtain a short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{U} \longrightarrow 0$$

such that $0 < \mathrm{rk}(\mathcal{U}), \mathrm{rk}(\mathcal{F}/\mathcal{U}) < \mathrm{rk}(\mathcal{F})$ with $\mathcal{F}/\mathcal{U}$ being torsion-free and $\mu(\mathcal{U}) = \mu(\mathcal{F}/\mathcal{U}) = \mu(\mathcal{F})$. We choose $\mathcal{U}$ such that $\mathcal{U}$ is of maximal rank subject to these properties.

Let $\mathcal{Q}$ be a torsion-free quotient of $\mathcal{F}/\mathcal{U}$ with $0 < \mathrm{rk}(\mathcal{Q}) < \mathrm{rk}(\mathcal{F}/\mathcal{U})$. There results a short exact sequence $0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0$ with $0 < \mathrm{rk}(\mathcal{V}), \mathrm{rk}(\mathcal{Q}) < \mathrm{rk}(\mathcal{F})$, and, observing $\mathrm{rk}(\mathcal{Q}) < \mathrm{rk}(\mathcal{F}/\mathcal{U})$, we conclude that $\mathrm{rk}(\mathcal{V}) > \mathrm{rk}(\mathcal{U})$. Since $\mathcal{F}$ is semistable, the choice of $\mathcal{U}$ in conjunction with Lemma 10.1.1 implies $\mu(\mathcal{V}) < \mu(\mathcal{F})$ and Lemma 10.1.1 yields $\mu(\mathcal{Q}) > \mu(\mathcal{F}) = \mu(\mathcal{F}/\mathcal{U})$. Consequently, [OSS80, (1.2.2)] shows that $\mathcal{F}/\mathcal{U}$ is stable.

Assume that $f \colon \mathcal{O}_{\mathbb{P}^2}(\ell) \longrightarrow \mathcal{F}/\mathcal{U}$ is a non-zero morphism. Since $\mathcal{O}_{\mathbb{P}^2}(\ell)$ is stable and $\mathsf{Im}\, f \subseteq \mathcal{F}/\mathcal{U}$ is a non-zero torsion-free sheaf, we conclude

$$\ell = \mu(\mathcal{O}_{\mathbb{P}^2}(\ell)) \leq \mu(\mathsf{Im}\, f) \leq \mu(\mathcal{F}/\mathcal{U}) = \mu(\mathcal{F}) = \frac{\dim_{\Bbbk} M_1}{\Delta_M} - 1 < 0.$$

We have proven:

(i) $\mu(\mathcal{U}) = \mu(\mathcal{F}) = \mu(\mathcal{F}/\mathcal{U})$, and

(ii) $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(\ell), \mathcal{F}/\mathcal{U}) = 0$ for all $\ell \geq 0$.

[5] This result holds for algebraically closed fields of arbitrary characteristic.

Thanks to (e) and (f) we find a subrepresentation $N \subseteq M$ such that $\widetilde{\Theta}(N) \cong \mathsf{Coker}\, \alpha_{\mathcal{U},-1}(1) \neq 0$. Moreover, (f) implies

$$\mu(N) = \mu(\widetilde{\Theta}(N)) = \mu(\mathsf{Coker}\, \alpha_{\mathcal{U},-1}(1)) \geq \mu(\widetilde{\Theta}(M)).$$

Note that $\mathsf{Coker}\, \alpha_{\mathcal{U},-1}(1) \subseteq \mathcal{U}(1)$ and $\mathrm{rk}(\mathcal{U}) < \mathrm{rk}(\mathcal{F})$ imply $\mathrm{rk}(\widetilde{\Theta}(N)) = \mathrm{rk}(\mathsf{Coker}\, \alpha_{\mathcal{U},-1}(1)) < \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\widetilde{\Theta}(M))$. In particular, $0 \neq N \neq M$ and M is not stable, a contradiction. It follows that $\widetilde{\Theta}(M) = \mathcal{F}(1)$ is stable.

□

We summarize our findings in the following result.

Theorem 10.7.3. *Let $0 \neq M \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$.*

(1) If $\Delta_M(2) \geq 0$, then $\widetilde{\Theta}(M)$ is semistable if and only if M is semistable.

(2) If $\Delta_M(2) > 0$, then $\widetilde{\Theta}(M)$ is stable if and only if M is stable.

Proof. This is just a combination of Proposition 10.2.1 and Proposition 10.7.2. □

Example 10.7.4. *(1) Exceptional Steiner bundles: According to [GR87, (4.1)], exceptional vector bundles on $\mathbb{P}^2(\mathbb{C})$ are stable. Thanks to [Bra08, (1.1)], this also holds for exceptional Steiner bundles on $\mathbb{P}^{r-1}(\mathbb{C})$. Returning to our context, we let $\mathcal{F}$ be an exceptional Steiner bundle on $\mathbb{P}(A_3)$. According to Proposition 8.3.1, there is a preprojective indecomposable representation $M \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ such that $\mathcal{F} \cong \widetilde{\Theta}(M)$. A consecutive application of Theorem 4.2.2 and Theorem 5.4.1 ensures that $\Delta_M(2) > 0$. Now Theorem 10.7.3 in conjunction with Remark 10.3.3 shows that $\mathcal{F}$ is stable.*

(2) "Regular" indecomposable Steiner bundles: Let (V_1, V_2) be a pair of $\Bbbk$-vector spaces such that $\Delta_{(V_1,V_2)}(2) > 0$ and $q_3(\underline{\dim}(V_1, V_2)) < 0$. The stable representations of the open set $\mathrm{rep}(K_3; V_1, V_2) \cap \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ may be interpreted as the stable points of the variety $\mathbb{P}(\mathrm{rep}(K_3; V_1, V_2))$ relative to the canonical action of $\mathrm{SL}(V_2) \times \mathrm{SL}(V_1)$. This can be used to show that the subset $\mathcal{S}(V_1, V_2)$ of stable representations is non-empty open (and hence dense) whenever $\Delta_{(V_1,V_2)} \geq 2$; see also Proposition 10.9.1 below for the case, where $\Bbbk = \mathbb{C}$.

10.8. Applications: Filtrations for Steiner bundles

Suppose that $M \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ is semistable. If $P_0(3)$ is a direct summand of M, then a two-fold application of the See-Saw Lemma implies $M_1 = 0$, whence $M \cong nP_0(3)$. Alternatively ψ_M is surjective, whence $\dim_{\Bbbk} M_2 \leq 3 \dim_{\Bbbk} M_1$. Hence the conditions of Theorem 10.7.3 may seem to limit its utility. The results of this section illustrate that a combination of Theorem 10.7.3 with techniques involving AR-theory provides many examples of stable Steiner bundles on $\mathbb{P}(A_3)$.

Theorem 10.8.1. *Let $E \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ be an elementary representation. Then $\widetilde{\Theta}(E)$ is a stable.*

Proof. Thanks to Proposition 10.3.4, the representation $0 \neq E$ is stable and $E \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ implies $\dim_{\Bbbk} E_1 < \dim_{\Bbbk} E_2$. In view of Theorem 10.7.3, it therefore suffices to consider the case, where $\dim_{\Bbbk} E_1 < \dim_{\Bbbk} E_2 \leq 2 \dim_{\Bbbk} E_1$. Thanks to [Rin16, Thm.], $\underline{\dim}\, E$ belongs to one of the σ_3-orbits generated by the dimension vectors $(1,1), (2,2)$. Since $\Delta_{\sigma_{K_3}(M)} = \Delta_M(2) < \Delta_M$ for every regular indecomposable $M \in \mathrm{rep}(K_3)$, there is a unique dimension vector $(x,y) \in \langle \sigma_3 \rangle . \underline{\dim}\, E$ such that $\Delta_{(x,y)} = y - x \in \mathbb{N}$ and $\Delta_{(x,y)}(2) = y - 2x \leq 0$. Hence the only dimension vectors in these orbits that satisfy $x < y \leq 2x$ are $(1,2)$ and $(2,4)$. By virtue of Theorem 5.4.1, representations with dimension vector $(1,2)$ do not belong to $\mathrm{rep}_{\mathrm{proj}}(K_3, 1)$, whence $\underline{\dim}\, E = (2,4)$. As elementary representations are bricks, $\widetilde{\Theta}(E)$ is a simple vector bundle of rank 2. We conclude with [OSS80, (1.2.10)], which holds in arbitrary characteristic (cf. also [Sch61a, Prop.1]), that $\widetilde{\Theta}(E)$ is stable. □

Let $\mathcal{C} \subseteq \Gamma(K_r)$ be a regular Auslander-Reiten component. We say that $\mathcal{C}$ is *(semi)stable*, provided $\mathcal{C} \cap \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ contains a (semi)stable vertex[6]. The following result shows that for such components the Harder-Narasimhan filtrations of most of the bundles $\widetilde{\Theta}(M)$ correspond to the quasi-composition series of M.

Proposition 10.8.2. *Let $\mathcal{C} \subseteq \Gamma(K_3)$ be a semistable Auslander-Reiten component. If $M \in (\tau_{K_3}^{-1}(M_{\mathcal{C},1}) \to)$ has quasi-length ℓ, then the Harder-Narasimhan filtration of $\widetilde{\Theta}(M)$ is given by*

$$0 \subsetneq \widetilde{\Theta}(M_{[1]}) \subsetneq \widetilde{\Theta}(M_{[2]}) \subsetneq \cdots \subsetneq \widetilde{\Theta}(M_{[\ell-1]}) \subsetneq \widetilde{\Theta}(M_{[\ell]}) = \widetilde{\Theta}(M)$$

with filtration factors $\widetilde{\Theta}(M_{[j]})/\widetilde{\Theta}(M_{[j-1]}) \cong \widetilde{\Theta}(\tau_{K_3}^{-j+1}(M_{[1]}))$ for all $j \in \{2, \ldots, \ell\}$.

Proof. We consider the quasi-composition series

$$0 \subsetneq M_{[1]} \subsetneq \cdots \subsetneq M_{[\ell-1]} \subsetneq M$$

of M. For $1 < j \leq \ell$ general AR-theory provides an exact sequence

$$0 \longrightarrow M_{[j-1]} \longrightarrow M_{[j]} \longrightarrow \tau_{K_3}^{-j+1}(M_{[1]}) \longrightarrow 0$$

in $\mathrm{rep}_{\mathrm{proj}}(K_3, 1)$. According to Corollary 8.2.6, application of $\widetilde{\Theta}$ yields an exact sequence

$$0 \longrightarrow \widetilde{\Theta}(M_{[j-1]}) \longrightarrow \widetilde{\Theta}(M_{[j]}) \longrightarrow \widetilde{\Theta}(\tau_{K_3}^{-j+1}(M_{[1]})) \longrightarrow 0$$

so that $\widetilde{\Theta}(M_{[j]})/\widetilde{\Theta}(M_{[j-1]}) \cong \widetilde{\Theta}(\tau_{K_3}^{-j+1}(M_{[1]}))$. Our assumption on M implies that the quasi-socle $M_{[1]}$ belongs to the cone $(\tau_{K_3}^{-1}(M_{\mathcal{C},1}) \to)$. It follows from Theorem 4.2.2(4) that $\tau_{K_3}^{-j+1}(M_{[1]}) \in \mathrm{rep}_{\mathrm{proj}}(K_3, 2)$, whence $\Delta_{\tau_{K_3}^{-j+1}(M[1])}(2) \geq 0$ by Theorem 5.4.1.

As $\mathcal{C}$ is semistable, Corollary 10.3.2 ensures that the same is true for $\tau_{K_3}^{-n}(M_{[1]})$ for all $n \in \mathbb{N}_0$. We apply Theorem 10.7.3 and conclude that $\widetilde{\Theta}(M_{[j]})/\widetilde{\Theta}(M_{[j-1]})$ is semistable. Proposition 10.3.1(3) yields

$$\mu(\widetilde{\Theta}(\tau_{K_3}^{-j+1}(M_{[1]}))) = \mu(\tau_{K_3}^{-j+1}(M_{[1]})) > \mu(\tau_{K_3}^{-j}(M_{[1]})) = \mu(\widetilde{\Theta}(\tau_{K_3}^{-j}(M_{[1]})))$$

for all $1 \leq j \leq \ell$. The statement now follows from the uniqueness of the Harder-Narasimhan filtration. □

[6] Recall from Corollary 10.3.2(3) that semistable vertices are quasi-simple.

Proposition 10.8.3. *Let $\mathcal{C} \subseteq \Gamma(K_3)$ be a regular component. If $M \in \mathcal{C} \cap \mathrm{rep}_{\mathrm{proj}}(K_3, d)$ has minimal type, then $\widetilde{\Theta}(N)$ is stable for every quasi-simple $N \in (M_{\mathcal{C},d} \to)$.*

Proof. By Corollary 5.4.4 the representation satisfies $M = M_{\mathcal{C},d}$ and is stable by Lemma 10.2.3. Let $N = \tau_{K_r}^{-\ell}(M)$ with $\ell \in \mathbb{N}_0$. If $\ell \in \mathbb{N}$, then Theorem 4.2.2 and Theorem 5.4.1 imply $N \in \mathrm{rep}_{\mathrm{proj}}(K_3, 2)$ and $\Delta_N(2) > 0$. In view of Corollary 10.3.2 and Theorem 10.7.3, it therefore suffices to show that $\widetilde{\Theta}(M)$ is stable. If $d = 2$, we can argue as before. Alternatively, $\widetilde{\Theta}(M) \in \mathrm{Vect}(\mathbb{P}^2)$ is simple (see Corollary 5.4.4) and of rank 2, so that [OSS80, (1.2.10)] implies the stability of $\widetilde{\Theta}(M)$. □

Theorem 10.8.4. *Let $\mathcal{C} \subseteq \Gamma(K_3)$ be a regular AR-component, $\mathcal{F}$ be an indecomposable Steiner bundle.*

(1) If $\mathcal{C}$ contains an elementary representation and $\mathcal{F} \in \widetilde{\Theta}(M_{\mathcal{C},1} \to)$, then $\mathcal{F}$ possesses a Harder-Narasimhan filtration by Steiner bundles, whose filtration factors are stable.

(2) If $\mathcal{F}$ has a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^a \to \mathcal{O}_{\mathbb{P}^2}^b \to \mathcal{F} \to 0$$

such that (a, b) is an imaginary root and $q_3(a, b) + \Delta_{(a,b)}(2) \geq 1$, then $\mathcal{F}$ possesses a filtration

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}$$

by Steiner bundles such that each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a stable Steiner bundle.

(3) Suppose $X \in \mathcal{C}$ is quasi-simple such that $q_3(\underline{\dim}\, X) + \Delta_X(2) \geq 1$. If $\mathcal{F} \in \widetilde{\Theta}(X \to)$, then $\mathcal{F}$ possesses a filtration

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}$$

by Steiner bundles such that each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a stable Steiner bundle.

Proof. (1) Let $E \in \mathcal{C}$ be elementary. By definition, E is quasi-simple, so that, for every $n \in \mathbb{N}_0$, the module $\tau_{K_3}^{-n}(M_{\mathcal{C},1})$ is elementary and contained in $\mathrm{rep}_{\mathrm{proj}}(K_3, 1)$. Given $\mathcal{F} \coloneqq \widetilde{\Theta}(M) \in \widetilde{\Theta}(M_{\mathcal{C},1} \to)$, we consider the quasi-composition series

$$0 = M_{[0]} \subsetneq M_{[1]} \subsetneq \cdots \subsetneq M_{[\mathrm{ql}(M)]} = M,$$

whose filtration factors belong to $\{\tau_{K_3}^{-\ell}(M_{\mathcal{C},1}) \mid \ell \in \mathbb{N}_0\}$. Application of $\widetilde{\Theta}$ provides a filtration

$$0 = \widetilde{\Theta}(M_{[0]}) \subsetneq \widetilde{\Theta}(M_{[1]}) \subsetneq \cdots \subsetneq \widetilde{\Theta}(M_{[\mathrm{ql}(M)]}) = \mathcal{F},$$

with filtration factors $\widetilde{\Theta}(M_{[j]})/\widetilde{\Theta}(M_{[j-1]}) \cong \widetilde{\Theta}(\tau_{K_3}^{-j+1}(M_{[1]}))$. Thanks to Theorem 10.8.1, the factors are stable and with strictly decreasing slopes (cf. Proposition 10.3.1), so that we have found the Harder-Narasimhan filtration of $\mathcal{F}$.

(2) By Theorem 8.2.5 we can find an indecomposable representation $M \in \mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ such that $\widetilde{\Theta}(M) \cong \mathcal{F}$. We have $b - a = \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\widetilde{\Theta}(M)) = \Delta_M$ and $a = c_1(\mathcal{F}) = c_1(\widetilde{\Theta}(M)) = \dim_{\Bbbk} M_1$. Hence

$$q_3(\underline{\dim}\, M) + \Delta_M(2) = q_3(a, b) + \Delta_{(a,b)}(2) \geq 1.$$

We conclude with Lemma 10.4.2 that there is a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that each quotient M_i/M_{i-1} is an elementary representation in $\mathrm{rep}_{\mathrm{proj}}(K_3, 1)$. Exactness of $\widetilde{\Theta}$ on $\mathrm{rep}_{\mathrm{proj}}(K_3, 1)$ yields a filtration

$$0 = \widetilde{\Theta}(M_0) \subsetneq \widetilde{\Theta}(M_1) \subsetneq \cdots \subsetneq \widetilde{\Theta}(M_n) = \widetilde{\Theta}(M) \cong \mathcal{F}$$

of Steiner bundles. By Theorem 10.8.1, each quotient $\widetilde{\Theta}(M_i)/\widetilde{\Theta}(M_{i-1}) \cong \widetilde{\Theta}(M_i/M_{i-1})$ is a stable Steiner bundle.

(3) This follows as in (2), using Corollary 10.4.3.

□

10.9. Applications: Restrictions to hyperplanes

In this section we study restrictions of Steiner bundles to hyperplanes.

Definition. Let $r \geq 3$. A closed subset $H \subseteq \mathbb{P}(A_r)$ of the form $H = \operatorname{im} \hat{\alpha}$ for some $\alpha \in \mathrm{Inj}_{\Bbbk}(A_{r-1}, A_r)$ is called *(linear) hyperplane of* $\mathbb{P}(A_r)$.

Let $\mathcal{F}$ be a stable (respectively, semistable) vector bundle of rank n on $\mathbb{P}(A_r)$ with $r \geq 4$. In [Har79, Problem 5] it was asked whether for almost all hyperplanes $H \subseteq \mathbb{P}(A_r)$ the restriction $\mathcal{F}|_H$ of $\mathcal{F}$ to H (see Remark 9.1.4) is stable (respectively, semistable). The question was answered affirmatively by Barth [Bar77] for $n = 2$ and for $n < r-1$ by Maruyama [Mar80] (in both cases with one exception). In the context of Steiner bundles, these rank conditions imply that only (direct sums of) the structure sheaf of $\mathbb{P}(A_r)$ can occur.

Consider the tangent bundle $\mathcal{T}_{\mathbb{P}(A_r)}(-1)$ of rank $r-1$ on $\mathbb{P}(A_r)$ and let $H = \operatorname{im} \hat{\alpha}$ be a hyperplane. Then $\mathcal{T}_{\mathbb{P}(A_r)}(-1) \cong \widetilde{\Theta}(P_1(r))$ is stable. On the other hand, $P_1(r-1) \subseteq \alpha^*(P_1(r))$, while

$$\mu(\alpha^*(P_1(r))) = \frac{1}{r-1} < \frac{1}{r-2} = \mu(P_1(r-1)).$$

Thanks to Lemma 9.1.2 and Proposition 10.2.1, $\mathcal{T}_{\mathbb{P}(A_r)}(-1)|_H$[7] is not semistable for any hyperplane. This example, which does not depend on the ground field, is well-known (cf. [Fle84]). Building on recent work of Coskun, Huizenga and Smith [CHS22], we will show below that general Steiner bundles on $\mathbb{P}(A_r)$ behave similarly. We assume throughout that $\Bbbk = \mathbb{C}$.

Proposition 10.9.1. *Let $r \geq 3$, (V_1, V_2) be a pair of vector spaces such that*

(a) $\Delta_{(V_1,V_2)} \geq r-1$, *and*

(b) $q_r(\underline{\dim}(V_1, V_2)) < 1$.

There exists a non-empty open subset $O \subseteq \mathrm{rep}(K_r; V_1, V_2) \cap \mathrm{rep}_{\mathrm{proj}}(K_r, 1)$ such that for every $M \in O$ the Steiner bundle $\widetilde{\Theta}(M)$ is stable.

[7] See also Remark 9.1.4.

Proof. Following [CHS22], we define $\psi_{r-1} \coloneqq \frac{r-2+\sqrt{(r-1)^2+2(r-1)-3}}{2}$, so that

$$\psi_{r-1}+1 = \frac{r+\sqrt{(r-1)^2+2(r-1)-3}}{2} = \frac{r+\sqrt{r^2-4}}{2} = L_r.$$

Since $q_r(\underline{\dim}(V_1,V_2)) < 1$, we have $V_1 \neq 0$ and our current assumptions imply (see also Corollary 1.5.2)

$$\frac{r-1}{\dim_{\Bbbk} V_1} \leq \frac{\Delta_{(V_1,V_2)}}{\dim_{\Bbbk} V_1} \quad \text{and} \quad \frac{\Delta_{(V_1,V_2)}}{\dim_{\Bbbk} V_1} = \frac{\dim_{\Bbbk} V_2}{\dim_{\Bbbk} V_1} - 1 < L_r - 1 = \psi_{r-1},$$

respectively. Thanks to [CHS22, (5.1)] and Theorem 8.2.5, there is a dense open subset

$$O \subseteq \operatorname{Hom}_{\mathbb{P}(A_r)}(V_1 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}(A_r)}(-1), V_2 \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}(A_r)}) \cong \operatorname{rep}(K_r; V_1, V_2)$$

such that $O \subseteq \operatorname{rep}_{\mathrm{proj}}(K_r, 1)$, and for every $M \in O$ the Steiner bundle $\widetilde{\Theta}(M)$ is stable. □

Remark 10.9.2. The statement is also true if $q_r(\underline{\dim}(V_1,V_2)) = 1$ and $\dim_{\Bbbk} V_1 \leq \dim_{\Bbbk} V_2$. In this case there exists a non-empty open subset $O \subseteq \operatorname{rep}(K_r; V_1, V_2)$ such that every representation in O is isomorphic to some $P_i(r)$. By Proposition 8.3.1 and [Bra08, (8.1)], the bundles $\widetilde{\Theta}(P_i(r))$ are stable.[8]

Corollary 10.9.3. *Assume that $q_r(\underline{\dim}(V_1,V_2)) < 1$ and $\Delta_{(V_1,V_2)}(r-1) \geq r-1$. A general Steiner bundle $\mathcal{F}$ of rank $\Delta_{(V_1,V_2)}$ and with first Chern class $\dim_{\Bbbk} V_1$ is stable and $\hat{\alpha}^*(\mathcal{F})$ is not semistable for each $\alpha \in \operatorname{Inj}_{\Bbbk}(A_{r-1}, A_r)$. In particular, $\mathcal{F}|_H$ is not semistable for each hyperplane $H \subseteq \mathbb{P}(A_r)$.*

Proof. Let $O \subseteq \operatorname{rep}(K_r; V_1, V_2)$ be the non-empty open set defined in Proposition 10.9.1. By Theorem 5.4.1 the set $O \cap \operatorname{rep}_{\mathrm{proj}}(K_r, r-1)$ is non-empty and open. Let $M \in O \cap \operatorname{rep}_{\mathrm{proj}}(K_r, r-1)$, then $\mathcal{F} \coloneqq \widetilde{\Theta}(M)$ is stable. Since $M \in \operatorname{rep}_{\mathrm{proj}}(K_r, r-1)$, we have

$$\alpha^*(M) \cong \Delta_M(r-1)P_0(r-1) \oplus (\dim_{\Bbbk} M_1)P_1(r-1)$$

for all $\alpha \in \operatorname{Inj}_{\Bbbk}(A_{r-1}, A_r)$. Thus, $\mu(\alpha^*(M)) < \mu(P_1(r-1))$, and the above arguments show that $\hat{\alpha}^*(\mathcal{F}) \cong \widetilde{\Theta}(\alpha^*(M))$ is not semistable. □

Corollary 10.9.3 provides negative answers to [Har79, Problem 5] for a fixed rank and Chern class on an open subset of $\operatorname{rep}(K_r; V_1, V_2)$.

Our next result shows that we can find an infinite family of Steiner bundles of fixed rank and Chern class that are counterexamples and even belong to different $\mathrm{GL}(A_r)$-orbits. Moreover, these bundles are isomorphic when we restrict them to any hyperplane.

Corollary 10.9.4. *Let (V_1, V_2) be a pair of vector spaces such that*

(a) $q_r(\underline{\dim}(V_1,V_1)) < -r^2$, *and*

(b) $\Delta_{(V_1,V_2)}(r-1) \geq r-1$.

[8] In [Bra08] Gieseker stability was considered. It can be shown, however, that the relevant result of loc. cit. also holds in our context.

There exists an infinite family $(\mathcal{F}_i)_{i\in I}$ of Steiner bundles that are pairwise not isomorphic such that for all $i, j \in I$ the following statements hold:

(1) $\mathcal{F}_i$ *is stable.*

(2) $c(\mathcal{F}_i) = \dim_{\Bbbk} V_1$, $\mathrm{rk}(\mathcal{F}_i) = \Delta_{(V_1,V_2)}$.

(3) $\mathcal{F}_i|_{\mathfrak{v}} \cong \mathcal{F}_j|_{\mathfrak{v}}$ *for each hyperplane* $\mathfrak{v} \in \mathrm{Gr}_{r-1}(A_r)$.

(4) $\mathcal{F}_i|_{\mathfrak{v}}$ *is not semistable for each hyperplane* $\mathfrak{v} \in \mathrm{Gr}_{r-1}(A_r)$.

(5) All $\mathcal{F}_i$ *lie in different orbits under the natural action of* $\mathrm{GL}(A_r)$ *on* $\mathrm{Coh}(\mathbb{P}(A_r))$.

Proof. Let $O \subseteq \mathrm{rep}(K_r; V_1, V_2)$ be the non-empty open set defined in Proposition 10.9.1. We have seen in Section 6.4 that the algebraic group $G := \mathrm{GL}(A_r) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_1)$ acts on $\mathrm{rep}(K_r; V_1, V_2)$. We consider the G-stable and open set $U := \bigcup_{g\in G} g.O \subseteq \mathrm{rep}(K_r; V_1, V_2)$ and set $\mathcal{U} := U \cap \mathrm{rep}(K_r; V_1, V_2) \cap \mathrm{rep}_{\mathrm{proj}}(K_r, r-1)$ (see Section 6.4). By Theorem 5.4.1 and assumption (b) the G-stable set $\mathcal{U}$ is non-empty and open.

Let $M \in \mathcal{U}$. We find $(a, h_2, h_1) \in G$ and $S \in O \cap \mathrm{rep}(K_r; V_1, V_2) \cap \mathrm{rep}_{\mathrm{proj}}(K_r, r-1)$ such that $M = (a, h_2, h_1).S$. By definition, the representations M and $(a, \mathrm{id}_{V_2}, \mathrm{id}_{V_1}).S = a.S$ are isomorphic. We obtain (see the proof of Corollary 9.1.5)

$$\mathcal{F} := \widetilde{\Theta}(M) \cong \widetilde{\Theta}(a.S) \cong a^*.\widetilde{\Theta}(S).$$

Since $S \in O$, we know that $\widetilde{\Theta}(S)$ is stable. Therefore, the same is true for $a^*.\widetilde{\Theta}(S) \cong \mathcal{F}$. This shows that every bundle arising in this way has property (1). Owing to Theorem 8.2.5, each bundle of this form satisfies (2). The arguments in the proof of Corollary 10.9.3 now show that (3) and (4) hold. Since $q_r(\underline{\dim}(V_1, V_2)) < -r^2$, we can apply Lemma 6.4.2 and conclude that $\mathcal{U}/G$ is not finite. By construction, elements of $\mathrm{rep}(K_r; V_1, V_2)$ belonging to different G-orbits correspond to Steiner bundles, whose isomorphism classes belong to different orbits under the action of $\mathrm{GL}(A_r)$. This shows (5). □

Example 10.9.5. *We fix* $r = 4$.

(1) $q_r(5, 18) = -11$ *and* $\Delta_{(5,18)}(3) = 3$. *We apply Corollary 10.9.3 and find a stable Steiner bundle of rank* 13 *on* $\mathbb{P}(A_4)$ *such that* $\mathcal{F}|_H$ *is not semistable for each hyperplane* $H \subseteq \mathbb{P}(A_4)$.

(2) $q_r(6, 21) = -27 < -4^2$ *and* $\Delta_{(6,21)}(3) = 3$. *We apply Corollary 10.9.4 and find an infinite family of pairwise non-isomorphic stable Steiner bundles* $(\mathcal{F}_i)_{i\in I}$ *of rank* 15 *on* $\mathbb{P}(A_4)$ *such that* $\mathcal{F}_i|_H$ *is not semistable for all* $i \in I$ *and all hyperplanes* $H \subseteq \mathbb{P}(A_4)$. *Moreover, we have* $\mathcal{F}_i|_H \cong \mathcal{F}_j|_H$ *for each hyperplane and all* $i, j \in I$, *while* $\mathcal{F}_i$ *are* $\mathcal{F}_j$ *are not in the same* $\mathrm{GL}(A_r)$*-orbit.*

Appendices

A. Vector bundles as locally trivial linear fibrations

We provide background information on the relationship between vector bundles and locally trivial fibrations. This chapter is based on [Le 97] and [Sha13b].

A.1. Preliminaries

Definition. (1) Let $p\colon E \longrightarrow X$ be a morphism of varieties. For $x \in X$ we denote by $E_x := p^{-1}(x)$ the *fiber* of x under p.

(2) A *linear fibration* or *vector space fibration* over X consists of a surjective morphism $p\colon E \longrightarrow X$ of varieties such that for every $x \in X$ there exists $n_x \in \mathbb{N}$ and an isomorphism of varieties $E_x = p^{-1}(x) \longrightarrow \mathbb{A}^{n_x}$.

Remark A.1.1. Let $p\colon E \longrightarrow X$ be a linear fibration. Then each fiber E_x obtains the structure of a $\Bbbk$-vector space via structure transport along $t_x\colon E_x \longrightarrow \mathbb{A}^{n_x}$. Usually, we suppress the isomorphism t_x and just say that the fiber E_x is a vector space that renders E_x a full affine subspace of E.

Definition. A *morphism* $f\colon (E,p) \longrightarrow (F,q)$ of linear fibrations over X is a morphism $f\colon E \longrightarrow F$ such that

(a) the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

commutes, and

(b) the map $f_x := f|_{E_x}\colon E_x \longrightarrow F_x\ ;\ e \mapsto f(e)$ is $\Bbbk$-linear for all $x \in X$ (note that $f(E_x) = f(p^{-1}(x)) \subseteq q^{-1}(x) = F_x$ by (a)).

Example A.1.2. *(1) Let V be a finite dimensional $\Bbbk$-vector space. The variety $E := X \times V$ together with the projection $\mathrm{pr}_X\colon E \longrightarrow X\ ;\ (x,v) \mapsto x$ is a linear fibration with fibers $E_x = \mathrm{pr}^{-1}(x) = \{x\} \times V$ for all $x \in X$. A set $\{x\} \times U \subseteq E_x$ is open if and only if U is open in the space $V = \mathbb{A}(V)$.*

(2) Let (E,p) be a linear fibration over X and $U \subseteq X$ be an open subset. Then

$$p|_{p^{-1}(U)}\colon (p^{-1}(U), \mathcal{O}_{E|p^{-1}(U)}) \longrightarrow U$$

is a linear fibration over U that we also denote by $(E|_U, p) := (p^{-1}(U), p|_{p^{-1}(U)})$.

Definition. Let (E,p) be a linear fibration over X.

(1) We call (E,p) *trivial*, if there exists a finite dimensional vector space V such that $(E,p) \cong (X \times V, \mathrm{pr}_X)$.

(2) We call (E,p) *locally trivial (fibration)*, if there is an open cover $X = \bigcup_{i\in I} U_i$ such that for each $i \in I$ the linear fibration $p|_{p^{-1}(U_i)} \colon p^{-1}(U_i) \longrightarrow U_i$ is isomorphic to the trivial fibration $(U_i \times W_i, \mathrm{pr}_{U_i})$ for a finite dimensional vector space W_i.

Definition. (1) A *morphism* $f \colon (E,p) \longrightarrow (F,q)$ of locally trivial fibrations over X is a morphism of linear fibrations. We denote the space of all morphisms from (E,p) to (F,q), which carries the structure of $\Bbbk$-vector space, by $\mathrm{Hom}_{\mathsf{Ltf}(X)}((E,p),(F,q))$.

(2) We denote by $\mathsf{Ltf}(X)$ the category of locally trivial fibrations over X.

Lemma A.1.3. *Assume that X is connected and let (E,p) be locally trivial over X. Then*

$$\mathrm{rk} \colon X \longrightarrow \mathbb{N}_0 \ ; \ x \mapsto \dim_{\Bbbk} E_x$$

is constant.

Proof. Let $X = \bigcup_{i\in I} U_i$ be a cover with trivializations $f_i \colon p^{-1}(U_i) \longrightarrow U_i \times V_i$. Let $x \in U_i$, then $\mathrm{rk}(x) = \dim_{\Bbbk} V_i$. Hence $\mathrm{rk}|_{U_i} \colon U_i \longrightarrow \mathbb{N}_0$ is constant. Let $\ell := \min \mathrm{rk}$ and $J := \{j \in J \mid \mathrm{im}\, \mathrm{rk}|_{U_j} = \{\ell\}\}$. Then $X = (\bigcup_{j\in J} U_j) \cup (\bigcup_{j\in I\setminus J} U_j)$ is a disjoint union of two open sets. Hence $I \setminus J = \varnothing$ by the connectedness of X. This shows that rk is constant. □

Definition. Let X be connected and (E,p) be a locally trivial fibration over X. We call $\mathrm{rk}(E) := \mathrm{rk}(E_x)$ for $x \in X$ the *rank* of (E,p).

From now on we assume that X is a connected variety.

A.2. Sub-fibrations and quotient-fibrations

Definition. Let (E,p) be a locally trivial fibration over X.

(1) A *sub-fibration*[1] of (E,p) is a subvariety $F \subseteq E$ such that, for each $x \in X$, the intersection $F \cap E_x$ is a subspace of E_x and the induced linear fibration $(F, p|_F)$ is a locally trivial over X.

(2) Let $F \subseteq (E,p)$ be a sub-fibration. We define $E/F := \coprod_{x\in X}(E_x/F_x) = \bigsqcup_{x\in X}(E_x/F_x) \times \{x\}$, $\pi \colon E \longrightarrow E/F \ ; \ e \in E_x \mapsto (e + F_x, x)$ and $p_{E/F} \colon E/F \longrightarrow X \ ; \ (e + F_x, x) \mapsto x$.

[1] In the literature one usually speaks of *subbundles*.

(3) Let $F \subseteq (E,p)$ be a sub-fibration, (G,g) be locally trivial and $\varphi\colon E \longrightarrow G$ be a morphism of fibrations such that $\varphi_x(F_x) = \{0_{G_x}\}$ for all $x \in X$. We denote by $\overline{\varphi}$ the map

$$\overline{\varphi}\colon E/F \longrightarrow G\ ;\ (e+F_x, x) \mapsto \varphi(e) = \varphi_x(e)$$

and note that $\varphi = \overline{\varphi} \circ \pi$.

Remark A.2.1. According to [CJ06, Proposition 3] for every sub-fibration $F \subseteq E$ the space F is closed in E.

Proposition A.2.2. *[Le 97, (1.7.1)] Let $F \subseteq (E,p)$ be a sub-fibration. There exists a unique way to make $(E/F, \pi_{E/F})$ a locally trivial fibration over X such that the following statements hold.*

(1) The map $\pi\colon E \longrightarrow E/F$ is a morphism of fibrations over X.

(2) For each morphism of locally trivial fibrations $\varphi\colon E \longrightarrow G$ such that $\varphi_x(F_x) = \{0_{G_x}\}$ for all $x \in X$, the map $\overline{\varphi}\colon E/F \longrightarrow G$ is a morphism of fibrations.

Proposition A.2.3. *[Le 97, (1.7.2)] Let $\varphi\colon (E,p) \longrightarrow (F,q)$ be a morphism between locally trivial fibrations. Suppose that $X \longrightarrow \mathbb{N}_0\ ;\ x \mapsto \mathrm{rk}(\varphi_x)$ is constant. Then $\ker\varphi$ and $\mathrm{im}\,\varphi$ are sub-fibrations of E and F, respectively.*

Definition. (1) Let $\varphi\colon (E',p') \longrightarrow (E,p)$ and $\psi\colon (E,p) \longrightarrow (E'',p'')$ be morphisms of locally trivial fibrations over X. We say that

$$E' \xrightarrow{\varphi} E \xrightarrow{\psi} E''$$

is *exact* at E, provided the sequence

$$E'_x \xrightarrow{\varphi_x} E_x \xrightarrow{\psi_x} E''_x$$

of $\Bbbk$-vector spaces is exact for all $x \in X$. Long exact sequences are defined in the obvious way.

(2) Let $\varphi\colon (E,p) \longrightarrow (F,q)$ be a morphism of locally trivial fibrations such that $X \longrightarrow \mathbb{N}_0\ ;\ x \mapsto \mathrm{rk}(\varphi_x)$ is constant. We apply Proposition A.2.3 and Proposition A.2.2 and obtain a locally trivial fibration $\pi\colon F \longrightarrow F/\mathrm{im}\,\varphi$. We write $\mathrm{coker}\,\varphi \coloneqq F/\mathrm{im}\,\varphi$ and have an exact sequence of locally trivial fibration

$$0 \longrightarrow \ker\varphi \longrightarrow E \xrightarrow{\varphi} F \xrightarrow{\pi} \mathrm{coker}\,\varphi \longrightarrow 0.$$

A.3. Locally trivial fibrations and vector bundles

Let (E,p) be a locally trivial fibration. For $U \subseteq X$ open we define

$$\mathcal{L}_E(U) = \mathcal{L}_{(E,p)}(U) \coloneqq \{s\colon U \longrightarrow E \mid s \text{ morphism of varieties}, p\circ s = \mathrm{id}_U\}.$$

Let $s,t \in \mathcal{L}_E(U)$ and $r \in \mathcal{O}_X(U)$. Given $x \in U$, we have $s(x), t(x) \in p^{-1}(x) = E_x$. Hence we obtain a map

$$s+t\colon U \longrightarrow E\ ;\ x \mapsto s(x) +_{E_x} t(x) \in E_x$$

that satisfies $p\circ(s+t) = \mathrm{id}_U$. Recall that $\mathcal{O}_X(U) \subseteq \mathrm{Map}(U,\Bbbk)$. Hence we define

$$r.s\colon U \longrightarrow E\ ;\ x \mapsto r(x)\cdot_{E_x} s(x) \in E_x$$

and note that $p\circ r.s = \mathrm{id}_U$.

Lemma A.3.1. *The maps $s+t, r.s\colon U \longrightarrow E$ are morphisms of varieties.*

Proof. We first consider the case that E is a trivial bundle $\mathrm{pr}\colon X\times V \longrightarrow X$. Note that

$$\mathcal{L}_{X\times V}(U) = \{(\mathrm{id}_U, r) \mid r\colon U \longrightarrow V \text{ morphism of varieties}\}.$$

We write $s = (\mathrm{id}_U, s')$, $t = (\mathrm{id}_U, t')$ with morphisms $s', t'\colon U \longrightarrow V$. We apply Lemma 3.1.4 and conclude that $s'+t'\colon U \longrightarrow V\ ;\ x \mapsto s'(x)+t'(x)$ is a morphism of varieties. Hence $s+t = (\mathrm{id}_U, s'+t')\colon U \longrightarrow U\times V$ is a morphism of varieties. By the same token, we obtain that $r.s$ is a morphism of varieties.

In the general case we find an open cover $X = \bigcup_{i\in I} U_i$ and trivializations $f_i\colon p^{-1}(U_i) \longrightarrow U_i \times V$ that restrict to trivializations $f_i\colon p^{-1}(U_i\cap U) \longrightarrow (U_i\cap U)\times V$. Let $i \in I$. We consider

$$s|_{U_i\cap U}\colon U_i\cap U \longrightarrow p^{-1}(U_i\cap U) \text{ and } t|_{U_i\cap U}\colon U_i\cap U \longrightarrow p^{-1}(U_i\cap U).$$

The morphisms $f_i\circ s|_{U_i\cap U}, f_i\circ t|_{U_i\cap U}\colon U_i\cap U \longrightarrow (U_i\cap U)\times V$ satisfy $\mathrm{pr}_{U_i\cap U}\circ(f_i\circ s|_{U_i\cap U}) = \mathrm{id}_{U_i\cap U} = \mathrm{pr}_{U_i\cap U}\circ(f_i\circ t|_{U_i\cap U})$. We conclude with the previous consideration that

$$f_i\circ s|_{U_i\cap U} + f_i\circ t|_{U_i\cap U}\colon U_i\cap U \longrightarrow (U_i\cap U)\times V; x \mapsto (f_i\circ s)(x) + (f_i\circ t)(x) = f_i(s(x)+t(x))$$

is a morphism of varieties. Note that $f_i\circ(s+t)|_{U_i\cap U} = f_i\circ s|_{U_i\cap U} + f_i\circ t|_{U_i\cap U}$ since f_i is $\Bbbk$-linear on fibers. Since f_i is an isomorphism, we conclude that $(s+t)|_{U_i\cap U}$ is a morphism of varieties for all $i\in I$. Now we apply Lemma 3.1.2 and conclude that $s+t\colon U \longrightarrow E$ is a morphism. By the same token, the map $r.s$ is a morphism of varieties. □

Let (E,p) be a locally trivial fibration over X. By Lemma A.3.1 the set $\mathcal{L}_E(U)$ obtains the structure of an $\mathcal{O}_X(U)$-module with 0-map given by

$$U \longrightarrow E\ ;\ u \mapsto 0_{E_u}.$$

We denote by $\mathcal{L}_E$ the corresponding $\mathcal{O}_X$-module with set of sections $\mathcal{L}_E(U)$ for $U\subseteq X$ open and natural restriction maps. Given a morphism $f\colon (E,p) \longrightarrow (F,q)$ of locally trivial fibrations over X we let

$$\mathcal{L}_f(U)\colon \mathcal{L}_E(U) \longrightarrow \mathcal{L}_F(U)\ ;\ s \mapsto f\circ s.$$

We note that this is well-defined since $q\circ(f\circ s) = p\circ s = \mathrm{id}_U$. This renders $\mathcal{L}$ a $\Bbbk$-functor from the category $\mathsf{Ltf}(X)$ of locally trivial fibrations over X into the category $\mathrm{Mod}\,\mathcal{O}_X$ of $\mathcal{O}_X$-modules. The next result shows that the elements in the essential image of $\mathcal{L}$ are vector bundles.

Lemma A.3.2. *Let (E,p) be a locally trivial fibration over X.*

(1) For each $x \in X$ we have a $\Bbbk$-linear map

$$\mathrm{ev}_x \colon \mathcal{L}_E(x) \longrightarrow E_x \ ; \ [(U,s)] + \mathfrak{m}_x(\mathcal{L}_E)_x \mapsto s(x).$$

(2) Let (F,q) be a locally trivial fibration, $\varphi \colon E \longrightarrow F$ be a morphism and $x \in X$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_E(x) & \xrightarrow{\mathrm{ev}_x} & E_x \\ {\scriptstyle \mathcal{L}(\varphi)(x)}\downarrow & & \downarrow{\scriptstyle \varphi_x} \\ \mathcal{L}_F(x) & \xrightarrow{\mathrm{ev}_x} & F_x \end{array}$$

in the category of finite dimensional vector spaces.

(3) Let $x \in X$, then ev_x is an isomorphism of vector spaces.

(4) The $\mathcal{O}_X$-module $\mathcal{L}_E$ is a vector bundle of rank $\mathrm{rk}(\mathcal{L}_E) = \mathrm{rk}(E)$.

Proof. (1) We consider the map

$$\mathrm{Ev}_x \colon (\mathcal{L}_E)_x \longrightarrow E_x \ ; \ [(U,s)] \mapsto s(x).$$

The $\Bbbk$-vector space E_x obtains the structure of an $\mathcal{O}_{X,x}$-module via

$$[(U,r)].e \coloneqq r(x) \cdot_{E_x} e \quad \text{for all } [(U,r)] \in \mathcal{O}_{X,x}, e \in E_x.$$

Direct computation shows that Ev_x is $\mathcal{O}_{X,x}$-linear. Clearly, $\mathfrak{m}_x(\mathcal{L}_E)_x \subseteq \ker \mathrm{Ev}_x$. Hence ev_x is well-defined.

(2) We have

$$\begin{aligned}(\mathrm{ev}_x \circ \mathcal{L}(\varphi)(x))([(U,s)] + \mathfrak{m}_x(\mathcal{L}_E)_x) &= \mathrm{ev}_x([(U,\varphi\circ s)] + \mathfrak{m}_x(\mathcal{L}_F)_x) = (\varphi \circ s)(x) \\ &= \varphi_x(s(x)) = (\varphi_x \circ \mathrm{ev}_x)([(U,s)] + \mathfrak{m}_x(\mathcal{L}_E)_x).\end{aligned}$$

(3) At first we consider the case that $E = X \times V$ is a trivial bundle. We have

$$\mathcal{L}_E(U) = \{s \colon U \longrightarrow U \times V \mid s = (\mathrm{id}_U, t), t \colon U \longrightarrow V \text{ morphism of varieties}\}.$$

Let $(x,v) \in \{x\} \times V = E_x$. We consider the constant map $t \colon U \longrightarrow V; u \mapsto v$. Then t is a morphism and $(\mathrm{id}_U, t) \in \mathcal{L}_E(U)$. We have $\mathrm{Ev}_x([(U,s)]) = (\mathrm{id}_U, t)(x) = (x, t(x)) = (x,v)$. Hence ev_x is surjective.

Now suppose that $[(U,s)] \in \ker \mathrm{Ev}_x$. We fix a basis $(v_1, \ldots, v_r)$ of V and write $s = (\mathrm{id}_U, t)$ with $t \colon U \longrightarrow V$. We apply Lemma 3.1.4 and obtain $t_1, \ldots, t_r \in \mathcal{O}_{X|U}(U) = \mathcal{O}_X(U)$ such that $t(u) = \sum_{i=1}^r t_i(u) v_i$ for all $u \in U$. We conclude

$$(x, 0_V) = 0_{E_x} = s(x) = (x, t(x)) = (x, \sum_{i=1}^r t_i(x) v_i).$$

Hence $t_i(x) = 0$ for all $i \in \{1, \dots, r\}$ and therefore $[(U, t_i)] \in \mathfrak{m}_x$. We consider the constant morphism $g_i \colon U \longrightarrow V; u \mapsto v_i$ for $1 \leq i \leq r$ and set $f_i := (\mathrm{id}_U, g_i) \in \mathcal{L}_E(U)$. We have

$$[(U, s)] = \sum_{i=1}^{r} [(U, t_i)] \cdot [(U, f_i)] \in \mathfrak{m}_x(\mathcal{L}_E)_x.$$

Hence $\ker \mathrm{Ev}_x = \mathfrak{m}_x(\mathcal{L}_E)_x$ and ev_x is an isomorphism.

Now we consider the general case. For $x \in X$ we find $x \in U_i \subseteq X$ open and an isomorphism of fibrations $f_i \colon p^{-1}(U_i) \longrightarrow U_i \times V$. By (2) and the above consideration, we conclude that

$$\mathrm{ev}_x \colon \mathcal{L}_{E|_{U_i}}(x) = \mathcal{L}_{p^{-1}(U_i)}(x) \longrightarrow p^{-1}(U_i)_x = E_x$$

is an isomorphism of vector spaces. Let $s \colon U \longrightarrow E \in (\mathcal{L}_E)|_{U_i}(U)$ be a section for $U \subseteq U_i$ open. Then $p \circ s = \mathrm{id}_U$ implies $\mathrm{im}\, s \subseteq p^{-1}(U) \subseteq p^{-1}(U_i)$. Hence $s \colon U \longrightarrow p^{-1}(U_i) \in \mathcal{L}_{E|U_i}(U)$. Let $\iota \colon p^{-1}(U_i) \longrightarrow E$ be the inclusion morphism. By the previous consideration the map

$$\hat{\iota} \colon \mathcal{L}_{E|U_i} \mapsto (\mathcal{L}_E)|_{U_i} \; ; \; s \mapsto \iota \circ s$$

is an isomorphism of $\mathcal{O}_{U_i}$-modules. We conclude that $\hat{\iota}(x) \colon (\mathcal{L}_E)|_{U_i}(x) \longrightarrow \mathcal{L}_{E|U_i}(x)$ is an isomorphisms of $\Bbbk$-vector spaces. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{E|U_i}(x) & \xrightarrow{\mathrm{ev}_x} & E_x \\ {\scriptstyle \hat{\iota}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{E_x}} \\ (\mathcal{L}_E)|_{U_i}(x) & \xrightarrow{\mathrm{ev}_x} & E_x. \end{array}$$

Hence $\mathrm{ev}_x \colon (\mathcal{L}_E)|_{U_i}(x) \longrightarrow E_x$ is an isomorphism. The statement follows since $(\mathcal{L}_E)|_{U_i}(x) = (\mathcal{L}_E)(x)$.

(4) Since X is connected, we conclude with (3) and Lemma A.1.3 that $\dim_{\Bbbk} \mathcal{L}_E(x) = \dim_{\Bbbk} E_x = \mathrm{rk}(E)$ does not depend on $x \in X$. Now we apply Theorem 7.2.1(1).

□

Example A.3.3. *Let V be an r-dimensional vector space and $1 \leq d < r$.*

(1) By definition we have

$$\begin{aligned} \mathcal{L}_{(\mathrm{Gr}_d(V) \times \Bbbk, \mathrm{pr}_{\mathrm{Gr}_d(V)})}(U) &= \{s \colon U \longrightarrow U \times \Bbbk \mid s = (\mathrm{id}_U, t), t \colon U \longrightarrow \Bbbk \textit{ morphism of varieties}\} \\ &\cong \{t \colon U \longrightarrow \Bbbk \mid t \textit{ morphism of varieties}\} = \mathcal{O}_{\mathrm{Gr}_d(V)}(U). \end{aligned}$$

Hence we may identify $\mathcal{O}_{\mathrm{Gr}_d(V)}$ and $\mathcal{L}_{(\mathrm{Gr}_d(V) \times \Bbbk, \mathrm{pr}_{\mathrm{Gr}_d(V)})}$. For $\mathfrak{v} \in \mathrm{Gr}_d(V)$, we obtain an isomorphism of $\Bbbk$-vector spaces

$$\mathrm{ev}_{\mathfrak{v}} \colon \mathcal{O}_{\mathrm{Gr}_d(V)}(\mathfrak{v}) \longrightarrow \Bbbk \; ; \; \overline{[(U, t)]} \mapsto t(\mathfrak{v}).$$

(2) We consider the universal vector bundle $\mathcal{U}_V$ of $\mathrm{Gr}_d(V)$. By definition, $\mathcal{U}_V := \mathcal{L}_{(E,p)}$, where

$$E := \{(\mathfrak{v}, x) \in \mathrm{Gr}_d(V) \times V \;\mid\; x \in \mathfrak{v}\} \;\textit{ and }\; p : E \longrightarrow \mathrm{Gr}_d(V) \; ; \; (\mathfrak{v}, x) \mapsto \mathfrak{v}$$

denote the incidence variety and the canonical projection, respectively. Consequently,

$$\mathcal{U}_V(U) = \{s : U \longrightarrow E \mid p \circ s = \mathrm{id}_U\} \cong \{t : U \longrightarrow V \mid t(\mathfrak{v}) \in \mathfrak{v} \quad \textit{for all } \mathfrak{v} \in U\}$$

for every open subset $U \subseteq \mathrm{Gr}_d(V)$. For $\mathfrak{v} \in \mathrm{Gr}_d(V)$, we obtain an isomorphism of $\Bbbk$-vector spaces

$$\mathrm{ev}_{\mathfrak{v}} \colon \mathcal{U}_V(\mathfrak{v}) \longrightarrow \mathfrak{v}; \overline{[(U,t)]} \mapsto t(\mathfrak{v}).$$

Corollary A.3.4. *The functor $\mathcal{L}$ is exact.*

Proof. Let

$$0 \longrightarrow E' \xrightarrow{\varphi} E \xrightarrow{\psi} E'' \longrightarrow 0$$

be an exact sequence of locally trivial fibrations and $x \in X$. From Lemma A.3.2 we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_{E'}(x) & \xrightarrow{\mathcal{L}(\varphi)(x)} & \mathcal{L}_E(x) & \xrightarrow{\mathcal{L}(\psi)(x)} & \mathcal{L}_{E''}(x) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ev}_x} & & \downarrow{\scriptstyle \mathrm{ev}_x} & & \downarrow{\scriptstyle \mathrm{ev}_x} & & \\ 0 & \longrightarrow & E'_x & \xrightarrow{\varphi_x} & E_x & \xrightarrow{\psi_x} & E''_x & \longrightarrow & 0. \end{array}$$

in $\mathrm{mod}\,\Bbbk$ with vertical isomorphisms. Since the lower row is exact, the same is true for the upper row. Now we apply Corollary 7.2.4. □

Theorem A.3.5. *[Sha13b, §6.1.3], [Le 97, (1.8.1)] The functor $\mathcal{L}$ induces an exact equivalence of categories*

$$\mathcal{L} \colon \mathsf{Ltf}(X) \longrightarrow \mathrm{Vect}(X)$$

such that for all locally trivial fibrations $(E,p),(F,q)$ the isomorphism $\mathrm{Hom}_{\mathsf{Ltf}(X)}((E,p),(F,q)) \cong \mathrm{Hom}_X(\mathcal{L}_{(E,p)}, \mathcal{L}_{(F,q)})$ is an isomorphism of $\Bbbk$-vector spaces.

Corollary A.3.6. *Let $\mathcal{F} \in \mathrm{Coh}(X)$ be a vector bundle of rank n. There exists (up to isomorphism) a unique locally trivial fibration (E,p) over X of rank n such that $\mathcal{F} \cong \mathcal{L}_E$.* □

Corollary A.3.7. *The following statements hold.*

(1) The vector bundle $\mathcal{U}_V$ is simple.

(2) There is an isomorphism of $\Bbbk$-vector spaces

$$\psi \colon V^* \longrightarrow \mathrm{Hom}_{\mathrm{Gr}_d(V)}(\mathcal{U}_V, \mathcal{O}_{\mathrm{Gr}_d(V)}) \;\; ; \;\; \lambda \mapsto \psi_\lambda,$$

where

$$(\psi_\lambda)_U \colon \mathcal{U}_V(U) \longrightarrow \mathcal{O}_{\mathrm{Gr}_d(V)}(U) \;\; ; \;\; (s \colon U \longrightarrow E) \mapsto ((\mathrm{id}_{\mathrm{Gr}_d(V)}, \lambda) \circ \iota \circ s \colon U \longrightarrow \Bbbk)$$

with $\iota \colon E \longrightarrow \mathrm{Gr}_d(V) \times V$ being the natural embedding.

(3) *We have* $\mathrm{Hom}_{\mathrm{Gr}_d(V)}(\mathcal{O}_{\mathrm{Gr}_d(V)}, \mathcal{U}_V) = 0$.

Proof. Recall from Example A.3.3 that $\mathcal{U}_V = \mathcal{L}_{(E,p)}$, where

$$E := \{(\mathfrak{v}, x) \in \mathrm{Gr}_d(V) \times V \mid x \in \mathfrak{v}\} \text{ and } p : E \longrightarrow \mathrm{Gr}_d(V)\ ;\ (\mathfrak{v}, x) \mapsto \mathfrak{v}.$$

(1) If $d = 1$, $\mathcal{U}_V$ is a line bundle, hence irreducible and in particular simple. From now on we assume that $d \geq 2$. Let $\varphi\colon (E,p) \longrightarrow (E,p)$ be a morphism of linear fibrations. We find a morphism $t\colon E \longrightarrow V$ such that $\varphi(\mathfrak{v}, v) = (\mathfrak{v}, t(\mathfrak{v}, v))$ for all $(\mathfrak{v}, v) \in E$. Let $v \in V$. The set $E_v = \{\mathfrak{v} \in \mathrm{Gr}_d(V) \mid v \in \mathfrak{v}\}$[2] is a closed irreducible subset of $\mathrm{Gr}_d(V)$. We conclude with [Sha13a, §1.5.2] that $t|_{E_v \times \{x\}}\colon E_v \times \{v\} \longrightarrow V$ is constant and find $x_v \in V$ such that $t(\mathfrak{v}, v) = x_v$ for all $\mathfrak{v} \in E_v$. Since $x_v \in \bigcap_{\mathfrak{v} \in E_v} \mathfrak{v} = \Bbbk v$, we find $\lambda_v \in \Bbbk$ such that

$$\varphi(\mathfrak{v}, v) = (\mathfrak{v}, \lambda_v v) \text{ for all } (\mathfrak{v}, v) \in E.$$

It remains to show that $\lambda_v = \lambda_w$ for all $v, w \in V \setminus \{0\}$. If $\Bbbk v = \Bbbk w$, then we have $E_v = E_w$ and find $\lambda \in \Bbbk$ such that $v = \lambda w$. For $\mathfrak{v} \in E_v$ we conclude with the linearity of $\varphi(\mathfrak{v}, -)$ that

$$(\mathfrak{v}, 0) = \varphi(\mathfrak{v}, 0) = \varphi(\mathfrak{v}, v - \lambda w) = (\mathfrak{v}, \lambda_v v - \lambda\lambda_w w) = (\mathfrak{v}, (\lambda_v - \lambda_w)(v)),$$

hence $\lambda_v = \lambda_w$. Now assume that $\Bbbk v \neq \Bbbk w$. Since $d \geq 2$, we find $\mathfrak{v} \in V$ such that $v, w \in \mathfrak{v}$. Again, we conclude with linearity

$$(\mathfrak{v}, \lambda_{v+w}(v + w)) = \varphi(\mathfrak{v}, v + w) = (\mathfrak{v}, \lambda_v v + \lambda_w w)$$

and $\Bbbk v \cap \Bbbk w = \{0\}$ yields $\lambda_v = \lambda_{v+w} = \lambda_w$. Hence $\varphi \in \Bbbk \,\mathrm{id}_{(E,p)}$.

(2) We consider

$$\overline{\psi}\colon V^* \longrightarrow \mathrm{Hom}_{\mathsf{Ltf}}((E,p), (\mathrm{Gr}_d(V) \times \Bbbk, \mathrm{pr}_{\mathrm{Gr}_d(V)}))\ ;\ \lambda \mapsto \overline{\psi}_\lambda$$

with $\overline{\psi}_\lambda\colon (E,p) \longrightarrow (\mathrm{Gr}_d(V) \times \Bbbk, \mathrm{pr}_{\mathrm{Gr}_d(V)})\ ;\ (\mathfrak{v}, v) \mapsto (\mathfrak{v}, \lambda(v))$. Note that we have a commutative diagram

$$\begin{array}{ccc} V^* & = & V^* \\ \downarrow{\scriptstyle \overline{\psi}} & & \downarrow{\scriptstyle \psi} \\ \mathrm{Hom}_{\mathsf{Ltf}}((E,p), (\mathrm{Gr}_d(V) \times \Bbbk, \mathrm{pr}_{\mathrm{Gr}_d(V)})) & \xrightarrow{\mathcal{L}} & \mathrm{Hom}_{\mathrm{Gr}_d(V)}(\mathcal{U}_V, \mathcal{O}_{\mathrm{Gr}_d(V)}). \end{array}$$

Hence it suffices to prove that ψ or $\overline{\psi}$ is an isomorphism of vector spaces. Note that $\overline{\psi}$ is injective, hence the same holds for ψ. If $d = 1$, we have $\mathcal{U}_V \cong \mathcal{O}_{\mathbb{P}(V)}(-1)$ (see [EH16, §3.2.3]) and [Ben16, (6.9.2)] implies that $\dim_\Bbbk \mathrm{Hom}_{\mathrm{Gr}_d(V)}(\mathcal{U}_V, \mathcal{O}_{\mathrm{Gr}_d(V)}) = r = \dim_\Bbbk V^*$. Hence we may assume from now on that $d \geq 2$. Let $\varphi\colon (E,p) \longrightarrow (\mathrm{Gr}_d(V) \times \Bbbk, \mathrm{pr}_{\mathrm{Gr}_d(V)})$ be a morphism of linear fibrations. As in (1) one shows that there are $\lambda(v) \in \Bbbk$ such that

$$\varphi(\mathfrak{v}, v) = (\mathfrak{v}, \lambda(v)) \text{ for all } (\mathfrak{v}, v) \in E.$$

Let $v, w \in V$, then $d \geq 2$ implies that we find $\mathfrak{v} \in \mathrm{Gr}_d(V)$ with $v, w \in \mathfrak{v}$. Using $\Bbbk$-linearity on fibers, we conclude

$$(\mathfrak{v}, \lambda(v + w)) = \varphi(\mathfrak{v}, v + w) = \varphi(\mathfrak{v}, v) + \varphi(\mathfrak{v}, w) = (\mathfrak{v}, \lambda(v) + \lambda(w)).$$

By the same token, we obtain $\lambda(\beta v) = \beta\lambda(v)$ for all $\beta \in \Bbbk$. This shows that $\lambda \in V^*$. Hence $\overline{\psi}$ is surjective and the same is true for ψ.

[2] For $v = 0$ we have $E_v = \mathrm{Gr}_d(V)$ and for $v \neq 0$ we have an isomorphism $E_v \longrightarrow \mathrm{Gr}_{d-1}(V/\Bbbk v)\ ;\ \mathfrak{v} \mapsto \mathfrak{v}/\Bbbk v$.

(3) Let $\varphi\colon (\mathrm{Gr}_d(V)\times\Bbbk, \mathrm{pr}_{\mathrm{Gr}_d(V)}) \longrightarrow (E,p)$ be a morphism of linear fibrations. Then there exists a morphism $f\colon \mathrm{Gr}_d(V)\times\Bbbk \longrightarrow V$ such that

$$\varphi(\mathfrak{v},\lambda) = (\mathfrak{v}, f(\mathfrak{v},\lambda))$$

for all $(\mathfrak{v},\lambda)\in \mathrm{Gr}_d(V)\times\Bbbk$. Let $\lambda\in\Bbbk$, then $\mathrm{Gr}_d(V)\times\{\lambda\}$ is an irreducible projective variety. As in (1) we conclude that $f\colon \mathrm{Gr}_d(V)\times\{\lambda\}\longrightarrow V$ is constant. Since $f(\mathfrak{v},\lambda)=f(\mathfrak{w},\lambda)\in\mathfrak{w}$ for all $\mathfrak{w}\in\mathrm{Gr}_d(V)$, we conclude $f(\mathfrak{v},\lambda)\in\bigcap_{\mathfrak{w}\in\mathrm{Gr}_d(V)}\mathfrak{w}=\{0\}$. Hence $\varphi(\mathfrak{v},\lambda)=(\mathfrak{v},0)$ for all $(\mathfrak{v},\lambda)\in\mathrm{Gr}_d(V)\times\Bbbk$.

□

Remark A.3.8. If we consider the elements $\mathcal{O}_{\mathrm{Gr}_d(V)}(U)$ and $\mathcal{U}_V(U)$ as morphisms $s\colon U\longrightarrow\Bbbk$, $t\colon U\longrightarrow V$ with $t(\mathfrak{v})\in\mathfrak{v}$ for all $\mathfrak{v}\in U$, respectively, then ψ from Corollary A.3.7(2) corresponds to the map

$$\psi\colon V^*\longrightarrow \mathrm{Hom}_{\mathrm{Gr}_d(V)}(\mathcal{U}_V,\mathcal{O}_{\mathrm{Gr}_d(V)})\ ;\ \lambda\mapsto\psi_\lambda,$$

with $\psi_\lambda(s)=\lambda\circ t$ for all $t\in\mathcal{U}_V(U)$.

B. Chern Classes for Grassmann varieties

Let V be a finite dimensional $\Bbbk$-vector space and $1 \leq d < r \coloneqq \dim_{\Bbbk} V$. In this section we recall the definition of the Chern polynomial $c(\mathcal{F})$ of a vector bundle $\mathcal{F}$ on $\mathrm{Gr}_d(V)$ and collect basic properties of the function $c(-)$. This chapter is based on [EH16]. We remark that the general assumption in [EH16] is that $\Bbbk$ is of characteristic zero. However, the cited results are true for algebraically closed fields of arbitrary characteristic. For all the details in a more general setting we refer the reader to [Ful98, §3.2, §14.6, §14.7].

B.1. The Chow ring

For $\ell \in \{0, \ldots, \dim \mathrm{Gr}_d(V) = d(r-d)\}$, we let $Z_\ell(\mathrm{Gr}_d(V))$ be the free abelian group with basis given by the ℓ-dimensional irreducible closed subsets of $\mathrm{Gr}_d(V)$. The elements of $Z_\ell(\mathrm{Gr}_d(V))$ are called *ℓ-cycles*, those of

$$Z(\mathrm{Gr}_d(V)) \coloneqq \bigoplus_{\ell=0}^{d(r-d)} Z_\ell(\mathrm{Gr}_d(V))$$

are referred to as *cycles*. The *Chow group* of $\mathrm{Gr}_d(V)$ is the factor group

$$A(\mathrm{Gr}_d(V)) \coloneqq Z(\mathrm{Gr}_d(V))/\mathrm{Rat}(\mathrm{Gr}_d(V))$$

given by the subgroup $\mathrm{Rat}(\mathrm{Gr}_d(V))$ defined in [EH16, §1.2.2]. If C is a cycle, we write $[C]$ for its residue class. The Chow group is graded by dimension, that is,

$$A(\mathrm{Gr}_d(V)) = \bigoplus_{i=0}^{d(r-d)} A_i(\mathrm{Gr}_d(V))$$

with $A_\ell(\mathrm{Gr}_d(V)) \coloneqq \{[C] \mid C \in Z_\ell(\mathrm{Gr}_d(V))\}$ the group of rational equivalence classes of ℓ-cycles. Following [EH16, §1.2.2] we set

$$A^i(\mathrm{Gr}_d(V)) \coloneqq A_{d(r-d)-i}(\mathrm{Gr}_d(V))$$

for $i \in \{0, \ldots, d(r-d)\}$. The class $[\mathrm{Gr}_d(V)] \in A(\mathrm{Gr}_d(V))$ is called the *fundamental class* of $\mathrm{Gr}_d(V)$. Since $\mathrm{Gr}_d(V)$ is projective, the group $A(\mathrm{Gr}_d(V))$ has the structure of a $\mathbb{Z}$-graded, commutative ring, see [EH16, §1.2.3].

Definition. The ring $A(\mathrm{Gr}_d(V))$ is called the *Chow ring* of $\mathrm{Gr}_d(V)$.

Let

$$\mathcal{V}: \quad 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{r-1} \subseteq V_r = V$$

be a complete flag in V, so that $\dim_{\Bbbk} V_i = i$. For $d \in \{1, \ldots, r\}$ we let

$$\mathbb{N}_0^d(r) \coloneqq \{a \in \mathbb{N}_0^d \mid r - d \geq a_1 \geq \cdots \geq a_d \geq 0\}.$$

For $a \in \mathbb{N}_0^d(r)$ we define the *Schubert variety* $\Sigma_a(\mathcal{V})$ via

$$\Sigma_a(\mathcal{V}) \coloneqq \{\mathfrak{v} \in \mathrm{Gr}_d(V) \mid \dim_{\Bbbk}(\mathfrak{v} \cap V_{r-d+i-a_i}) \geq i \quad \text{for all } i \in \{1, \ldots, d\}\}.$$

Then $\Sigma_a(\mathcal{V})$ is smooth, closed, irreducible and of codimension $|a| \coloneqq \sum_{i=1}^d a_i$ in $\mathrm{Gr}_d(V)$, cf. [EH16, §4.1]. Its class

$$\sigma_a \coloneqq [\Sigma_a(\mathcal{V})]$$

is the *Schubert class*. We have the following Theorem.

Theorem B.1.1. *(cf. [EH16, (4.7)]) The Schubert classes $\{\sigma_a \mid a \in \mathbb{N}_0^d(r)\}$ form a basis of the $\mathbb{Z}$-module $A(\mathrm{Gr}_d(V))$.*

It follows that

$$\{\sigma_a \mid a \in \mathbb{N}_0^d(r),\ |a| = i\}$$

is a basis of $A^i(\mathrm{Gr}_d(V))$ for every $i \in \{0, \ldots, d(r-d)\}$. In particular, we have isomorphisms of abelian groups $A^0(\mathrm{Gr}_d(V)) = \mathbb{Z}\sigma_{(0,\ldots,0)} \cong \mathbb{Z}$, $A^1(\mathrm{Gr}_d(V)) \cong \mathbb{Z}\sigma_{(1,0,\ldots,0)} \cong \mathbb{Z}$ and $A(\mathrm{Gr}_d(V)) \cong \mathbb{Z}^{\binom{r}{d}}$ (cf. [EH16, (4.13)]). Given $\ell \in \{0, \ldots, r-d\}$, we write

$$\Sigma_\ell \coloneqq \Sigma_{(\ell,0,\ldots,0)},$$

so that

$$\Sigma_\ell(\mathcal{V}) = \{\mathfrak{v} \in \mathrm{Gr}_d(V) \mid \mathfrak{v} \cap V_{r-d+1-\ell} \neq 0\}.$$

The classes σ_ℓ are referred to as *special Schubert classes*. We have

$$\sigma_\ell \in A^\ell(\mathrm{Gr}_d(V))$$

for all $\ell \in \{0, \ldots, r-d\}$.

B.2. The Chern polynomial

Let $\mathcal{F}$ be a vector bundle on $\mathrm{Gr}_d(V)$. We denote by

$$c(\mathcal{F}) = 1 + \sum_{i=1}^{d(r-d)} c_i(\mathcal{F}) t^i \in A(\mathrm{Gr}_d(V))[t]$$

the *Chern polynomial* of $\mathcal{F}$. The coefficient $c_i(\mathcal{F}) \in A^i(\mathrm{Gr}_d(V))$ is called the i-th *Chern class* of $\mathcal{F}$. For the definition of $c_i(\mathcal{F})$ see [EH16, §5.2, §5.5]. We recall the for us most important properties of $c(-)$.

Theorem B.2.1 (Whitney sum formula). *(cf. [EH16, (5.3)]) Let*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

be a short exact sequence of vector bundles on $\mathrm{Gr}_d(V)$. *Then* $c(\mathcal{F}) = c(\mathcal{E}) \cdot c(\mathcal{G}) \in A(\mathrm{Gr}_d(V))$ *and, in particular* $c_1(\mathcal{F}) = c_1(\mathcal{E}) + c_1(\mathcal{G})$.

Theorem B.2.2 (Vanishing). *(cf. [EH16, (5.13)]) Let* $\mathcal{F}$ *be a vector bundle on* $\mathrm{Gr}_d(V)$, *then* $c_i(\mathcal{F}) = 0$ *for all* $i > \mathrm{rk}(\mathcal{F})$.

Theorem B.2.3 (Sums of line bundles). *(cf. [EH16, (5.4)]) If* $\mathcal{F}$ *is a direct sum of line bundles* $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, *then*

$$c(\mathcal{F}) = \prod_{i=1}^{n} c(\mathcal{L}_i).$$

We denote by $\mathrm{Pic}(\mathrm{Gr}_d(V))$ the *Picard group* of $\mathrm{Gr}_d(V)$. By definition $\mathrm{Pic}(\mathrm{Gr}_d(V))$ is the abelian group whose elements consist of isomorphism classes of line bundles with addition $[\mathcal{L}] + [\mathcal{L}'] := [\mathcal{L} \otimes_{\mathrm{Gr}_d(V)} \mathcal{L}']$.

Proposition B.2.4. *(cf. [EH16, (1.30)]) The map*

$$c_1 \colon \mathrm{Pic}(\mathrm{Gr}_d(V)) \longrightarrow A^1(\mathrm{Gr}_d(V)) = \mathbb{Z}\sigma_1 \; ; \; \mathcal{L} \mapsto c_1(\mathcal{L})$$

is an isomorphism of groups.

B.3. Projective space

In the following we consider the special case $d = 1$, i.e. $\mathrm{Gr}_d(V) = \mathbb{P}(V)$. Then $\mathbb{N}_0^d(r) = \{0, \ldots, r-1\}$, every cycle is a special Schubert cycle and $A^i(\mathbb{P}(V)) = \mathbb{Z}\sigma_i \cong \mathbb{Z}$ for all $i \in \{0, \ldots, r-1\}$. Therefore, we may write

$$c(\mathcal{F}) = \sum_{i=0}^{r-1} c_i(\mathcal{F}) t^i = \sum_{i=0}^{r-1} (c_i \sigma_i) t^i = \sum_{i=0}^{r-1} c_i t^i$$

with uniquely determined $c_0, \ldots, c_{r-1} \in \mathbb{Z}$. According to Pieri's formula (cf. [EH16, §4.2.4]), we have

$$\sigma_a \cdot \sigma_b = \sum_{c=a+b} \sigma_c = \begin{cases} \sigma_{a+b}, \text{ if } a+b \leq r-1 \\ 0, \text{if} a+b \geq r, \end{cases}$$

and conclude $A(\mathbb{P}(V)) \cong \mathbb{Z}[t]/(t^r)$ (see also [EH16, (2.1)]). Let $\ell \in \{0, \ldots, r-1\}$. Note that

$$\Sigma_\ell(\mathcal{V}) = \{\mathfrak{v} \in \mathbb{P}(V) \mid \mathfrak{v} \cap V_{r-\ell} \neq \{0\}\} = \{\mathfrak{v} \in \mathbb{P}(V) \mid \mathfrak{v} \subseteq V_{r-\ell}\}.$$

Following [EH16, §5.6.1] we call $\sigma_1 = [\Sigma_1(\mathcal{V})] = [\{\mathfrak{v} \in \mathbb{P}(V) \mid \mathfrak{v} \subseteq V_{r-1}\}] \in A^1(\mathbb{P}(V))$ the *hyperplane class*. By [EH16, (5.3)] we have

$$c(\mathcal{O}_{\mathbb{P}(V)}(d)) = 1 + d \cdot t$$

for all $d \in \mathbb{Z}$, where $\mathcal{O}_{\mathbb{P}(V)}(d)$ denotes the *d-th Serre twist* on $\mathbb{P}(V)$. As a consequence $c(\mathcal{O}_{\mathbb{P}(V)}(d)) \in A(\mathbb{P}(V)) \cong \mathbb{Z}[t]/(t^r)$ is invertible and by Theorem B.2.1 also $c(\mathcal{F})$ for every direct sum of Serre twists. To define the Chern polynomial for coherent sheaves on $\mathbb{P}(V)$, we need the following consequence of Hilbert's Syzygy Theorem.

Theorem B.3.1. *(cf. [Ben16, (6.12.5)]) Every coherent $\mathcal{O}_{\mathbb{P}(V)}$-module has a resolution of length at most r in which each term is a finite direct sum of Serre-twists.*

Definition. Let $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}(V))$ be a coherent sheaf and

$$0 \longrightarrow \mathcal{F}_r \longrightarrow \mathcal{F}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

be a resolution such that each $\mathcal{F}_i$ is a direct sum of Serre twists. We define

$$c'(\mathcal{F}) \coloneqq \prod_{i=0}^{r} c(\mathcal{F}_i)^{(-1)^i} \in A(\mathbb{P}(V)),$$

where $c(0) \coloneqq 1$. Note that this is well-defined since each $c(\mathcal{F}_i)$ is invertible by the previous considerations. Moreover, the definition does not depend on the choice of the resolution of the form as in Theorem B.3.1 (cf. [EH16, §14.2.1]).

Remark B.3.2. Assume that $\mathcal{F}$ is a vector bundle, then the Whitney sum formula implies that $c(\mathcal{F}) = c'(\mathcal{F})$. Therefore, we may define $c(\mathcal{F}) \coloneqq c'(\mathcal{F})$ for all $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}(V))$.

Theorem B.3.3. *(cf. [Ben16, (7.3.7)]) The assignment $c\colon \mathrm{Coh}(\mathbb{P}(V)) \longrightarrow A(\mathbb{P}(V))$ is the unique function that satisfies the following properties.*

(1) If $0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$ is a short exact sequence of coherent sheaves then

$$c(\mathcal{F}_2) = c(\mathcal{F}_1) \cdot c(\mathcal{F}_3).$$

(2) $c(\mathcal{O}_{\mathbb{P}(V)}(d)) = 1 + d \cdot t$ for all $d \in \mathbb{Z}$.

Definition. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(V)$. We denote by

$$c(\mathcal{F}) = 1 + \sum_{i=1}^{r} c_i(\mathcal{F}) t^i \in \mathbb{Z}[t]/(t^r)$$

the *Chern polynomial* of $\mathcal{F}$. The coefficient $c_i(\mathcal{F}) \in \mathbb{Z}$ is called the i-th *Chern class* of $\mathcal{F}$.

Remark B.3.4. In contrast to Theorem B.2.2, a coherent sheaf on $\mathbb{P}(V)$ that is not a vector bundle may have Chern classes $c_i(\mathcal{F}) \neq 0$ with $i > \mathrm{rk}(\mathcal{F})$.

C. Principal bundles

We recall the definition of principal bundles locally trivial in the Zariski topology and record basic results with detailed proofs. This chapter is based on [Ser58] and [Cra20]. Throughout this chapter G denotes an algebraic group. For each G-variety E under consideration, the action of G on E is a right action $E \times G \longrightarrow E\,;\ (e,g) \mapsto e.g$. Recall that by the definition of a G-variety the map $E \times G \longrightarrow E\,;\ (e,g) \mapsto e.g$ is a morphism of varieties.

C.1. Preliminaries

Definition. Let X be a variety.

(1) Let E, E' be G-varieties. A *G-equivariant morphism* $f\colon E \longrightarrow E'$ is a morphism f of varieties such that $f(e.g) = f(e).g$ for all $g \in G$, $e \in E$.

(2) A *G-fibration (over X)* is a pair (E,p), where E is a G-variety and $p\colon E \longrightarrow X$ is a surjective morphism of varieties such that $p(e.g) = p(e)$ for all $e \in E$ and all $g \in G$.

(3) A *morphism* $f\colon (E,p) \longrightarrow (F,q)$ of G-fibrations over X is a G-equivariant morphism $f\colon E \longrightarrow F$ of varieties such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

commutes.

Definition. (a) A G-fibration (E,p) over X is called *trivial* if it is isomorphic to $(X \times G, \mathrm{pr}_X)$ in the category of G-fibrations over X with morphisms being the G-equivariant morphisms. The G-fibration structure on $(X \times G, \mathrm{pr}_X)$ is given by

$$(x,g).g' := (x, gg') \text{ for all } x \in X, g, g' \in G.$$

(b) A G-fibration (E,p) over X is called *principal G-bundle locally trivial in the Zariski topology* if there is an open cover $X = \bigcup_{i\in I} U_i$ such that for all $i \in I$ we have an isomorphism of G-fibrations over U_i

$$f_i\colon (p^{-1}(U_i), p|_{p^{-1}(U_i)}) \longrightarrow (U_i \times G, \mathrm{pr}_{U_i}).$$

Remark C.1.1. Let $p\colon E \longrightarrow X$ be a principal G-bundle locally trivial in the Zariski topology and $x \in X$. We find $i \in I$ such that $x \in U_i$. We consider the trivialization

$$f_i\colon p^{-1}(U_i) \longrightarrow U_i \times G.$$

Let $e, e' \in p^{-1}(x) \subseteq p^{-1}(U_i)$. We find $g, g' \in G$ such that $f_i(e) = (x, g)$, $f_i(e') = (x, g')$ and conclude

$$f_i(e.g^{-1}g') = f_i(e).g^{-1}g' = (x, g') = f_i(e'),$$

hence $e.g^{-1}g' = e'$. This shows that G acts transitively on each fiber. Let $h \in G$ be such that $e.h = e$. We obtain

$$(x, g) = f_i(e) = f_i(e.h) = (x, g).h = (x, gh)$$

and conclude $h = 1_G$. In conclusion, G acts freely and transitively on each fiber.

Lemma C.1.2. *Let E be a G-variety, X be a variety and $p\colon E \longrightarrow X$ be a map that is constant on G-orbits. Assume there exists an open cover $(U_J)_J$ of X such that for each J there exists a G-equivariant isomorphism of varieties $\chi_J\colon U_J \times G \longrightarrow p^{-1}(U_J)$ and a commutative diagram*

$$\begin{array}{ccc} U_J \times G & \xrightarrow{\chi_J} & p^{-1}(U_J) \\ & {\scriptstyle \mathrm{pr}_{U_J}} \searrow \quad \swarrow {\scriptstyle p} & \\ & U_J. & \end{array}$$

Then $p\colon E \longrightarrow X$ a principal G-bundle locally trivial in the Zariski topology. In particular, p is a morphism of varieties.

Proof. Note that p being constant on G-orbits implies that $p^{-1}(U_J)$ is G-invariant, hence we can talk about χ_J being G-equivariant. The diagram above shows that

$$\mathrm{pr}_{U_J} \circ \chi_J^{-1} = p|_{p^{-1}(U_J)}\colon p^{-1}(U_J) \longrightarrow U_J \subseteq X$$

is a morphism of varieties. Since $(p^{-1}(U_J))_J$ is an open cover of E, we conclude with Lemma 3.1.2 that $p\colon E \longrightarrow X$ is a morphism. Note that the diagram also shows that $p|_{p^{-1}(U_J)}\colon p^{-1}(U_J) \longrightarrow U_J$ is surjective and therefore the same is true for $p\colon E \longrightarrow X$. □

C.2. Properties of principal bundles

Let X be a G-variety. We denote by X/G the set of orbits and by $\pi_X\colon X \longrightarrow X/G$ the orbit map. We define a space with functions $(X/G, \mathcal{O}_{X/G})$ by letting $U \subseteq X/G$ be open if and only if $\pi_X^{-1}(U)$ is open in X and define

$$\mathcal{O}_{X/G}(U) \coloneqq \{r\colon U \longrightarrow \Bbbk \mid r \circ \pi_X \in \mathcal{O}_X(\pi_X^{-1}(U))\}.$$

By construction $\pi_X\colon X \longrightarrow X/G$ is a morphism of spaces with functions. Since for $U \subseteq X$ open we have $\pi_X^{-1}(\pi_X(U)) = \bigcup_{g\in G} U.g$, the map is also open.

Definition. If the space with functions $(X/G, \mathcal{O}_{X/G})$ is a variety, we call the space a *geometric quotient*.

Lemma C.2.1. *(1) Let X be a variety, then $X \times G$ obtains the structure of a G-variety via $(x,g).g' := (x, gg')$ and we have an isomorphism $(X \times G)/G \cong X$ of spaces with functions. In particular, $(X \times G)/G \cong X$ is a geometric quotient.*

(2) Assume that $\varphi\colon X \longrightarrow Y$ is a G-equivariant isomorphism of varieties, then the bijection $\overline{\varphi}\colon X/G \longrightarrow Y/G\ ;\ xG \mapsto \varphi(x)G$ is an isomorphism of spaces with functions. In particular, X/G is a geometric quotient if and only if Y/G is a geometric quotient.

Proof. (1) We consider the open (see [Cra20, §1.1.E]) and surjective morphism of varieties $p\colon X \times G \longrightarrow X; (x,g) \mapsto x$. Let $U \subseteq X$ be such that $p^{-1}(U)$ is open, then $p(p^{-1}(U)) = U$ is open. We conclude that U is open if and only if $p^{-1}(U)$ is open in $X \times G$. We consider the mutually inverse maps

$$f\colon (X \times G)/G \longrightarrow X\ ;\ (x,g)G \mapsto x, \text{and}$$
$$g\colon X \longrightarrow (X \times G)/G\ ;\ x \mapsto (x, 1_G)G.$$

Let $U \subseteq X$ be open, then $f^{-1}(U) = \pi_{X\times G}(p^{-1}(U))$ is open. Hence f is continuous. We let $r\mathcal{O}_X(U)$ be a regular map. We would like to show that $r \circ f\colon f^{-1}(U) \longrightarrow \Bbbk \in \mathcal{O}_{X\times G/G}(f^{-1}(U))$. By definition this is equivalent to $r \circ f \circ \pi_{X\times G} \in \mathcal{O}_{X\times G}(\pi^{-1}_{X\times G}(f^{-1}(U)))$. We note that $f \circ \pi_{X\times G} = p$ and therefore

$$r \circ f \circ \pi_{X\times G} = r \circ p \in \mathcal{O}_{X\times G}(p^{-1}(U)) = \mathcal{O}_{X\times G}(\pi^{-1}_{X\times G}(f^{-1}(U))).$$

Hence f is a morphism of spaces with functions. Note that $g = \pi_{X\times G} \circ \iota_X\colon X \longrightarrow (X \times G)/G$ is a morphism of spaces with functions, where $\iota_X\colon X \longrightarrow X \times G\ ;\ x \mapsto (x, 1_G)$.

(2) We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow{\overline{\varphi}} & Y/G. \end{array}$$

Let $U \subseteq Y/G$ be open. Since π_X is open, we conclude that $\overline{\varphi}^{-1}(U) = \pi_X(\varphi^{-1}(\pi_Y^{-1}(U)))$ is open. Let $r\colon U \longrightarrow \Bbbk \in \mathcal{O}_{Y/G}(U)$. We show that $r \circ \overline{\varphi} \in \mathcal{O}_{X/G}(\overline{\varphi}^{-1}(U))$. By definition this is equivalent to $r \circ \overline{\varphi} \circ \pi_X \in \mathcal{O}_X(\pi_X^{-1}(\overline{\varphi}^{-1}(U))) = \mathcal{O}_X((\pi_Y \circ \varphi)^{-1}(U))$. Since $r \circ \overline{\varphi} \circ \pi_X = r \circ (\pi_Y \circ \varphi)$ and $\pi_Y \circ \varphi$ is a morphism of spaces with functions, we conclude $r \circ \overline{\varphi} \circ \pi_X \in \mathcal{O}_X((\pi_Y \circ \varphi)^{-1}(U)) = \mathcal{O}_X(\pi_X^{-1}(\overline{\varphi}^{-1}(U)))$. This shows that $\overline{\varphi}$ is a morphism of spaces with functions. By considering the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi^{-1}} & X \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} \\ Y/G & \xrightarrow{\overline{\varphi}^{-1}} & X/G \end{array}$$

we conclude in the same fashion that $\overline{\varphi}^{-1}$ is a morphism of spaces with functions.

□

Corollary C.2.2. *Let $p\colon E \longrightarrow X$ be a principal G-bundle locally trivial in the Zariski topology, then p induces an isomorphism $E/G \cong X$ of spaces with functions. In particular, E/G is a geometric quotient.*

Proof. Let $X = \bigcup_{i\in I} U_i$ be a cover of X such that for each $i \in I$ we have a commutative diagram

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{f_i} & U_i \times G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_{U_i}} & \\ & U_i. & \end{array}$$

with f_i being an G-equivariant isomorphism. We apply Lemma C.2.1(1),(2) and conclude that

$$q_i\colon U_i \longrightarrow (U_i \times G)/G \longrightarrow p^{-1}(U_i)/G\ ;\ x \mapsto (x, 1_G)G \mapsto f_i^{-1}(x, 1_G)G$$

is an isomorphism of varieties. Note that $p^{-1}(U_i)/G = \pi_E(p^{-1}(U_i))$ is an open subset of E/G and $E/G = \bigcup_{j\in I} p^{-1}(U_j)/G$ is an open cover.

We consider q_i as morphism $q_i\colon U_i \longrightarrow p^{-1}(U_i)/G \subseteq E/G$. Let $x \in U_j \cap U_i$, then $p(f_j^{-1}(x, 1_G)) = \mathrm{pr}_{U_j}(x, 1_G) = \mathrm{pr}_{U_i}(x, 1_G) = p(f_i^{-1}(x, 1_G))$ and Remark C.1.1 implies

$$q_j(x) = f_j^{-1}(x, 1_G)G = f_i^{-1}(x, 1_G)G = q_i(x).$$

Therefore, we have $q_j|_{U_j\cap U_i} = q_i|_{U_j\cap U_i}$ and Lemma 3.1.2 yields a morphism $q\colon X \longrightarrow E/G$ such that $q|_{U_i} = q_i$ for all $i \in I$. We consider the family of morphisms

$$q_i^{-1}\colon p^{-1}(U_i)/G \longrightarrow U_i \subseteq X\ ;\ eG \mapsto \mathrm{pr}_{U_i}(f_i(e)).$$

Let $eG \in p^{-1}(U_i)/G \cap p^{-1}(U_j)/G$, i.e. $e \in p^{-1}(U_i) \cap p^{-1}(U_j) = p^{-1}(U_i \cap U_j)$. Then

$$q_i^{-1}(eG) = \mathrm{pr}_{U_i}(f_i(e)) = p(e) = \mathrm{pr}_{U_j}(f_j(e)) = q_j^{-1}(eG).$$

Using Lemma 3.1.2 we obtain a morphism of spaces with functions $q^{-1}\colon E/G \longrightarrow X$ of spaces with functions such that $q^{-1}|_{p^{-1}(U_i)/G} = q_i^{-1}$. This is the inverse of q. □

C.3. Example

In this section give an example of a principal bundle locally trivial in the Zariski topology. This section is based on Section 2.4 of [Cra20]. Let V be a finite dimensional $\Bbbk$-vector space, $1 \leq d < r = \dim_\Bbbk V$ and $\mathfrak{v} \subseteq V$ be a space of dimension d. We consider the Plücker embedding

$$\mathrm{pl}_{V,d}\colon\ \mathrm{Gr}_d(V) \longrightarrow \mathbb{P}(\bigwedge^d V)\ ;\ \langle w_1, \ldots, w_d\rangle \longrightarrow \Bbbk(w_1 \wedge \ldots \wedge w_d),$$

the quasi-affine variety (see Example 3.2.1)

$$\mathrm{Inj}_\Bbbk(\mathfrak{v}, V) \coloneqq \{f \in \mathrm{Hom}_\Bbbk(\mathfrak{v}, V) \mid \ker f = \{0\}\}$$

and the map

$$\mathsf{im}\colon\ \mathrm{Inj}_\Bbbk(\mathfrak{v}, V) \longrightarrow \mathrm{Gr}_d(V)\ ;\ f \mapsto \mathrm{im}\, f.$$

We note that $\mathrm{GL}(\mathfrak{v})$ acts on $\mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V)$ via $f.g = f \circ g$. Moreover, im is constant on $\mathrm{GL}(\mathfrak{v})$-orbits. We fix a basis $(v_1, \dots, v_r)$ of V and denote by $\mathcal{S}_{d,r} \subseteq \mathcal{P}(\{1, \dots, r\})$ the collection of subsets with d elements. Given $\{j_1 < j_2 < \cdots < j_d\} = J \in \mathcal{S}_{d,r}$ we define $v_J \coloneqq v_{j_1} \wedge v_{j_2} \cdots \wedge v_{j_d}$ and obtain the basis $\{v_J \mid J \in \mathcal{S}_{d,r}\}$ of $\bigwedge^d V$. Recall (see [Per08, (III.8.5)]) that the collection of open sets

$$D_J \coloneqq \{\Bbbk w \in \mathbb{P}(\bigwedge^d V) \mid (v_J)^*(w) \neq 0\}$$

is an affine cover of $\mathbb{P}(\bigwedge^d V)$. We fix $J \in \mathcal{S}_{d,r}$, set $J' \coloneqq \{1, \dots, r\} \setminus J$ and define $V_J \coloneqq \langle v_j \mid j \in J\rangle \in \mathrm{Gr}_d(V)$ and $V_{J'} \coloneqq \langle v_j \mid j \in J'\rangle \in \mathrm{Gr}_{r-d}(V)$. Moreover, we denote by $\pi_J \colon V \longrightarrow V_J \,;\, v_J + v_{J'} \mapsto v_J$ the projection onto V_J.

Lemma C.3.1. *The following statements hold for $J \in \mathcal{S}_{d,r}$.*

(1) We have

$$\begin{aligned} O_J \coloneqq \mathrm{pl}_{V,d}^{-1}(D_J) &= \{\mathfrak{u} \in \mathrm{Gr}_d(V) \mid \ker(\pi_J|_{\mathfrak{u}} \colon \mathfrak{u} \longrightarrow V_J) = \{0\}\} \\ &= \{\mathfrak{u} \in \mathrm{Gr}_d(V) \mid \mathfrak{u} \oplus V_{J'} = V\}. \end{aligned}$$

(2) We have $\mathsf{im}^{-1}(O_J) = \{f \in \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V) \mid \pi_J \circ f \in \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V_J)\}$.

(3) The set $\mathsf{im}^{-1}(O_J) \subseteq \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V)$ *is open.*

Proof. (1) We may assume that $J = \{1, \dots, d\}$. Let $\mathfrak{u} \in \mathrm{Gr}_d(V)$ and $(b_1, \dots, b_d)$ be a basis of $\mathfrak{u}$. We write $b_l = \sum_{i=1}^r \lambda_{il} v_i$ for $1 \leq l \leq d$ and obtain a matrix $A = (\lambda_{il}) \in \mathrm{Mat}_d(\Bbbk)$. We assume that $\mathfrak{u} \in \mathrm{pl}_{V,d}^{-1}(D_J)$ and obtain

$$0 \neq (v_J)^*(b_1 \wedge \cdots \wedge b_d) = \det(v_i^*(b_l))_{1 \leq i,l \leq d} = \det(\lambda_{il})_{1 \leq i,l \leq d} = \det(A).$$

Let $u = \sum_{l=1}^d \beta_l b_l \in \ker \pi_J \cap \mathfrak{u}$. We have $u = \sum_{i=1}^r (\sum_{l=1}^d \lambda_{il}\beta_l) v_i$ and obtain $0 = \pi_J(u) = \sum_{i \in J}^d (\sum_{l=1}^d \lambda_{il}\beta_l) v_i$. This implies $\sum_{l=1}^d \lambda_{il}\beta_l = 0$ for all $i \in J = \{1, \dots, d\}$ and $\det(A) = 0$ gives us $(\beta_1, \dots, \beta_d) = 0$. This shows $u = 0$ and therefore $\ker(\pi_J|_{\mathfrak{u}} \colon \mathfrak{u} \longrightarrow V_J) = \{0\}$.

Now we assume that

$$\{0\} = \ker(\pi_J|_{\mathfrak{u}} \colon \mathfrak{u} \longrightarrow V_J) = \{\sum_{i=1}^d \beta_i b_i \mid \forall i \in J \colon \sum_{l=1}^d \lambda_{il}\beta_l = 0\}.$$

Hence $\det(A) \neq 0$ and therefore $0 \neq \det(A) = (v_J)^*(b_1 \wedge \cdots \wedge b_d)$. This shows $\mathrm{pl}_{V,d}(\mathfrak{u}) = \Bbbk(b_1 \wedge \cdots \wedge b_d) \in D_J$. We have shown that $\mathrm{pl}_{V,d}^{-1}(D_J) = \{\mathfrak{u} \in \mathrm{Gr}_d(V) \mid \ker(\pi_J|_{\mathfrak{u}} \colon \mathfrak{u} \longrightarrow V_J) = \{0\}\}$. Now we assume that $\mathfrak{u} \oplus V_{J'} = V$ and consider $x \in \ker(\pi_J|_{\mathfrak{u}})$. We write $x = x_J + x_{J'}$ and obtain $0 = \pi_J|_{\mathfrak{u}}(x) = x_J$ and therefore $x = x_{J'} \in V_{J'} \cap \mathfrak{u} = \{0\}$. If $\ker(\pi_J|_{\mathfrak{u}}) = \{0\}$, then $\mathfrak{u} \cap V_{J'} = \ker(\pi_J|_{\mathfrak{u}}) = \{0\}$. Since $\dim_{\Bbbk} \mathfrak{u} + \dim_{\Bbbk} V_{J'} = \dim_{\Bbbk} V$, we conclude $\mathfrak{u} \oplus V_{J'} = V$.

(2) Let $f \in \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V)$. We have $f \in (\mathrm{pl}_{V,d} \circ \mathsf{im})^{-1}(D_J)$ if and only if $\mathrm{im}\, f \in \mathrm{pl}_{V,d}^{-1}(D_J)$. By (1) this is equivalent to the injectivity of $\pi_J|_{\mathrm{im}\, f} \colon \mathrm{im}\, f \longrightarrow V_J$, i.e. $\ker(\pi_J \circ f) = \{0\}$.

(3) We consider the lower semicontinuous map (cf. Example 3.2.1)

$$\mathrm{rk}\colon \mathrm{Hom}_{\Bbbk}(\mathfrak{v}, V_J) \longrightarrow \mathbb{N}_0; f \mapsto \mathrm{rk}(f),$$

and the open set $\mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V_J) = \mathrm{rk}^{-1}(\mathbb{N}_{>\dim \mathfrak{v}-1}) \subseteq \mathrm{Hom}_{\Bbbk}(\mathfrak{v}, V_J)$. Since $(\pi_J)_*\colon \mathrm{Hom}_{\Bbbk}(\mathfrak{v}, V) \longrightarrow \mathrm{Hom}_{\Bbbk}(\mathfrak{v}, V_J)\ ;\ f \mapsto \pi_J \circ f$ is a morphism and therefore continuous, we conclude with (2) that

$$(\pi_J)_*{}^{-1}(\mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V_J)) = \mathrm{pl}_{V,d}^{-1}(O_J)$$

is open.

□

Theorem C.3.2. *The map* $\mathsf{im}\colon \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V) \longrightarrow \mathrm{Gr}_d(V)$ *is a principal* $\mathrm{GL}(\mathfrak{v})$*-bundle locally trivial in the Zariski topology.*

Proof. Let $J \in \mathcal{S}_{d,r}$. We fix a $\Bbbk$-linear isomorphism $\zeta\colon V_J \longrightarrow \mathfrak{v}$. Given $(\mathfrak{w}, g) \in O_J \times \mathrm{GL}(\mathfrak{v})$ we have a $\Bbbk$-linear isomorphism $\pi_J|_{\mathfrak{w}}\colon \mathfrak{w} \longrightarrow V_J$. Hence

$$\Psi(\mathfrak{w}, g) \coloneqq \pi_J{}_{|\mathfrak{w}}^{-1} \circ \zeta^{-1} \circ g \in \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V)$$

is a well-defined. Let $v \in \ker(\pi_J \circ \pi_J{}_{|\mathfrak{w}}^{-1} \circ \zeta^{-1} \circ g)$. We obtain $\pi_J{}_{|\mathfrak{w}}^{-1}(\zeta^{-1}(g(v))) \in V_{J'} \cap \mathfrak{w} = \{0\}$ and therefore $0 = \zeta^{-1}(g(v))$ and $v = 0$. This shows that $\pi_J \circ \Psi(\mathfrak{w}, g) \in \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V_J)$. Hence $\Psi(\mathfrak{w}, g) \in \mathsf{im}^{-1}(O_J)$ by Lemma C.3.1(2). We obtain a morphism of varieties

$$\Psi\colon O_J \times \mathrm{GL}(\mathfrak{v}) \longrightarrow \mathsf{im}^{-1}(O_J)\ ;\ (\mathfrak{w}, g) \mapsto \pi_J{}_{|\mathfrak{w}}^{-1} \circ \zeta^{-1} \circ g.$$

By [EH16, §3.2.2] the map

$$\Theta\colon \mathrm{Hom}_{\Bbbk}(V_J, V_{J'}) \longrightarrow O_J\ ;\ h \mapsto \{v + h(v) \mid v \in V_J\}$$

is an isomorphism of varieties. Let $f \in \mathsf{im}^{-1}(O_J)$, then $\pi_J \circ f\colon \mathfrak{v} \longrightarrow V_J$ is an isomorphism and

$$\Phi(f) \coloneqq (\Theta((\pi_{J'} \circ f) \circ (\pi_J \circ f)^{-1}), \zeta \circ \pi_J \circ f) \in O_J \times \mathrm{GL}(\mathfrak{v})$$

is well-defined. We obtain a morphism of varieties

$$\Phi\colon \mathsf{im}^{-1}(O_J) \longrightarrow O_J \times \mathrm{GL}(\mathfrak{v}).$$

In the following we show that Ψ and Φ are mutually inverse to each other. Let $f \in \mathsf{im}^{-1}(O_J)$, then

$$(\Psi \circ \Phi)(f) = \Psi(\Theta((\pi_{J'} \circ f) \circ (\pi_J \circ f)^{-1}), \zeta \circ \pi_J \circ f) = \pi_J{}_{|\Theta((\pi_{J'} \circ f) \circ (\pi_J \circ f)^{-1})}^{-1} \circ \pi_J \circ f.$$

We have

$$\pi_J{}_{|\Theta((\pi_{J'} \circ f) \circ (\pi_J \circ f)^{-1})}\colon \{v + ((\pi_{J'} \circ f) \circ (\pi_J \circ f)^{-1})(v) \mid v \in V_J\} \longrightarrow V_J\ ;\ v + ((\pi_{J'} \circ f) \circ (\pi_J \circ f)^{-1})(v) \mapsto v$$

and obtain $(\Psi \circ \Phi)(f) = \pi_J \circ f + (\pi_{J'} \circ f) \circ (\pi_J \circ f)^{-1} \circ \pi_J \circ f = \pi_J \circ f + \pi_{J'} \circ f = f$.

Let $(\mathfrak{m}, g) \in O_J \times \mathrm{GL}(\mathfrak{v})$. Since $\pi_J \circ \pi_J|_{\mathfrak{w}}^{-1} = \mathrm{id}_{V_J}$, we obtain

$$x \coloneqq (\pi_{J'} \circ \pi_J|_{\mathfrak{w}}^{-1} \circ \zeta^{-1} \circ g) \circ (\pi_J \circ \pi_J|_{\mathfrak{w}}^{-1} \circ \zeta^{-1} \circ g)^{-1} = \pi_{J'} \circ \pi_J|_{\mathfrak{w}}^{-1}.$$

Moreover, we have

$$\pi_J|_{\mathfrak{w}} \colon \mathfrak{w} = \Theta(\Theta^{-1}(\mathfrak{w})) \longrightarrow V_J \; ; \; v + \Theta^{-1}(\mathfrak{w})(v) \mapsto v$$

and conclude with $\operatorname{im} \Theta^{-1}(\mathfrak{w}) \subseteq V_{J'}$ that $\pi_{J'}(v + \Theta^{-1}(\mathfrak{w})(v)) = \Theta^{-1}(\mathfrak{w})(v)$ for all $v \in V_J$. Hence

$$\begin{aligned}\Theta(x) &= \{v + x(v) \mid v \in V_J\} = \{v + (\pi_{J'} \circ \pi_J|_{\mathfrak{w}}^{-1})(v) \mid v \in V_J\} \\ &= \{v + \Theta^{-1}(\mathfrak{w})(v) \mid v \in V_J\} = \Theta(\Theta^{-1}(\mathfrak{w})) = \mathfrak{w}.\end{aligned}$$

Finally we obtain,

$$(\Phi \circ \Psi)(\mathfrak{w}, g) = \Phi(\pi_J|_{\mathfrak{w}}^{-1} \circ \zeta^{-1} \circ g) = (\Theta(x), \zeta \circ \pi_J \circ \pi_J|_{\mathfrak{w}}^{-1} \circ \zeta^{-1} \circ g) = (\mathfrak{w}, g).$$

We have proven that Ψ is an isomorphism of varieties that is $\mathrm{GL}(\mathfrak{v})$-equivariant by definition. Furthermore, we have

$$\mathsf{im}(\Psi(\mathfrak{w}, g)) = \operatorname{im}(\pi_{J|_{\mathfrak{w}}}^{-1} \circ \zeta^{-1} \circ g) = \operatorname{im} \pi_{J|_{\mathfrak{w}}}^{-1} = \mathfrak{w} = \mathrm{pr}_{O_J}(\mathfrak{w}, g)$$

and therefore a commutative diagram

$$\begin{array}{ccc} O_J \times \mathrm{GL}(\mathfrak{v}) & \xrightarrow{\Psi} & \mathsf{im}^{-1}(O_J) \\ & {\scriptstyle \mathrm{pr}_{O_J}} \searrow \quad \swarrow {\scriptstyle \mathsf{im}} & \\ & O_J. & \end{array}$$

Now we apply Lemma C.1.2. □

Corollary C.3.3. *Let $\mathfrak{v} \in \mathrm{Gr}_d(V)$. We have an isomorphism of spaces with functions*

$$\overline{\mathrm{im}} \colon \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V)/\mathrm{GL}(\mathfrak{v}) \longrightarrow \mathrm{Gr}_d(V) \; ; \; f\,\mathrm{GL}(\mathfrak{v}) \mapsto \operatorname{im} f.$$

In particular, $\mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V)/\mathrm{GL}(\mathfrak{v})$ is a geometric quotient and $\mathrm{Gr}_d(V)$ is irreducible.

Proof. We apply Corollary C.2.2 to Theorem C.3.2 and conclude that $\overline{\mathrm{im}}$ is an isomorphism. It therefore suffices to show that $\mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V)/\mathrm{GL}(\mathfrak{v})$ is irreducible. This is trivial since $\mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V) \subseteq \mathrm{Hom}_{\Bbbk}(\mathfrak{v}, V)$ is an open set and $\pi \colon \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V) \longrightarrow \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V)/\mathrm{GL}(\mathfrak{v})$ is a surjective morphism. □

C.4. Another Example

This example is well-known. Since we did not find a proof in the literature, we provide a proof, although the result will not be needed in this thesis.

Recall that from [Mum99, (I.§6.4)] that a prevariety X is separated (and therefore a variety) if and only if for each prevariety Y and all morphisms $\varphi_1, \varphi_2 \colon Y \longrightarrow X$ the set $\{y \in Y \mid \varphi_1(y) = \varphi_2(y)\} \subseteq Y$ is closed. Let E, X be varieties and $p \colon E \longrightarrow X$ be a morphism. We define

$$E \times_X E \coloneqq \{(e, f) \in E \times E \mid p(e) = p(f)\} \subseteq E \times E$$

and let $\mathrm{pr}_i\colon E \times E \longrightarrow E$ be the projection onto the i-th coordinate for $i \in \{1,2\}$. Since $E \times E$ is a variety,

$$E \times_X E = \{(e,f) \in E \times E \mid (p \circ \mathrm{pr}_1)(e,f) = (p \circ \mathrm{pr}_2)(e,f)\}$$

is closed in $E \times E$ and obtains the structure of a variety.

Let $X = \bigcup_{i\in I} U_i$ be an open cover and $p\colon E \longrightarrow X$ be a morphism, then $p^{-1}(U_i) \times p^{-1}(U_i)$ is open in $E \times E$. Let $(e,f) \in E \times_X E$, then there exists i such that $p(e) = p(f) \in U_i$. Hence $(e,f) \in p^{-1}(U_i) \times p^{-1}(U_i)$ and therefore $(e,f) \in p^{-1}(U_i) \times_X p^{-1}(U_i) \coloneqq (p^{-1}(U_i) \times p^{-1}(U_i)) \cap E \times_X E$. This shows that $E \times_X E = \bigcup_{i\in I} p^{-1}(U_i) \times_X p^{-1}(U_i)$.

Let E be a G-variety and $p\colon E \longrightarrow X$ be a map constant on G-orbits such that G acts transitively and freely on fibers. Given $(e,f) \in E \times_X E = \{(a,b) \in E \times E \mid p(a) = p(b)\}$ there exists a unique element $g_{(e,f)} \in G$ such that $e = f.g_{(e,f)}$. We define

$$\Gamma\colon E \times_X E \longrightarrow G\,;\ (e,f) \mapsto g_{(e,f)}.$$

The following result is taken from [Ser58, Proposition 2].

Lemma C.4.1. *Let E be a G-variety, X be a variety and $p\colon E \longrightarrow X$ be a G-fibration. The following statements are equivalent.*

(1) The G-fibration p is a principal G-bundle locally trivial in the Zariski topology.

(2) (i) The group G acts freely and transitively on fibers and the map $\Gamma\colon E \times_X E \longrightarrow G\,;\ (e,f) \mapsto g_{(e,f)}$ is a morphism of varieties.

(ii) There exists an open cover $X = \bigcup_{i\in I} U_i$ together with morphisms $s_i\colon U_i \longrightarrow p^{-1}(U_i)$ such that $p \circ s_i = \mathrm{id}_{U_i}$ for all $i \in I$.

Proof. (1) $\Rightarrow$ (2). By Remark C.1.1 G acts freely and transitively on fibers. By assumption we have an open cover $X = \bigcup_{i\in I} U_i$ and G-equivariant isomorphisms $f_i\colon p^{-1}(U_i) \longrightarrow U_i \times G$ with commutative diagrams

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{\quad f_i \quad} & U_i \times G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_{U_i}} & \\ & U_i. & \end{array}$$

We have a morphism $s_i\colon U_i \longrightarrow p^{-1}(U_i)\,;\ x \mapsto f_i^{-1}(x, 1_G)$ and note that $(p \circ s_i)(x) = \mathrm{pr}_{U_i}(x, 1_G) = x$ for all $x \in U_i$. This shows (ii).

Since taking inverses and multiplication in G are morphisms, we have for each $i \in I$ a morphism

$$\Gamma_i'\colon p^{-1}(U_i) \times p^{-1}(U_i) \longrightarrow G\,;\ (e,e') \mapsto (\mathrm{pr}_G \circ f_i)(e')^{-1} \cdot (\mathrm{pr}_G \circ f_i)(e).$$

Let $(e,e') \in p^{-1}(U_i) \times_X p^{-1}(U_i)$. Since $\mathrm{pr}_{U_i} \circ f_i = p$, we find $g,h \in G$ and $u \in U_i$ such that $f_i(e) = (u,g)$ and $f_i(e') = (u,h)$. We have $(\mathrm{pr}_G \circ f_i)(e')^{-1} \cdot (\mathrm{pr}_G \circ f_i)(e) = h^{-1}g$ and

$$e'.(h^{-1}g) = f_i^{-1}(f_i(e')).(h^{-1}g) = f_i^{-1}(f_i(e').(h^{-1}g)) = f_i^{-1}((u,h).(h^{-1}g)) = f_i^{-1}(u,g) = e,$$

and conclude $g_{(e,e')} = h^{-1}g = (\mathrm{pr}_G \circ f_i)(e')^{-1} \cdot (\mathrm{pr}_G \circ f_i)(e)$. Hence restriction of Γ_i' to $p^{-1}(U_i) \times_X p^{-1}(U_i)$ gives a morphism

$$\Gamma_i\colon p^{-1}(U_i) \times_X p^{-1}(U_i) \longrightarrow G\,;\ (e,e') \mapsto (\mathrm{pr}_G \circ f_i)(e')^{-1} \cdot (\mathrm{pr}_G \circ f_i)(e) = g_{(e,e')}.$$

Since $E\times_X E = \bigcup_{i\in I} p^{-1}(U_i)\times_X p^{-1}(U_i)$ and $\Gamma|_{p^{-1}(U_i)\times_X p^{-1}(U_i)} = \Gamma_i$, we conclude with Lemma 3.1.2 that Γ is a morphism.

(2) $\Rightarrow$ (1). Let $X = \bigcup_{i\in I} U_i$ be an open cover with corresponding local sections $s_i\colon U_i \longrightarrow p^{-1}(U_i)$. Let $i \in I$. Since p is constant on G-orbits, we have a G-equivariant morphism

$$\chi_i\colon U_i \times G \longrightarrow p^{-1}(U_i)\ ;\ (y,g) \mapsto s_i(y).g.$$

We define a morphism

$$f_i\colon p^{-1}(U_i) \longrightarrow U_i \times G\ ;\ e \mapsto (p(e), \Gamma(e, s_i(p(e)))).$$

Let $(y,g) \in U_i \times G$, then p being constant on G-orbits and $p \circ s_i = \mathrm{id}_{U_i}$ imply

$$\begin{aligned}(f_i \circ \chi_i)(y,g) &= f_i(s_i(y).g) = (p(s_i(y).g), \Gamma(s_i(y).g, s_i(p(s_i(y).g))))\\ &= (y, \Gamma(s_i(y).g, s_i(y))) = (y,g).\end{aligned}$$

For $e \in p^{-1}(U_i)$ we have

$$(\chi_i \circ f_i)(e) = \chi_i(p(e), \Gamma(e, s_i(p(e)))) = s_i(p(e)).\Gamma(e, s_i(p(e))) = e.$$

This shows that f_i is a G-equivariant isomorphism. □

As before we let V be a finite dimensional $\Bbbk$-vector space with d-dimensional subspace $\mathfrak{v} \subseteq V$. We consider the parabolic subgroup

$$P_{\mathfrak{v}} := \{g \in \mathrm{GL}(V) \mid g(\mathfrak{v}) = \mathfrak{v}\}.$$

The group acts on $\mathrm{GL}(V)$ via

$$\mathrm{GL}(V) \times P_{\mathfrak{v}} \longrightarrow \mathrm{GL}(V)\ ;\ (g,h) \mapsto g \circ h$$

making $\mathrm{GL}(V)$ a $P_{\mathfrak{v}}$-variety.

Theorem C.4.2. *Let $\mathfrak{v} \in \mathrm{Gr}_d(V)$. The map*

$$\varphi_{\mathfrak{v}}\colon\ \mathrm{GL}(V) \longrightarrow \mathrm{Gr}_d(V)\ ;\ g \mapsto g(\mathfrak{v})$$

is a principal $P_{\mathfrak{v}}$-bundle locally trivial in the Zariski topology.

Proof. Let $(v_1, \ldots, v_d)$ be a basis of $\mathfrak{v}$. Note that we have a commutative diagram

$$\begin{array}{ccc}\mathrm{Gr}_d(V) & \xrightarrow{\mathrm{pl}_{V,d}} & \mathrm{im}\,\mathrm{pl}_{V,d} \subseteq \mathbb{P}(\bigwedge^d V)\\ & \nwarrow^{\varphi_{\mathfrak{v}}} & \uparrow{\scriptstyle \psi}\\ & & \mathrm{GL}(V)\end{array}$$

with ψ being the morphism

$$\psi\colon\ \mathrm{GL}(V) \longrightarrow \mathbb{P}(\bigwedge^d V); g \mapsto \Bbbk(g(v_1) \wedge \cdots \wedge g(v_d))$$

induced from the morphism

$$\mathrm{GL}(V) \times \bigwedge^d V \longrightarrow \bigwedge^d V \; ; \; (g, w_1 \wedge \cdots \wedge w_d) \mapsto g(w_1) \wedge \cdots \wedge g(w_d).$$

Hence $\varphi_{\mathfrak{v}}$ is a morphism (see Section 3.1). Moreover, $\varphi_{\mathfrak{v}}$ is a $P_{\mathfrak{v}}$-fibration. The map

$$\mathrm{GL}(V) \times \mathrm{GL}(V) \longrightarrow \mathrm{GL}(V) \; ; \; (g, h) \mapsto h^{-1}g$$

is a morphism. Let $e, f \in \mathrm{GL}(V)$ such that $e(\mathfrak{v}) = \varphi_{\mathfrak{v}}(e) = \varphi_{\mathfrak{v}}(f) = f(\mathfrak{v})$, then $f^{-1} \circ e \in P_{\mathfrak{v}}$ and $f.(f^{-1} \circ e) = e$. Therefore, we have $g_{(e,f)} = f^{-1} \circ e$. Hence

$$\mathrm{GL}(V) \times_{\mathrm{Gr}_d(V)} \mathrm{GL}(V) \longrightarrow G \; ; \; (e, f) \mapsto g_{(e,f)} = f^{-1} \circ e$$

is a morphism. This also shows that $P_{\mathfrak{v}}$ acts transitively on fibers, so that (i) of Lemma C.4.1 is satisfied, since the action is clearly free. Let $J \in \mathcal{S}_{d,r}$. Recall that

$$O_J = \{\mathfrak{u} \in \mathrm{Gr}_d(V) \mid \pi_J \circ \iota_{\mathfrak{u}} \in \mathrm{Inj}_{\Bbbk}(\mathfrak{u}, V_J)\},$$

therefore

$$\begin{aligned}\varphi_{\mathfrak{v}}^{-1}(O_J) &= \{g \in \mathrm{GL}(V) \mid \pi_J \circ \iota_{g(\mathfrak{v})} \in \mathrm{Inj}_{\Bbbk}(g(\mathfrak{v}), V_J)\} \\ &= \{g \in \mathrm{GL}(V) \mid \pi_J \circ g \circ \iota_{\mathfrak{v}} \in \mathrm{Inj}_{\Bbbk}(\mathfrak{v}, V_J)\}.\end{aligned}$$

In the following we construct a section $s_j \colon O_J \longrightarrow \varphi_{\mathfrak{v}}^{-1}(O_J)$ for

$$\varphi_{\mathfrak{v}} \colon \varphi_{\mathfrak{v}}^{-1}(O_J) \longrightarrow O_J.$$

We fix an isomorphism $\zeta \colon V \longrightarrow V$ such that $\zeta(V_J) = \mathfrak{v}$. Recall from [EH16, §3.2.2] that we have an isomorphism of varieties

$$\Theta \colon \mathrm{Hom}_{\Bbbk}(V_J, V_{J'}) \longrightarrow O_J \; ; \; f \mapsto \{f(v) + v \mid v \in V_J\}.$$

Let $\mathfrak{u} \in O_J$ and $v_J + v_{J'} \in \ker(\Theta^{-1}(\mathfrak{u}) \circ \pi_J + \mathrm{id}_V) \subseteq V$[1]. We conclude with $\operatorname{im} \Theta^{-1}(\mathfrak{u}) \subseteq V_{J'}$ and

$$0 = (\Theta^{-1}(\mathfrak{u}) \circ \pi_J + \mathrm{id}_V)(v_J + v_{J'}) = v_J + (\Theta^{-1}(\mathfrak{u})(v_J) + v_{J'}) \in V_J \oplus V_{J'}$$

that $v_J = 0$ and therefore $0 = \Theta^{-1}(\mathfrak{u})(v_J) + v_{J'} = v_{J'}$. Hence $s_J(\mathfrak{u}) \coloneqq (\Theta^{-1}(\mathfrak{u}) \circ \pi_J + \mathrm{id}_V) \circ \zeta^{-1} \in \mathrm{GL}(V)$. Now we show that $s_J(\mathfrak{u}) \in \varphi_{\mathfrak{v}}^{-1}(O_J)$. Let $v \in \ker(\pi_J \circ s_J(\mathfrak{u}) \circ \iota_{\mathfrak{v}}) \subseteq \mathfrak{v}$, then $\operatorname{im} \Theta^{-1}(\mathfrak{u}) \subseteq V_{J'} = \ker \pi_J$ and $\zeta^{-1}(v) \in V_J$ imply

$$\begin{aligned}0 &= (\pi_J \circ s_J(\mathfrak{u}) \circ \iota_{\mathfrak{v}})(v) = (\pi_J \circ (\Theta^{-1}(\mathfrak{u}) \circ \pi_J + \mathrm{id}_V) \circ \zeta^{-1} \circ \iota_{\mathfrak{v}})(v) \\ &= (\pi_J \circ \zeta^{-1})(v) = \zeta^{-1}(v).\end{aligned}$$

Hence $v = 0$ and $s_J(\mathfrak{u}) \in \varphi_{\mathfrak{v}}^{-1}(O_J)$. We obtain a morphism

$$s_J \colon O_J \longrightarrow \varphi_{\mathfrak{v}}^{-1}(O_J) \; ; \; \mathfrak{u} \mapsto (\Theta^{-1}(\mathfrak{u}) \circ \pi_J + \mathrm{id}_V) \circ \zeta^{-1}.$$

Moreover, we obtain with $\zeta^{-1}(\mathfrak{v}) = V_J$

$$\begin{aligned}\varphi_{\mathfrak{v}}(s_J(\mathfrak{u})) = s_J(\mathfrak{u})(\mathfrak{v}) &= \{((\Theta^{-1}(\mathfrak{u}) \circ \pi_J + \mathrm{id}_V) \circ \zeta^{-1})(v) \mid v \in \mathfrak{v}\} \\ &= \{(\Theta^{-1}(\mathfrak{u}) \circ \pi_J + \mathrm{id}_V)(v) \mid v \in V_J\} \\ &= \{\Theta^{-1}(\mathfrak{u})(v) + v \mid v \in V_J\} \\ &= \Theta(\Theta^{-1}(\mathfrak{u})) = \mathfrak{u}.\end{aligned}$$

We have proven that $\varphi_{\mathfrak{v}} \circ s_J = \mathrm{id}_{O_J}$. Hence (ii) of Lemma C.4.1 holds and $\varphi_{\mathfrak{v}} \colon \mathrm{GL}(V) \longrightarrow \mathrm{Gr}_d(V) \; ; \; g \mapsto g(\mathfrak{v})$ is a principal $P_{\mathfrak{v}}$-bundle locally trivial in the Zariski topology. □

[1] Note that we suppress the embedding $\iota_{J'} \colon V_{J'} \to V$ and consider $\Theta^{-1}(\mathfrak{u})$ as a map $V_J \to V$.

Bibliography

[AM15] E. Arrondo and S. Marchesi. "Jumping pairs of Steiner bundles". In: *Forum Math.* 27 (2015), pp. 3233–3267. DOI: 10.1515/forum-2013-0095.

[ARS95] M. Auslander, I. Reiten, and S. O. Smalø. *Representation Theory of Artin Algebras*. Cambridge Stud. Adv. Math. 36. Cambridge University Press, 1995. DOI: 10.1017/CBO9780511623608.

[ASS06] I. Assem, A. Skowronski, and D. Simson. *Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory*. Vol. 1. London Math. Soc. Stud. Texts 65. Cambridge University Press, 2006. DOI: 10.1017/CBO9780511614309.

[Ati56] M. Atiyah. "On the Krull-Schmidt theorem with application to sheaves". In: *Bull. Soc. Math. France* 84 (1956), pp. 307–317. DOI: 10.24033/bsmf.1475.

[Bar77] W. Barth. "Some properties of stable rank-2 vector bundles on $\mathbb{P}_n$". In: *Math. Ann.* 226 (1977), pp. 125–150. DOI: 10.1007/BF01360864.

[BE82] E. Ballico and F. Ellia. "Fibrés homogènes sur $\mathbb{P}^n$". In: *C. R. Math. Acad. Sci. Paris* 294 (1982), pp. 403–406.

[Ben16] D. J. Benson. *Representations of Elementary Abelian p-Groups and Vector Bundles*. Cambridge Tracts in Mathematics 208. Cambridge University Press, 2016. DOI: 10.1017/9781316795699.

[Ben91] D. J. Benson. *Representations and cohomology. II: Cohomology of groups and modules*. Cambridge Stud. Adv. Math. 31. Cambridge University Press, 1991. DOI: 10.1017/CBO9780511623622.

[BF24] D. Bissinger and R. Farnsteiner. *Representations of Kronecker quivers and Steiner bundles on Grassmannians*. 2024. arXiv: 2403.00079 [math.RT].

[Bis04] I. Biswas. "On the stability of homogeneous vector bundles". In: *J. Math. Sci. Univ. Tokyo* 11 (2004), pp. 133–140.

[Bis18] D. Bissinger. "Representations of regular trees and invariants of AR-components for generalized Kronecker quivers". In: *Algebr. Represent. Theory* 21 (2018), pp. 331–358. DOI: 10.1007/s10468-017-9716-x.

[Bis20] D. Bissinger. "Representations of constant socle rank for the Kronecker algebra". In: *Forum Math.* 32 (2020), pp. 23–43. DOI: 10.1515/forum-2018-0143.

[Bis23] D. Bissinger. “Dimension vectors with the equal kernels property”. In: *J. Pure Appl. Algebra* 227 (2023), Paper No. 107424, 23. DOI: 10.1016/j.jpaa.2023.107424.

[Bis24] D. Bissinger. *Shift orbits for elementary modules of Kronecker quivers*. 2024. arXiv: 2403.01824 [math.RT].

[BLV15] R.-O. Buchweitz, G. J. Leuschke, and M. Van den Bergh. “On the derived category of Grassmannians in arbitrary characteristic”. In: *Compos. Math.* 151 (2015), pp. 1242–1264. DOI: 10.1112/S0010437X14008070.

[Bon81] K. Bongartz. “Tilted algebras”. In: *Representations of Algebras*. Lecture Notes in Math. 903. Springer-Verlag, 1981, pp. 26–38. DOI: 10.1007/BFb0092979.

[Bra05] M. C. Brambilla. “Simplicity of generic Steiner bundles”. In: *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* 8 (2005), pp. 723–735. URL: http://eudml.org/doc/195093.

[Bra08] M. C. Brambilla. “Cokernel bundles and Fibonacci bundles”. In: *Math. Nachr.* 281 (2008), pp. 499–516. DOI: 10.1002/mana.200510620.

[BS92] G. Bohnhorst and H. Spindler. “The stability of certain vector bundles on $\mathbb{P}^n$”. In: *Complex algebraic varieties*. Lecture Notes in Math. 1507. Springer-Verlag, 1992, pp. 39–50. DOI: 10.1007/BFb0094509.

[CE99] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton Mathematical Series 19. Princeton University press, 1999.

[CFP12] J. F. Carlson, E. M. Friedlander, and J. Pevtsova. “Representations of elementary abelian p-groups and bundles on Grassmannians”. In: *Adv. Math.* 229 (2012), pp. 2985–3051. DOI: 10.1016/j.aim.2011.11.004.

[CFP15] J. F. Carlson, E. M. Friedlander, and J. Pevtsova. “Elementary subalgebras of Lie algebras”. In: *J. Algebra* 442 (2015), pp. 155–189. DOI: 10.1016/j.jalgebra.2014.10.015.

[Che13] B. Chen. “Dimension vectors in regular components over wild Kronecker quivers”. In: *Bull. Sci. Math.* 137 (2013), pp. 730–745. DOI: 10.1016/j.bulsci.2013.04.002.

[CHS22] I. Coskun, J. Huizenga, and G. Smith. *Stability and cohomology of kernel bundles on projective space*. 2022. arXiv: 2204.10247 [math.AG].

[CJ06] W. Crawley-Boevey and B. T. Jensen. “A note on sub-bundles of vector bundles”. In: *Glasg. Math. J.* 48 (2006), pp. 459–462. DOI: 10.1017/S0017089506003259.

[Cra20] W. Crawley-Boevey. *Noncommutative Algebra 3: Geometric methods for representations of algebras*. Lecture notes, "https://www.math.uni-bielefeld.de/~wcrawley/20noncommalg3/NA3-12.pdf". 2020.

[Cra96] W. Crawley-Boevey. "Subrepresentations of general representations of quivers". In: *Bull. London Math. Soc.* 28 (1996), pp. 363–366. DOI: 10.1112/blms/28.4.363.

[DK93] I. Dolgachev and M. Kapranov. "Arrangements of hyperplanes and vector bundles on $\mathbb{P}^n$". In: *Duke Math. J.* 71 (1993), pp. 633–664. DOI: 10.1215/S0012-7094-93-07125-6.

[Dré87] J.-M. Drézet. "Fibrés exceptionnels et variétés de modules de faisceaux semi-stables sur $\mathbb{P}_2(\mathbb{C})$." In: *J. Reine Angew. Math.* 380 (1987), pp. 14–58. DOI: 10.1515/crll.1987.380.14.

[Dré99] J.-M. Drézet. "Variétés de modules alternatives". In: *Ann. Inst. Fourier (Grenoble)* 49 (1999), pp. 57–139. DOI: 10.5802/aif.1669.

[EH16] D. Eisenbud and J. Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016. DOI: 10.1017/CBO9781139062046.

[Ele79] G. Elencwajg. "Des fibrés uniformes non homogènes". In: *Math. Ann.* 239 (1979), pp. 185–192. DOI: 10.1007/BF01420375.

[Ell17] P. Ellia. "On jumping lines of vector bundles on $\mathbb{P}^n$". In: *Ann. Univ. Ferrara Sez. VII Sci. Mat.* 63 (2017), pp. 103–112. DOI: 10.1007/s11565-017-0282-2.

[Far11] R. Farnsteiner. *Categories of modules given by varieties of p-nilpotent operators*. 2011. arXiv: 1110.2706 [math.RT].

[Fle84] H. Flenner. "Restrictions of semistable bundles on projective varieties". In: *Comment. Math. Helv.* 59 (1984), pp. 635–650. DOI: 10.1007/BF02566370.

[Ful98] W. Fulton. *Intersection Theory*. Springer-Verlag, 1998. DOI: 10.1007/978-1-4612-1700-8.

[Gat21] A. Gathmann. *Algebraic Geometry*. Lecture notes, https://agag-gathmann.math.rptu.de/class/alggeom-2021/alggeom-2021.pdf. 2021.

[Gec13] M. Geck. *An Introduction to Algebraic Geometry and Algebraic Groups*. Oxf. Grad. Texts Math. 10. Oxford University Press, 2013.

[GR87] A. L. Gorodentsev and A. N. Rudakov. "Exceptional vector bundles on projective spaces". In: *Duke Math. J.* 54 (1987), pp. 115–130. DOI: 10.1215/S0012-7094-87-05409-3.

[GW10] U. Görtz and T. Wedhorn. *Algebraic Geometry I. Schemes. With Examples and Exercises*. Advanced Lectures in Mathematics. Vieweg+Teubner, 2010. DOI: 10.1007/978-3-8348-9722-0.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics 52. Springer-Verlag, 1977. DOI: 10.1007/978-1-4757-3849-0.

[Har79] R. Hartshorne. "Algebraic vector bundles on projective spaces: a problem list". In: *Topology* 18 (1979), pp. 117–128. DOI: 10.1016/0040-9383(79)90030-2.

[Hil15] L. Hille. *Moduli of representations, quiver Grassmannians, and Hilbert schemes*. 2015. arXiv: 1505.06008 [math.RT].

[HU91] D. Happel and L. Unger. "A family of infinite-dimensional non-selfextending bricks for wild hereditary algebras". In: *Representations of finite-dimensional algebras*. CMS Conf. Proc. 11. 1991, pp. 181–189.

[Hul80] K. Hulek. "On the classification of stable rank-r vector bundles over the projective plane". In: *Vector bundles and differential equations*. Progr. Math. 7. Birkhäuser Boston, 1980, pp. 113–144. DOI: 10.1007/978-1-4684-9415-0_6.

[Hum12] J. E. Humphreys. *Linear algebraic groups*. Graduate Texts in Mathematics 21. Springer-Verlag, 2012. DOI: 10.1007/978-1-4684-9443-3.

[Huy06] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Math. Monogr. Oxford: Clarendon Press, 2006. DOI: 10.1093/acprof:oso/9780199296866.001.0001.

[JP15] M. Jardim and D. M. Prata. "Vector bundles on projective varieties and representations of quivers". In: *Algebra and Discrete Math.* 20 (2015), pp. 217–249. URL: https://admjournal.luguniv.edu.ua/index.php/adm/article/view/142.

[Kac80] V. G. Kac. "Infinite root systems, representations of graphs and invariant theory". In: *Invent. Math.* 56 (1980), pp. 57–92. DOI: 10.1007/BF01403155.

[Kac82] V. G. Kac. "Infinite root systems, representations of graphs and invariant theory, II". In: *J. Algebra* 78 (1982), pp. 141–162. DOI: 10.1016/0021-8693(82)90105-3.

[Kar93] B. V. Karpov. "Semistable sheaves on a two-dimensional quadric and Kronecker modules". In: *Russian Academy of Sciences. Izvestiya Mathematics* 40 (1993), pp. 33–66. DOI: 10.1070/IM1993v040n01ABEH001860.

[Kem93] G. R. Kempf. *Algebraic varieties*. London Math. Society Lecture Note Series 172. Cambridge University Press, 1993. DOI: 10.1017/CBO9781107359956.

[Ker94] O. Kerner. "Representations of wild quivers". In: *Representation theory of algebras and related topics*. CMS Conf. Proc. 19. American Mathematical Society, 1994, pp. 65–107.

[KL91] O. Kerner and F. Lukas. "Regular modules over wild hereditary algebras". In: *Representations of finite-dimensional algebras*. CMS Conf. Proc. 11. American Mathematical Society, 1991, pp. 191–208.

[KL96] O. Kerner and F. Lukas. "Elementary modules". In: *Math. Z.* 223 (1996), pp. 421–434.

[Kun97] E. Kunz. *Einführung in die algebraische Geometrie*. Vieweg Stud. Aufbaukurs Math. 87. Vieweg+Teubner, 1997. DOI: 10.1007/978-3-322-80313-9.

[Lan08] A. Langer. "Lectures on torsion-free sheaves and their moduli". In: *Algebraic Cycles, Sheaves, Shtukas, and Moduli*. Trends Math. Birkhäuser Verlag, 2008, pp. 69–103. DOI: 10.1007/978-3-7643-8537-8.

[Le 97] J. Le Potier. *Lectures on Vector Bundles*. Cambridge Stud. Adv. Math. 54. Cambridge University Press, 1997.

[Luk24] F. Lukas. *Prüfer modules in filtration categories of semi-bricks*. 2024. arXiv: 2402.13142 [math.RT].

[Luk92] F. Lukas. "Elementare Moduln über wilden erblichen Algebren". PhD thesis. Düsseldorf, 1992.

[Mar12] S. Marchesi. "Jumping spaces in Steiner bundles". PhD thesis. Università degli Studi di Milano, Universidad Complutense de Madrid, 2012. URL: https://air.unimi.it/handle/2434/170793.

[Mar80] M. Maruyama. "Boundedness of semistable sheaves of small ranks". In: *Nagoya Math. J.* 78 (1980), pp. 65–94. URL: http://projecteuclid.org/euclid.nmj/1118786090.

[MM21] S. Marchesi and R. M. Miró Roig. "Uniform Steiner bundles". In: *Ann. Inst. Fourier* 71 (2021), pp. 447–472. DOI: 10.5802/aif.3403.

[MS09] R. M. Miró-Roig and H. Soares. "Cohomological characterisation of Steiner bundles". In: *Forum Math.* 21 (2009), pp. 871–891. DOI: 10.1515/FORUM.2009.043.

[Mum99] D. Mumford. *The red book of varieties and schemes*. Lecture Notes in Math. 1358. Springer-Verlag, 1999. DOI: 10.1007/978-3-662-21581-4.

[OSS80] C. Okonek, M. Schneider, and H. Spindler. *Vector bundles on Complex Projective Spaces. With an Appendix by S. I. Gelfand*. Modern Birkhäuser Classics. Birkhäuser-Verlag, 1980. DOI: 10.1007/978-3-0348-0151-5.

[Per08] D. Perrin. *Algebraic Geometry: An Introduction*. Universitext. Springer-Verlag, 2008. DOI: 10.1007/978-1-84800-056-8.

[Rei24] M. Reineke. "Dimension expanders via quiver representations". In: *J. Comb. Algebra* 8 (2024), pp. 111–119. DOI: 10.4171/JCA/79.

[Rin16] C. M. Ringel. *The elementary 3-Kronecker modules*. 2016. arXiv: 1612.09141 [math.RT].

[Rin17] C. M. Ringel. "The eigenvector variety of a matrix pencil". In: *Linear Algebra Appl.* 531 (2017), pp. 447–458. DOI: 10.1016/j.laa.2017.05.004.

[Rin18] C. M. Ringel. "Kronecker modules generated by modules of length 2". In: *Representations of algebras*. Contemp. Math. 705. Amer. Math. Soc., 2018, pp. 189–214. DOI: 10.1090/conm/705/14194.

[Rin76] C. M. Ringel. "Representations of K-species and bimodules". In: *J. Algebra* 41 (1976), pp. 269–302. DOI: 10.1016/0021-8693(76)90184-8.

[Rin78] C. M. Ringel. "Finite dimensional hereditary algebras of wild representation type". In: *Math. Z.* 161 (1978), pp. 235–255. DOI: 10.1007/BF01214506.

[Rud90] A. N. Rudakov. "Exceptional collections, mutations, and helices". In: *Helices and vector bundles*. London Math. Soc. Lecture Note Series 148. Cambridge University Press, 1990. DOI: 10.1017/CBO9780511721526.001.

[Sch61a] R. L. E. Schwarzenberger. "Vector bundles on algebraic surfaces". In: *Proc. Lond. Math. Soc.* 11 (1961), pp. 601–622. DOI: 10.1112/plms/s3-11.1.601.

[Sch61b] R. L. E. Schwarzenberger. "Vector bundles on the projective plane". In: *Proc. Lond. Math. Soc.* 11 (1961), pp. 623–640. DOI: 10.1112/plms/s3-11.1.623.

[Sch92] A. Schofield. "General representations of quivers". In: *Proc. Lond. Math. Soc.* 65 (1992), pp. 46–64. DOI: 10.1112/plms/s3-65.1.46.

[Ser58] J.-P. Serre. "Espaces fibrés algébriques". In: *Séminaire Claude Chevalley* 3 (1958), pp. 1–37. URL: http://www.numdam.org/item/SCC_1958__3__A1_0/.

[Sha13a] I. R. Shafarevich. *Basic Algebraic Geometry 1. Varieties in projective space*. Springer-Verlag, 2013. DOI: 10.1007/978-3-642-37956-7.

[Sha13b] I. R. Shafarevich. *Basic Algebraic geometry 2. Schemes and complex manifolds*. Springer-Verlag, 2013. DOI: 10.1007/978-3-642-38010-5.

[SY11] A. Skowroński and K. Yamagata. *Frobenius algebras. I. Basic represention theory*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2011. DOI: 10.4171/102.

[SY17] A. Skowroński and K. Yamagata. *Frobenius algebras. II. Tilted and Hochschild Extension Algebras*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2017. DOI: 10.4171/174.

[Wor13a] J. Worch. "Categories of modules for elementary abelian p-groups and generalized Beilinson algebras". In: *J. London Math. Soc.* 88 (2013), pp. 649–668. DOI: 10.1112/jlms/jdt039.

[Wor13b] J. Worch. “Module categories and Auslander-Reiten theory for generalized Beilinson algebras”. PhD thesis. University of Kiel, 2013. URL: https://macau.uni-kiel.de/receive/diss_mods_00013419?lang=de.

Erklärung

Die nachfolgend aufgeführten Arbeiten sind während meiner Habilitationszeit in Kiel entstanden. Die Habilitationsschrift basiert auf den Artikeln (2), (3), (4) und (5).

(1) D. Bissinger. *Indecomposable Jordan types of Loewy length 2*, *J. Algebra*, 2020, DOI: https://doi.org/10.1016/j.jalgebra.2020.03.010.

(2) D. Bissinger. *Dimension vectors with the equal kernels property*, *J. Pure Appl. Algebra 227*, 2023, DOI: https://doi.org/10.1016/j.jpaa.2023.107424.

(3) D. Bissinger. *Shift orbits for elementary representations of Kronecker quivers*, Preprint, 2024, https://arxiv.org/abs/2403.01824. Zur Publikation eingereicht.

(4) D. Bissinger und Rolf Farnsteiner. *Representations of Kronecker quivers and Steiner bundles on Grassmannians*, Preprint, 2024, https://arxiv.org/abs/2403.00079. Zur Publikation eingereicht.

(5) D. Bissinger. *An adjoint pair of reflection functors and uniform Steiner bundles on projective space $\mathbb{P}^n$*, *In Vorbereitung*.

(6) D. Bissinger und Rolf Farnsteiner. *Representations of Kronecker quivers and Steiner bundles on Grassmannians, II*, *In Vorbereitung*.

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